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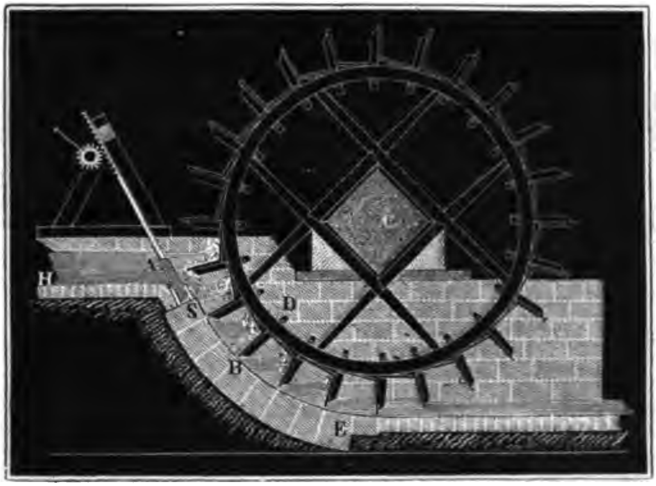


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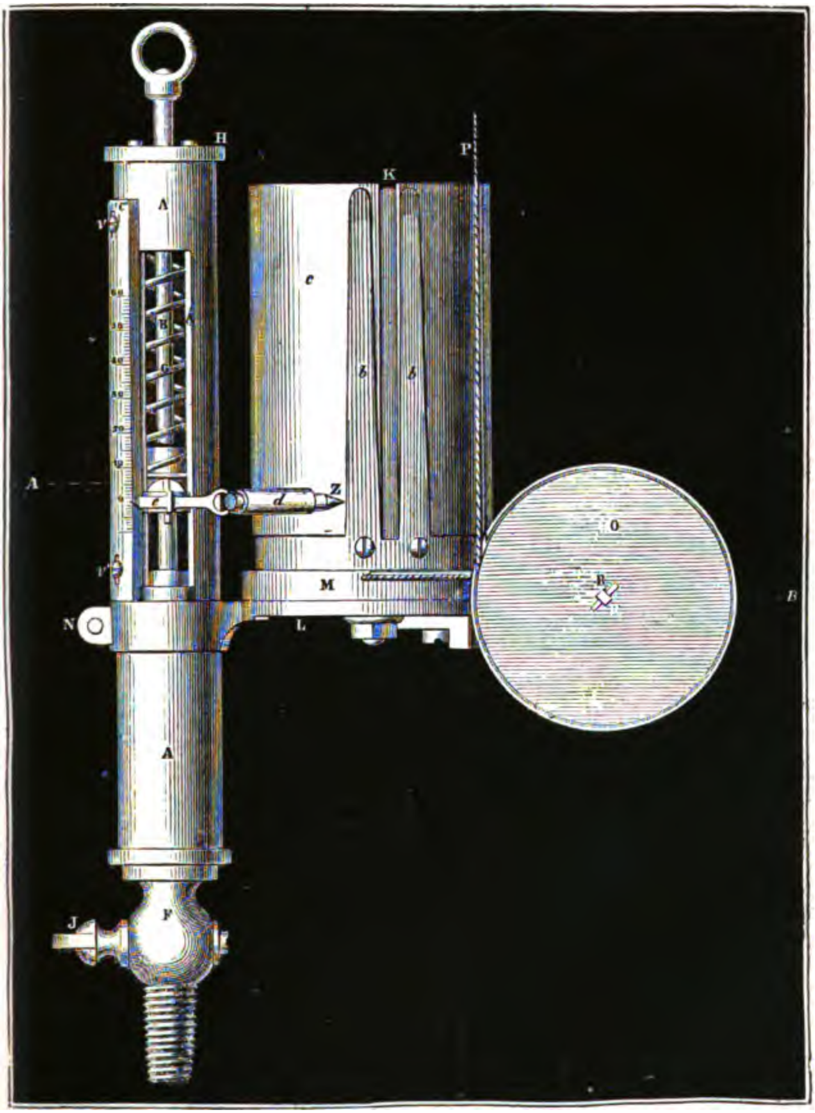


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VOL. I.

LONDON:

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1847.

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# PRINCIPLES

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BY JULIUS WEISBACH,

PROFESSOR OF MECHANICS AND APPLIED-MATHEMATICS IN THE ROYAL MINING ACADEMY OF FREIBERG.

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## VOL. I.

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# NOTICE BY THE TRANSLATOR.

From the well earned reputation of Professor Weisbach, as a teacher and original investigator, from the interest which attaches itself, especially at the present moment, to all that pertains to Mechanics or Engineering, from the able manner in which he has treated both the theoretical and practical portion of his subject, and from the variety and abundance of examples which illustrate the principles and formulæ, it is presumed that a translation of his excellent work may not be unacceptable to the English reader. "The Mechanics of Machinery and Engineering" has met with deserved favour in Germany, and is not unknown to many of the profession in our own country. The aim and objects of the undertaking are fully explained by the author in his Preface; and it is hoped that the same end may be obtained here, by affording, through the medium of a translation, valuable aid to the acquirement of professional knowledge.

With respect to the execution of the present volume, the translator will merely observe, that he has, in most instances, maintained a rigid adherence to the text, and in his endeavour to give a faithful interpretation of the author's language, which in a work of this nature is of the first importance, he trusts that no obscurity will be perceptible.

The translation is much indebted to the revisal of the proofs by Professor Gordon, who proposes, in an Appendix, to give

such a *résumé* of our present knowledge of the strength of materials as will render this part of the subject more available to practical men than has been accomplished by the manner of treating it chosen by Professor Weisbach.

The examples have been all reduced to the English measure, and formulæ given for that purpose when required. The numerical calculations have been worked out with care. Though they have not, as a verification of their accuracy, been subjected to that triple control which enables us to place implicit reliance on the original, yet it is believed that they will in general be found correct.

In the passage of the volume through the press, some errors have unavoidably appeared, a list of which will be found at the end. The work will be complete in two volumes; the second, being in the course of printing, may be expected shortly.

J. R.

London, October, 1847.

# AUTHOR'S PREFACE.

In giving to the public the First Volume of my Elementary Treatise on Mechanics, for Engineers and Machinists, I feel some degree of diffidence. Although I am conscious that, in composing the book, I have proceeded with the greatest care and circumspection; I am nevertheless apprehensive that it cannot satisfy all, since the views, wishes and requirements of different individuals differ so widely. One reader may probably consider this or that chapter too long and too minutely treated, which another may find too short. Some will miss higher science in the treatment of certain subjects, which others would have wished to see treated in a still more popular manner. But many years of study, much experience in teaching, and continued observation, have led me to the method, according to which I have composed the present work, and I consider it the most appropriate for the intended purpose.

My chief aim in writing this work was the attainment of the greatest simplicity in enunciation and proof; and with this to give the demonstration of all problems, important in their practical application, by the lower mathematics only. If we consider the great variety of knowledge which the Engineer and Machinist have to acquire, who wish to do credit to their pro-

fession, we, their instructors, should make it our duty to render well-grounded study of science easy by simplicity of explanation, by the use of only the best known and easiest auxiliary sciences, and by eschewing every thing that is unnecessary.

I have therefore avoided, in the present work, the use of the differential and integral calculus; for although there exists now more frequent opportunities, than formerly, for learning these methods, it is still unquestionable, that without constant practice, the readiness of using them is very soon lost; and that there are, consequently, many excellent practical men who have entirely forgotten how to apply them. Some popular authors give the results of the more difficult problems without proofs. I cannot approve of this, and have preferred to give the proofs of practically important problems in an elementary way, although this may sometimes appear rather long and tedious. There are, therefore, but few formulæ in the work unaccompanied by their derivation. A general acquaintance with some doctrines of natural philosophy, but especially an intimate knowledge of pure elementary mathematics are of course necessary for the understanding of the present work.

I have taken especial pains to preserve the right medium between *generalizing* and *individualizing;* for, although I am fully aware of the advantages of generalization, I must still adhere to the opinion, that in an elementary work, too much generalizing is to be avoided. Simple cases are, in practice, of more frequent occurrence than complicated. It is also undeniable, that in treating a general case, the knowledge which might be gained by the treatment of a specific case, is frequently lost, and that it is not unfrequently easier to deduce the compound from the simple, than to expiscate the special from the general.

The "Mechanics of Engineering and Machinery" must not be mistaken for a work on the construction of machines, but it is

to be considered merely as an introduction to or preparatory science to this. This Treatise of Mechanics is to stand in the same relation to the construction of machines, as descriptive geometry stands to the drawing of machines. After mechanics and descriptive geometry have been learned, the instructions on the construction and the drawing of machines may with advantage be united in one course.

The propriety of dividing the Mechanics of Engineering into a theoretical and practical part, may perhaps be doubted. If it be borne in mind, that this work is to furnish instructions on all mechanical relations, in architecture and the science of machines, the utility, or rather necessity of this division must be apparent. In order to form a complete opinion of a building or machine, the most various doctrines of mechanics (*i. e.* the doctrines of friction, strength of materials, inertia, impact, efflux, &c.), must be taken into consideration; the material for the study of the mechanics of a building, or machine must, therefore, be collected from all parts of mechanics. Now, as it is much more useful practically to be able to study the doctrines relative to every individual machine in connexion, than to have to collect them from all departments of mechanical science, the utility of the adopted division seems to be beyond all doubt.

Having practical application always in view, I have endeavoured to illustrate each doctrine, as much as possible, with appropriate examples, taken from practice; and I can truly assert, that this book excels most works of a similar nature, by the great number and appropriate choice of worked out examples. The large number of carefully executed figures will, no doubt, likewise assist the attainment of the above-mentioned object; and I cannot omit here to express how greatly satisfied I feel with

the manner in which the publishers have performed their part in the getting up of the book.

Finally, it is necessary to point out to the reader of the work, that he will find much that is new and peculiar to the author. Passing over smaller matters, which occur in almost every chapter, I will mention only the following more comprehensive subjects. A general and easy method of determining the centre of gravity of even surfaces and even-sided polyeders, will be found in the paragraphs 107, 112, and 113; an approximate formula for the catenary in the paragraphs 147 and 148; supplements to the theory of the friction of axes in the paragraphs 167, 168, 169, 172, and 173. The doctrine of impact especially has received essential additions by the paragraphs 262 and 263, since the impact of imperfectly elastic bodies has been hitherto insufficiently treated; and the case, where a perfectly elastic body comes in contact with a partially elastic body, has been passed over entirely. The largest number of additions, and some entirely new laws will be found in the Hydraulics, the author having made this branch of the sciences his particular study during a number of years. The laws of the incomplete contraction of the fluid vein, discovered by the author, appear here for the first time in a course of mechanics. The chief results of the author's experiments on the flow of water through slides, cocks, clacks and valves, are likewise given, as well as his principal observations made during his latest experiments on the flow of water through prolonged oblique tubes, and through angular, curved, and long straight tubes, though he has not yet been able to publish the third number of his "Untersuchungen im Gebiete der Mechanik und Hydraulik," which is to contain the results of these experiments. The chapters on flowing water and the gauging and impulse of water have likewise received additional matter from the author.

But now, after completion of the first volume, I cannot refrain from wishing that several subjects had been treated in a different manner, although I have not been able to discover any essential errors or imperfections in it. Should the reader here and there find omissions, I must refer him to the second volume, which will contain supplements, most of which have been intentionally reserved for that volume, as has been mentioned in several passages of that now published.

It will give me great satisfaction and pleasure, if the purpose which I had in view in writing this work has been attained in some measure. I wished to supply the practical man with useful advice, the instructor in mechanics with a guide for teaching, and the young engineer and machinist with a welcome auxiliary for the acquirement of the science of mechanics.

JULIUS WEISBACH.

Freiberg, March, 1846.

# COMPARATIVE TABLES OF ENGLISH, FRENCH, AND GERMAN MEASURES AND WEIGHTS.

### I. Measures of length.

| English (and Russian) foot | French metre. | Paris foot. | Prussian, Danish, and Rhenish foot. |
|---|---|---|---|
| 1 | .3047945 | .9382928 | .9711361 |
| 3.280899 | 1 | 3.078444 | 3.186199 |
| 1.065765 | .3248394 | 1 | 1.035003 |
| 1.029722 | .3138535 | 0.9661806 | 1 |

### II. Square measure.

| Square foot. | Sq. metre. | Square foot. | Square foot. |
|---|---|---|---|
| 1 | .09289969 | 0.8803934 | 0.9431053 |
| 10.76430 | 1 | 9.476817 | 10.15187 |
| 1.135856 | .1055207 | 1 | 1.071232 |
| 1.060327 | .09850405 | 0.9335049 | 1 |

### III. Cubic measure.

| cubic foot. | cubic metre. | cubic foot. | cubic foot. |
|---|---|---|---|
| 1 | .02831531 | 0.8260668 | .9158836 |
| 35.31658 | 1 | 29.17385 | 32.34587 |
| 1.210556 | .03427727 | 1 | 1.108728 |
| 1.091842 | 0.9019342 | .03091584 | 1 |

### IV. Measures of capacity.

| English gallon, = 277.27384 cubic inches. | French litre, = .001 cubic metre. | Prussian quart, = 64 cubic inches. |
|---|---|---|
| 1 | 4.543458 | 3.967977 |
| .2200967 | 1 | 0.8733386 |
| .2520176 | 1.145031 | 1 |
| **Bushel, = 8 gallons.** | **Hectolitre, = 100 litres.** | **Scheffel, = 3072 cubic inches.** |
| 1 | 0.3634767 | .6613296 |
| 2.751208 | 1 | 1.819455 |
| 1.512105 | 0.5496150 | 1 |

### V. Weights.

| English pound, avoirdupoid. | French. | | Prussian, Hanoverian, Brunswick, and Hessian pound. |
|---|---|---|---|
| | Kilogram. | Livre, poids de marc. | |
| 1 | .4535976 | .9266439 | .9698245 |
| 2.204597 | 1 | 2.042879 | 2.138072 |
| 1.079163 | .4895058 | 1 | 1.046599 |
| 1.031114 | .4677110 | .9554758 | 1 |
| **Hundred weight. = 112 lbs. avd.** | **Quintal métrique. = 100 kilo.** | **Quintal, = 100 livres (old measure)** | **Centner, = 110 lbs.** |
| 1 | 0.5080293 | 1.037841 | 0.9874577 |
| 1.968390 | 1 | 2.042877 | 1.943702 |
| 0.9635386 | 0.4095058 | 1 | 0.9514536 |
| 1.012702 | 0.5144821 | 1.051023 | 1 |

L. G.

# PRINCIPLES

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## SECTION I.

PHORONOMY; OR, THE PURE MATHEMATICAL SCIENCE OF MOTION.

### CHAPTER I.

SIMPLE MOTION.

§ 1. *Rest and Motion.*—Every body takes up a certain position in space; a body is at rest when it does not change its position; and is in motion, on the other hand, when it successively passes from one position into others. The rest and motion of a body are either absolute or relative, according as we refer its position to a space which itself is at rest or in motion, or considered to be in either state.

Upon the earth there is no rest, for all bodies upon the earth share in its motion about the sun, and about its own axis; but if we suppose the earth to be at rest, then all those terrestrial bodies are at rest which, with reference to the earth, do not change their position.

§ 2. *Kinds of Motion.*—The continual succession of positions which a body in its motion gradually takes up, forms a space which is called the trajectory, or path of the moving body. The path of a moving point is a line; that of a geometrical body, is another body; but by this latter is generally understood that line

which a certain point of the body, viz: the centre, describes in its motion.

When the path is a straight line, the motion is rectilinear; when a curved line, the motion is curvilinear.

With reference to time, motion is uniform or variable.

§ 3. A motion is uniform, when equal spaces are described by it in equal and arbitrarily small times; and variable, when this equality does not take place. When the spaces, described in equal times, increase continuously with the time, a variable motion is called—accelerated; and when the spaces decrease—retarded.

Periodic differs from uniform motion in this, that equal spaces are described within certain intervals only, which are called periods.

The apparent diurnal revolution of the fixed stars, and the progressive motion of the hands of a watch are instances of uniform motion. Falling and upwardly projected bodies, the sinking of the surface of water by its flow from vessels are instances of variable motion. The oscillations of a pendulum, the play of the piston of a steam engine, &c., are illustrations of periodic motion.

§ 4. *Uniform Motion.*—Velocity is the path of a body's strength or magnitude of a motion. The greater the space is which a body describes in a given time, the stronger is its motion, and the greater is its velocity. In uniform motion, the velocity is invariable; and in a variable one, it changes at every instant. The measure of the velocity at any determinate point of time, is the path which a body actually describes, or would describe in a unit of time or second, if from that moment the motion were to become uniform, and the velocity to remain invariable. In general this measure is called simply—velocity.

§ 5. When a body describes the path $\sigma$ at each particle of time, and a second consists of very many ($n$) such particles; the path during one second is the velocity, or rather the measure of the velocity:

$$c = n . \sigma.$$

After a time, $t$ (seconds) $n . t$ particles have elapsed, but in each a space $\sigma$ has been described; the whole space, therefore, which corresponds to the time $t$ is:

$$s = n . t . \sigma = n . \sigma . t, \text{ i. e.}$$

$$\text{I. } s = ct.$$

so that in uniform motion, the space ($s$) is a product of the velocity ($c$) and the time ($t$).

Inversely

$$\text{II. } c = \frac{s}{t} \text{ and III. } t = \frac{s}{c}.$$

*Example* 1. A locomotive, going with a velocity of 30 feet, passes over in 2 hours = 120 min. = 7200 seconds, a space ($s$) 30 × 7200 = 216000 feet.—2. If a time, $3\frac{1}{2}$ minutes = 210 seconds be required to raise up a ton weight from a shaft 1200 feet deep, its mean velocity ($v$) must be taken $= \frac{1200}{210} = \frac{40}{7} = 5\frac{5}{7} = 5{,}714 \ldots$ feet.—3. A horse, moving with a velocity of 6 feet, requires to perform one mile, or 24,000 feet, a time $\frac{24000}{6} = 4000$ seconds.

§ 6. If two different uniform motions be compared with each other, the following is arrived at:

The spaces are $s = ct$ and $s_1 = c_1 t_1$, therefore their ratio is $\frac{s}{s_1} = \frac{ct}{c_1 t_1}$. If $t_1 = t$, then $\frac{s}{s_1} = \frac{c}{c_1}$, or if $c_1 = c$, then $\frac{s}{s_1} = \frac{t}{t_1}$, lastly if $s_1 = s$, $\frac{c}{c_1} = \frac{t_1}{t}$.

The spaces described in different uniform motions in equal times are to each other as the velocities; the spaces described with equal velocities are as the times; and lastly, the velocities corresponding to equal spaces are inversely proportional to the times.

§ 7. *Uniformly variable Motion.*—A motion is uniformly variable when its velocity either increases or diminishes by a certain amount in equal and arbitrarily small times. It is either uniformly accelerated or uniformly retarded, according as in the first a gradual increase, or in the second a gradual diminution of velocity takes place.

In *vacuo*, the motion of a falling body is uniformly accelerated; were the air to exert no influence upon it, the motion of a body vertically projected would be uniformly retarded.

§ 8. The strength or magnitude of the change in the velocity of a body is called acceleration; it is either positive or negative, according as there is increase or diminution of the velocity. The greater this increase or diminution within a given time, the greater is the acceleration. In uniformly variable motion, the acceleration is invariable, and may be measured by the increase or diminution of velocity which takes place in a second of time. In every other motion, the measure of the acceleration is the increase or diminution which a body would acquire if, from the moment in which the acceleration begins, it lose its variability and the motion pass into a uniformly variable one.

The measure is very commonly called the *velocity*.

§ 9. If the velocity of a uniformly accelerated motion increase ($\kappa$) in infinitely small particles of time, and a second of time is made up of such particles, the increment of velocity, or the acceleration, in one second is: $p = n\,\kappa$,
and the increment after $t$ seconds $= n\,t.\,\kappa = n\,\kappa.\,t = pt$.

If the initial velocity (the moment from which the time is counted) $= c$, the terminal velocity, *i. e.* the velocity acquired after the time ($t$) is: $v = c + pt$.

For motion, commencing without velocity, $c = o$, therefore $v = p\,t$, and for uniformly retarded motion, having a negative acceleration ($p$), $v = c - p\,t$.

*Example* 1. The accceleration of a body falling freely in vacuo $= 32.2$ feet; it acquires, therefore, after 3 seconds, a velocity $v = pt = 32.2 \times 3 = 96.6$ feet. —2. A sphere rolling down an inclined plane, with an initial velocity $c = 25$ feet, acquires in its course, at each second, 5 feet additional velocity; its velocity, therefore, after $2\frac{1}{2}$ seconds: $v = 25 + 5 \times 2.5 = 25 + 12.5 = 37.5$ feet, &c.; proceeding from the last point uniformly, it will pass over 37.5 feet in every second. —3. A locomotive going with a 30 feet velocity is so retarded, that in each second it loses 3.5 feet of velocity; its acceleration is $-3.5$; its velocity, therefore, after 6 seconds is $v = 30 - 3.5 \times 6 = 30 - 21 = 9$ feet.

§ 10. *Uniformly accelerated Motion.*—The velocity of every motion may be regarded as invariable within a small particle of time $\tau$. We may, therefore, put the space described in such time $\sigma = v\,.\,\tau$, and we obtain the whole space in the finite time $t$, by the measurement of these small spaces. Now for all these small spaces, the time $\tau$ is one and the same; this sum, therefore, may be put equal to the product of these particles of time, and the sum of the velocities corresponding to equal intervals.

In uniformly accelerated motion, the sum $(o + v)$ of the velocities, in the first and last moment, is as great as the sum $p\tau + (v - p\,\tau)$ of the velocities in the second and last but one; also, equal to the sum $2\,p\,\tau + (v - 2\,p\,\tau)$ of the velocities in the third and last but two; and this sum is equal to the terminal velocity $v$. Here, therefore, the sum of all the velocities is equal to the product $\left(v\,.\,\frac{n}{2}\right)$ of the terminal ~~velocity $v$, and half the~~ number of all the particles of time. The space described is the product $(v\,.\,\frac{n}{2}\,.\,\tau)$ of the terminal velocity $v$, and ~~half the num~~ber and th magnitude of the particles. Now the magnitude ($\tau$) of such a particle, multiplied by the number, gives the time $t$; the space,

therefore, described in the time $t$ with a uniform accelerated motion is $s = \frac{v\,t}{2}$.

The space, therefore, described in uniformly accelerated motion is as in uniform motion when its velocity is half as great as the terminal velocity of the former.

*Example* 1. If a body in 10 seconds has acquired a velocity $v$ by uniform accelerated motion of 26 feet, this space described in equal times is $s = \frac{26.10}{2} = 130$ feet. —2. A carriage which in its motion goes over 25 feet in $2\frac{1}{4}$ seconds, proceeds at the end with a velocity $v = \frac{2\,.\,25}{2{,}25} = \frac{50\,.\,4}{9} = 22.22$.

§ 11. The two fundamental formulæ of uniformly accelerated motion:

$$\text{I. } v = p\,t \text{ and II. } s = \frac{v\,t}{2},$$

which express that the velocity is a product of the acceleration and the time; and the space, half the velocity and the time; include two other principal formulæ which are obtained, if from both equations $v$ be eliminated once, and $t$ twice. It follows, namely:

$$\text{III. } s = \frac{p\,t^2}{2} \text{ and IV. } s = \frac{v^2}{2\,p}.$$

From this, the space described is a product of half the acceleration, and the square of the time; and the quotient of the square of the velocity by twice the acceleration.

These four formulæ give, by inversion, after one or other of the magnitudes contained have been separated, eight other formulæ.

*Example* 1. A body moving with an acceleration of 15.625 feet, describes in $1\frac{1}{2}$ second a space $\frac{15.625 \times (1.5)^2}{2} = 15.625 \times \frac{9}{8} = 17.578$ feet.—2. A body transported with an acceleration $p = 4.5$ into a velocity $v = 16.5$ feet, has described a space $s = \frac{(16\,.\,5)^2}{2\,.\,4{,}5} = 30.25$ feet.

§ 12. By a comparison of two uniformly accelerated motions, we arrive at the following:

The velocities are $v = p\,t$ and $v_1 = p_1 t_1$, the spaces on the other hand are $s = \frac{p\,t^2}{2}$ and $s_1 = \frac{p_1 t_1^{\,2}}{2}$; from this it follows:

$$\frac{v}{v_1} = \frac{p\,t}{p_1 t_1} \text{ and } \frac{s}{s_1} = \frac{p\,t^2}{p_1 t_1^{\,2}} = \frac{v\,t}{v_1 t_1} = \frac{v^2 p_1}{v_1^{\,2} p}.$$

If we put $t_1 = t$, we have $\frac{s}{s_1} = \frac{v}{v_1} = \frac{p}{p_1}$; the spaces described are to each other as the terminal velocities; or, as the accelerations.

If further, we take $p_1 = p$, it gives $\frac{v}{v_1} = \frac{t}{t_1}$ and $\frac{s}{s_1} = \frac{t^2}{t_1^2} = \frac{v^2}{v_1^2}$; so that, in like accelerations, and also in one and the same uniformly accelerated motion, the terminal velocities are proportional to the times and the spaces described to the squares of the times, as also to the squares of the terminal velocities.

Further, if $v_1 = v$ it gives $\frac{p}{p_1} = \frac{t_1}{t}$, and $\frac{s}{s_1} = \frac{t}{t_1}$; in equal velocities, the accelerations are inversely, and the spaces directly proportional to the times.

Lastly, $s_1 = s$ gives $\frac{p}{p_1} = \frac{t_1^2}{t^2} = \frac{v^2}{v_1^2}$; with equal spaces, the accelerations are inversely as the squares of the times, and directly as the squares of the terminal velocities.

§ 13. For a uniformly accelerated motion commencing with a velocity ($c$) we have § 9:

$$\text{I. } v = c + p\,t,$$

and as the space $c\,t$ belongs to the invariable velocity ($c$), and the space $\frac{p\,t^2}{2}$ to the acceleration $p$:

$$\text{II. } s = c\,t + \frac{p\,t^2}{2}.$$

If we eliminate $p$ from both equations, we have:

$$\text{III. } s = \frac{c + v}{2}\,t,$$

and substituting the value of $t$,

$$\text{IV. } s = \frac{v^2 - c^2}{2p}.$$

*Example* 1. A body propelled with an initial velocity $c =$ 3 feet, and with an acceleration $p =$ 5 feet, describes in 7 seconds, a space $s = 3.7 + 5.\frac{7^2}{2} = 21 + 122.5 = 143.5$ feet.—2. Another body which in 3 minutes = 180 seconds, changes its velocity from 2½ feet into 7½ feet, performs in this time a distance $\frac{2.5 + 7.5}{2} . 180 = 900$ feet.

§ 14. For a uniformly retarded motion with an initial velocity $c$, these formulæ are applicable:

$$\text{I. } v = c - p\,t,$$

$$\text{II.}\quad s = c\,t - \frac{p\,t^2}{2},$$

$$\text{III.}\quad s = \frac{c+v}{2}\,.\,t,$$

$$\text{IV.}\quad s = \frac{c^2 - v^2}{2p},$$

they are derived from the former §, when $p$ is made negative. Whilst in uniformly accelerated motion, the velocity increases without limit; in a uniformly retarded one, the velocity at a certain point of time becomes null, and afterwards negative, *i. e.* it goes on in an inverse direction.

If in the first formula we put $v = o$, $p\,t = c$, the time at which the velocity becomes null is, $t = \frac{c}{p}$; if we substitute this value of $t$ in the second equation, we have the space which the body has described at the point of time $= \frac{c^2}{2p}$.

If the time be greater than $\frac{c}{p}$, the space is less than $\frac{c^2}{2p}$; if it be $= \frac{2c}{p}$, the space becomes null, and the body returns to the point from which it set out. If the time be greater than $\frac{2c}{p}$, then $s$ becomes negative, and the body is on the opposite side of its initial point.

*Example.* A body which rolls up an inclined plane with an initial velocity of 40 ft., by which it suffers a retardation of 8 feet per second, ascends only $\frac{40}{8} = 5$ seconds and $\frac{40^2}{2.8} = 100$ feet in height, then rolls back and returns after 10 seconds with a velocity of 40 feet, to its initial point; and after 12 seconds, arrives at a distance $40 \times 12 - 4 \times (12)^2 = 96$ feet below this point if the plane extend itself backwards.

§ 15. *Free descent of Bodies*—The free or vertical descent of bodies in vacuo, offers the most important example of uniformly accelerated motion. The acceleration of this motion brought about by gravity is designated by the letter $g$, and has the mean value of:

9,81 metres
30,20 Paris feet.
32,2? English feet.
31,03 Vienna feet.
31.25 Prussian feet.

If either of these values of $g$ be substituted in the formula :

$$v = gt, \quad s = g\,\frac{t^2}{2} \text{ and } s = \frac{v^2}{2g}, \quad v = \sqrt{2gs},$$

all questions, with reference to the free descent of bodies, may be answered. For the English measure :

$$v = 32.2\,.\,t = 8.0\cancel{2}^{3}\sqrt{s}\,;\; s = 16.1\,.\,t^2 = .0155\,v^2$$
$$\text{and } t = 0.031\,v = 2.48\cancel{7}^{9}\sqrt{s}.$$

*Example* 1. A body acquires in its free descent of 4 seconds a velocity $v = 32.2 \times 4 = 128.8$ feet, and describes in this time a space $s =$ ~~15.625~~ $\times\ 4^2 =$ ~~250~~ feet.—2. A body falling from a height $s = 9$ feet, has a velocity $v = 8.02\sqrt{9} = 24.06$ feet.—3. A body projected vertically with a velocity of 10 feet, ascends to a height $s =$ ~~0.016~~. $10^2 = 1.6$ feet, and requires for it a time $t = 0.031.10 = 0.3$, or about one-third of a second.

16.1 × 4² = 257.6

0.0155

§ 16. The following Table will show the relations of the motion to the time in the free descent of bodies.

| Time in seconds. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Velocity. | 0 | $1g$ | $2g$ | $3g$ | $4g$ | $5g$ | $6g$ | $7g$ | $8g$ | $9g$ | $10g$ |
| Space. | 0 | $1\frac{g}{2}$ | $4\frac{g}{2}$ | $9\frac{g}{2}$ | $16\frac{g}{2}$ | $25\frac{g}{2}$ | $36\frac{g}{2}$ | $49\frac{g}{2}$ | $64\frac{g}{2}$ | $81\frac{g}{2}$ | $100\frac{g}{2}$ |
| Difference. | 0 | $1\frac{g}{2}$ | $3\frac{g}{2}$ | $5\frac{g}{2}$ | $7\frac{g}{2}$ | $9\frac{g}{2}$ | $11\frac{g}{2}$ | $13\frac{g}{2}$ | $15\frac{g}{2}$ | $17\frac{g}{2}$ | $19\frac{g}{2}$ |

The last horizontal column of this table gives the spaces which the freely falling body describes in the single seconds. We see that these spaces are to each other as the odd numbers 1, 3, 5, 7, &c., whilst the times and velocities are as the natural numbers 1, 2, 3, 4, &c., and the spaces fallen through as their squares 1, 4, 9, 16, &c. For example, the velocity after six seconds, is $6\,g = 193{,}2$ feet, that is the body would if it proceeded from this time uniformly upon an horizontal plane, offering no impediment, pass over in each second a space $6\,g = 193{,}2$ feet. This space it describes in the course of the following and seventh second, but not in reality, for according to the last column it amounts to

$13 \cdot \frac{g}{2} = 13 \times 16{,}1 = 209{,}3$ feet, in the eighth second it is $15 \cdot \frac{g}{2} = 15 \cdot 16{,}1 = 241$ feet, &c.

*Remark.* Many writers designate the space of 16 feet, which a body freely descending will describe in one second, by $g$, and term it properly the acceleration of gravity. They have then for the free descent of bodies, the following formulæ:

$$v = 2gt = 2\sqrt{gs},$$
$$s = gt^2 = \frac{v^2}{4g},$$
$$t = \frac{v}{2g} = \sqrt{\frac{s}{g}}.$$

This custom, which is met with in Germany only, is disappearing by degrees, and in consequence of its being frequently misunderstood and the many mistakes which arise therefrom, this is much to desired.

§ 17. If the free descent of a body go on with a certain initial velocity ($c$) the formulæ are of the following kind:

$$v = c + gt = c + 32{,}2\,t, \text{ also } v = \sqrt{c^2 + 2gs} = \sqrt{c^2 + 64{,}4.s},$$
$$s = ct + g\frac{t^2}{2} = ct + 155\,t^2, \text{ also } s = \frac{v^2 - c^2}{2g} = 0{,}0155\,(v^2 - c^2).$$

If on the other hand the body be projected vertically to a height with the velocity $c$, then:

$$v = c - gt = c - 32{,}2\,t, \text{ also } v = \sqrt{c^2 - 2gs} = \sqrt{c^2 - 64{,}4s},$$
$$s = ct - g\frac{t^2}{2} = ct - 16{,}1\,t^2, \text{ also } s = \frac{c^2 - v^2}{2g} = 0{,}0155\,(c^2 - v^2).$$

If we consider a given velocity $c$ as the terminal velocity acquired by a free descent, then the corresponding space fallen through $\frac{c^2}{2g} = 0{,}0155 \cdot c^2$ is called the height ~~of~~ due to the velocity. By the introduction of this quantity, some of the foregoing formulæ may be expressed more simply. If the height of velocity $\left(\frac{c^2}{2g}\right)$ of the initial velocity $c$ be put $= h$, and that of the terminal velocity $\frac{v^2}{2g} = h_1$, we have the following for falling bodies:

$$h_1 = h + s,\ s = h_1 - h,$$

and for ascending: $h_1 = h - s,\ s = h - h_1.$

The space of fall or ascent is, therefore, equal to the difference of the heights of velocity.

*Example.* The velocities are 5 and 11 feet, the height of velocity $= 0{,}0155\,(5)^2 = 0{,}3875$ feet, and $0{,}0155 \cdot 11^2 = 1{,}8755$ feet; the space which is described during the passage from one velocity to the other: $s = 1{,}8755 - 0{,}3875 = 1{,}4880$ feet.

§ 18. From the formula $s = h - h_1$ it also follows that a body vertically projected has at each point that velocity which it would have, but in an inverse direction, were it to have fallen from the height still remaining to that point, and which it then actually possesses in its following descent.

*Example.* A body is thrown up with a 15 feet velocity, and strikes in its rise against an elastic impediment, which for the moment throws it back with the same velocity with which it struck. How great then is this velocity, and the time of ascending and descending? To the velocity ($c = 15$ ft.) corresponds the height of ascent $h = 3{,}49$ ft.; the height of velocity at the moment of impact is $h_1 = 3{,}49 - 2{,}00 = 1{,}49$, and consequently this velocity $= 8{,}03\sqrt{1{,}6} = 9{,}636$ ft. The time to attain the whole height (3,49 ft.) is $t = 0{,}032 . v = 0{,}032 . 15 = \cancel{0{,}480}''$, for the height 1,49 ft. $t_1 = \cancel{0{,}032 . 10} = \cancel{0{,}320}''$; there remains then for the time required to rise to the height of 2 ft., or the time from the commencement to impact: $t - t_1 = \cancel{0{,}480} - 0{,}320 = 0{,}160''$, and the whole time of rising and falling $= 2 . 160'' = 0{,}320$ ~~ft.~~ This is also but the $\frac{0{,}320}{0{,}960}$th = 3rd part of the time, which would be necessary for rising and falling if the body were to rise and fall unimpeded. This fall finds its application in the forging of hot iron, because in the gradual cooling of that metal it is desirable that the blows of the hammer follow as quickly as possible in a short time. When the hammer is thrown back by an elastic arrangement, it will give in the same time, in the proportions above laid down, thrice the number of blows to what it would give were its rise unimpeded.

*Remark* 1. The transformation of the velocity into height of velocity, and the reverse, is very often required in practical mechanics, and especially in hydraulics. A table where this is set down is of great use to the practical man.

*Remark* 2. The foregoing formulæ are only strictly correct for a free descent in vacuo; they may be used with tolerable accuracy for a fall in air, if the falling bodies have a weight great in proportion to their volume, and if the velocities do not come out very great. For the rest, they are also used under other circumstances and relations in many other descents, as will hereafter be shown.

§ 19. *Variable motions in particular.*—For variable motions especially, in which the perodic are also included, the formulæ

$$1.\ \kappa = p\,\tau, \text{ and} \qquad dv = p\,dt$$
$$2.\ \sigma = v\,\tau \qquad ds = v\,dt$$

hold good:—the increment of velocity ($\kappa$) acquired in a very small time $\tau$ (element of time), is a product of the acceleration $p$ and this time; and the space $\sigma$ described in the element of time $\tau$ is a product of the velocity ($v$) and the time $\tau$. By inversion:

$$3.\ p = \frac{\kappa}{\tau} \text{ and} \qquad \frac{dv}{dt}$$

$$4.\ v = \frac{\sigma}{\tau}. \qquad \frac{ds}{dt}$$

Acceleration is the quotient of the increment of velocity by the

element of the time $\tau$ in which it is acquired. Velocity is the ratio of the element of space to that of the time.

The two last formulæ may be used for the measurement of the acceleration and velocity. *Ex.* From the motion given by the formula $s = at^2$ when $a$ is the space described after the first second, it follows: if $t$ increase by $\tau$ and $s$ by $\sigma$, $s + \sigma = a(t + \tau)^2$. Now $(t + \tau)^2 = t^2 + 2\,t\,\tau + \tau^2$, or because $\tau$ is small $= t^2 + 2\,t\,\tau$, it therefore follows $s + \sigma = at^2 + 2\,at\,\tau$, or $\sigma = 2\,at\,\tau$; lastly, $v = \dfrac{\sigma}{\tau} = 2\,at$. By the same hypotheses, we learn from the last formula $v + \kappa = 2\,a\,(t + \tau) = 2\,a\,t + 2\,a\,\tau$, so that $\kappa = 2\,a\,\tau$ and the acceleration $p = \dfrac{\kappa}{\tau} = 2\,a$. We have, therefore, in this way found from the formulæ for the spaces, formulæ for the velocity and acceleration.

§ 20. The velocity $c = \dfrac{s}{t}$ differs from the velocity $v = \dfrac{\sigma}{\tau}$ of an element of time, and is given when the space, which in a certain time or period of a periodic motion is described, is divided by the time itself. This is called the mean velocity, and may be also regarded as that velocity which a body must have in order to describe uniformly in a given time ($t$) a given space ($s$), which in reality is described variably. So, for example, in uniformly variable motion, the velocity is equal to half the sum $\left(\dfrac{c + v}{2}\right)$ of the initial and terminal velocities; for, according to § 13, the space is equal to this sum $\left(\dfrac{c + v}{2}\right)$ multiplied by the time ($t$).

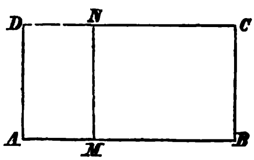
While a handle turns uniformly in a circle, the load attached to it, the piston of an air or water-pump for instance, moves variably up and down; the velocity of this at its lowest and highest point is at a minimum, viz., null; at half the height a maximum, viz., equal to the velocity of the handle. In half a revolution, the mean velocity equals the whole height of ascent, *i. e.*, the diameter of the circle which the handle describes, divided by the time of half a revolution. The diameter $= 2\,r$ and the time $= t$, then the mean velocity of the load $= \dfrac{2r}{t}$. The handle in this time describes the semicircle $\pi\,r$; its velocity therefore $= \dfrac{\pi\,r}{t}$, and consequently

the mean velocity of the load $= \frac{2}{\pi} = \frac{2}{3.141} = 0,6366$ times as great as the invariable velocity of the handle.

§ 21. *Graphical representation.*—The laws of motion found above may be expressed by geometrical figures, or as it is said, graphically represented. Graphical representations, especially facilitate the conception, sustain the thoughts, prevent mistakes, and serve not unfrequently for the discovery of a quantity, and on that account are of great use in mechanics.

In uniform motion the space ($s$) is the product ($ct$) of the velocity and the time, and in geometry the area of a rectangular figure is the product of the height and base. We can, therefore, represent the space ($s$) uniformly described by a rectangle $A B C D$, Fig. 1, whose base $A B$ is the time ($t$) and whose height ($A D = B C$) is the velocity ($c$), provided that the time be expressed in the same unit of length as the velocity, and that the second of time and the foot be represented by one and the same line.

FIG. 1.




§ 22. Whilst in uniform motion, the velocity ($M N$) at any other time ($A M$) of the motion is one and the same, it differs at every instant in a variable one; this motion, therefore, can only be represented by a quadrilateral figure $A B C D$, Fig. 2, which has $A B$ the time for base, and for the other limits, three lines $A B$, $B C$, $C D$, of which the first two are equal to the initial and terminal velocities, and the last is determined by the extremity ($N$) of the different velocities at the intervals ($M$). The line $C D$ is either straight or curved, according to the different kinds of variable motion from the commencement, ascending or descending, or lastly concave or convex towards the base. But in every case the area of this figure must be put equal to the variably described space ($s$); for each area of space $ABCD$, Fig. 3, may be considered as decomposable into many small rectangular strips, like $M\ O\ P\ N$, of which each is a product of a part ($M\ O$)

FIG. 2.

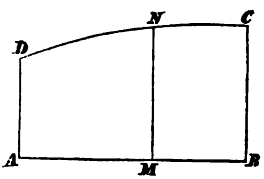
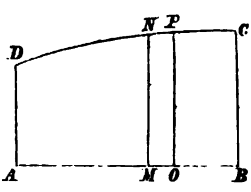



FIG. 3.



of the base, and its corresponding height ($M N$ or $O P$), and the spaces described in a certain time composed of particles of which each is a product of that particle and its corresponding velocity.

§ 23. In uniformly variable motion, the increase or diminution $(v-c)$ of the velocity $(=p t$, § 13) is proportional to the time. If in the Figures 4 and 5, the line $D E$ be drawn parallel to the

FIG. 4.

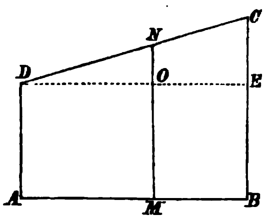
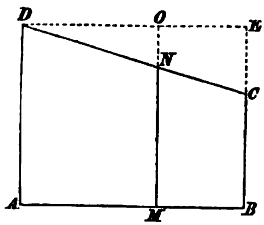



FIG 5.




base $A B$, and $B E$ and $M O =$ to the initial velocity $A D$ be cut off from the lines $M N$ and $B C$, there remain the lines $C E$ and $N O$ for the increase or diminution of the velocity, for which from the above we have the proportion

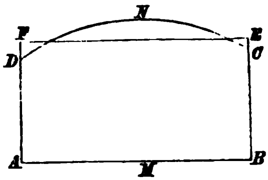
$$N O : C E = D O : D E.$$

Such a proportion requires that $N$ as well as each point of the line $C D$ lie in the straight line connecting $C$ and $D$, and also that the line $C D$ limiting the different velocities ($M N$) be a right line.

In consequence of this, the uniformly accelerated and uniformly retarded space described may be represented by the area of a trapezium $A B C D$, which has for the height $A B$ the time ($t$), and for the parallel bases the initial and terminal velocities $A D$ and $B C$. The formula of § 13, $s = \frac{c+v}{2} . t$ is in perfect accordance with this. In uniformly accelerated motion, the fourth side $D C$ ascends from its initial point, and in uniformly retarded motion descends. In a uniformly accelerated motion beginning with a velocity null, the trapezium becomes a triangle whose area is $\frac{1}{2} B C \times A B = \frac{1}{2} ct$.

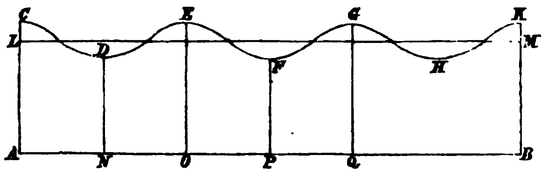
§ 24. The *mean velocity* of a variable motion is the quotient of the *space divided by the time*; multiplied therefore by the time, it gives as a product the trajectory, and consequently may be also considered as the height $A F = B E$ of the parallelogram $A B E F$ Fig. 6, which has the time ($t$) for the base $A B$, and an area equal

FIG. 6.




to the four-sided figure $A\,B\,C\,N\,D$ which measures the trajectory or space passed through. The mean velocity is, therefore, likewise obtained by transforming the four-sided figure $A\,B\,C\,N\,D$ into a parallelogram $A\,B\,E\,F$ of the same length. Its determination is of importance, particularly in periodic motions, which occur in nearly all machines. The law for these motions is represented by a curved line $C\,D\,E\,F\,G\,H\,K$, Fig. 7.

FIG. 7.




If the line $L\,M$ running parallel with $A\,B$ cuts off the same space as the curved line, and is as it were the axis round which $C\,D\,E\,F$.... coils itself, then the distance $A\,L = B\,M$ between the two parallel lines $A\,B$ and $L\,M$ is the *mean* velocity of the periodic motion, whilst $A\,C$, $O\,E$, $B\,K$, &c., is the *maximum*, and $N\,D$, $P\,F$, &c., the *minimum* velocity of a period $A\,O$, $O\,Q$, $Q\,B$, &c.

§ 25. The acceleration also, or the increase of velocity during a second of time may be easily shown in the figure. In the case of uniformly variable motion it remains unchangeable; it is hence the difference $P\,Q$, Figs. 8 and 9, between two velocities $O\,P$ and

FIG. 8.

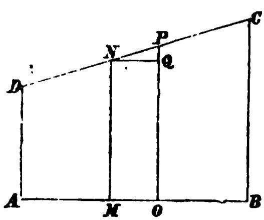
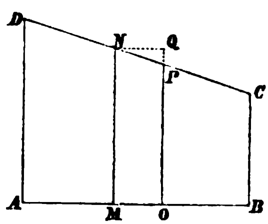



FIG. 9.




$M\,N$, the one of which appertains to a longer time by one second ($M\,O$) than the other. If the motion is not uniformly variable, and the line of velocity $C\,D$ therefore a curve, then for each second of time ($M$) the acceleration varies, and is consequently

not the real difference $PQ$ between the two velocities $OP$ and $MN = OQ$, Figs. 10 and 11; but it is the increase $RQ$ of the

FIG. 10.

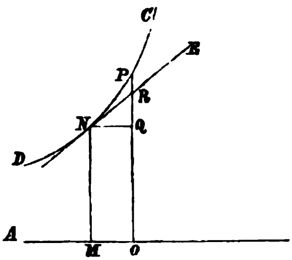
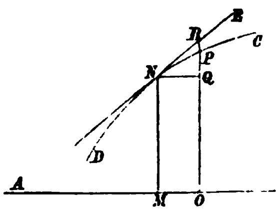



FIG. 11.




velocity $MN$, which would occur if commencing at the moment $M$ the motion became uniformly accelerated, and the curved line of velocity $NC$ passed into the straight line $NE$. Now the tangent or line of contact $NE$, is that straight line in the direction of which a curve $DN$ proceeds, when from a certain point ($N$) it ceases to change its direction; hence the new line of velocity coincides with the tangent, and the perpendicular line $OR$ which cuts it, is accordingly the velocity which would take place after the lapse of a second, supposing the motion to have become uniformly accelerated from the commencement of that period, and lastly, the difference $RQ$ between this velocity and the primary velocity ($MN$) is the acceleration for that moment which is determined by the point $M$ in the time line $AB$.

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## CHAPTER II.

### COMPOUND MOTION.

§ 26. *Compound Motion.*—One and the same body may at the same time have two or more motions; every (relative) motion consists of the motion within a certain space, and of the motion of this space within or in relation to a second space. Each point upon the surface of the earth has thus two motions, for it revolves daily once round the axis of the earth, and simultaneously with the earth once yearly round the sun. A person walking on board a ship has two motions in relation to the shore, his own motion and

that of the water; water flowing from a hole in the bottom or side of a vessel, whilst the latter is moving along in a carriage, has two motions, the motion from the vessel and the motion with the vessel, &c.

Hence we distinguish *simple* and *compound* motion. Those rectilinear motions are called *simple*, of which other rectilinear or curvilinear motions, consequently called compound, are made up or may be imagined to be made up.

The combination of several simple motions to form one single motion, and the resolution of a compound motion into several simple motions will be treated of in the sequel.

§ 27. When simple motions occur in the direction of one and the same straight line, their sum or difference gives the resulting compound motion, the former, when the motions take place in the same direction; the latter, when their directions are opposite. The truth of this axiom becomes directly obvious, when the contemporary spaces of the simple motions are united into one. The contemporary spaces $c_1 t$ and $c_2 t$ correspond with the uniform motions and their velocities $c_1$ and $c_2$; if these motions go on in the same direction, then after $t$ seconds the space becomes $s = c_1 t + c_2 t = (c_1 + c_2) t$, and consequently the resulting velocity with which the compound motion proceeds is the sum of the velocities of the simple motions. When the directions of both motions are opposite, then $s = c_1 t - c_2 t = (c_1 - c_2) t$, here, therefore, the resulting velocity is equal to the difference of the simple velocities.

*Example* 1. To a person moving with a velocity of four feet upon the deck of a ship, in the same direction with the motion of the ship itself, which has a velocity of six feet, the objects on the shore appear to pass by with a velocity of $4 + 6 = 10$ feet.—2. The water which flows from the lateral opening of a vessel with a velocity of 25 feet, whilst the vessel containing it is moved in an opposite direction with a velocity of 10 feet, has, in relation to the other objects at rest, only a velocity of $25 - 10 = 15$ feet.

§ 28. The same relations obtain with variable motions. If one and the same body have, in addition to the primary velocities, $c_1$ and $c_2$ the constant accelerations $p_1$ and $p_2$, then the corresponding spaces are $c_1 t_1$; $c_2 t_1, p_1 \frac{t^2}{2}, p_2 \frac{t^2}{2}$, if the velocities and the accelerations are in the same direction, the whole space corresponding to these simple motions, will be:

$$s = (c_1 + c_2) t + (p_1 + p_2) \frac{t^2}{2}.$$

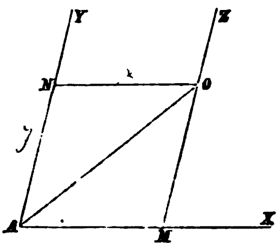
If $c_1 + c_2 = c$ and $p_1 + p_2 = p$, we then obtain $s = ct + p\,\frac{t^2}{2}$, and it follows, consequently, that not only the velocity of the resulting or compound motion is made up of the sum of the simple velocities, but that also the sum of the accelerations of the simple motions gives the resulting acceleration.

*Example.* A magnet falls more quickly to the earth than another body, when a mass of iron is immediately below it. The acceleration which the magnet experiences, in consequence of this iron, may be considered invariable when the height from which it falls is small and the mass of iron very considerable, viz.: an extensive layer of magnetic iron ore. If this acceleration were 5 feet, then the magnet would fall with an increased velocity of ~~31,25~~ + 5 = ~~36,25~~ feet in the first second, therefore it would fall 18⅛ feet instead of 15⅝ feet. 387 31,2
19,5

§ 29. *Parallelogram of the velocities.*—If a body has at the same time two motions differing from each other in direction, it will assume a medium direction between them; and if these motions are of different kinds, the one, viz. uniform, and the other uniformly increasing, then the direction will vary in every part of the motion, and the motion itself become curvilinear.

The place $O$, Fig. 12, which a body moving simultaneously in the directions $AX$ and $AY$ will occupy after a certain time ($t$), is found when the fourth corner of the parallelogram $AM$ $ON$, determined by the contemporaneous trajectories $AM = x$ and $AN = y$, as well as by the angle $XAY$, or the distance by which the directions of motion deviate from each other, is known. The correctness of this mode of procedure becomes evident when the trajectories $x$ and $y$ are supposed described one after the other, and not at the same time. In compliance with the one motion, the body describes the trajectory $AM = x$; and in compliance with the other, the trajectory proceeding from $M$ in the direction of $AY$, therefore in a line $MZ$ parallel to $AY$, or the trajectory $AN = y$. If $MO = AN$, then $O$ is the position of the body corresponding to both motions $x$ and $y$ simultaneously, which in accordance with the construction is the fourth corner of the parallelogram $AMON$. We may likewise imagine that the space $AM = x$ is passed over in a line $AX$, which with all its points proceeds at the same time in the direction $AY$, and therefore carries with it $M$ in a parallel

FIG. 12.

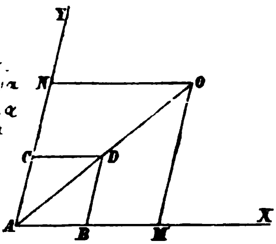


direction to $A\,Y$, and causes this point to perform the trajectory $M\,O = A\,N = y$.

§ 30. If both the motions in the directions $A\,X$ and $A\,Y$ take place uniformly and with the velocities $c_1$ and $c_2$, then the spaces will become after a certain time $(t)$: $x = c_1\,t$ and $y = c_2\,t$; their ratio ~~relationship~~ $\frac{y}{x} = \frac{c_2}{c_1}$ is, therefore, the same at all time, a peculiarity which is only proper to the straight line $A\,O$, Fig. 13. Hence it follows that the compound motion proceeds in a straight line. If, with the velocities $A\,B = c_1$ and $A\,C = c_2$, the parallelogram $A\,B\,C\,D$ is constructed, its fourth corner gives the position $D$, in which the body will be placed after the lapse of one second. But as the resulting motion is rectilinear, it follows that it must always occur in the direction of the diagonal of that parallelogram which is constructed by the velocities. If the trajectory $A\,O$ which is actually passed through in the time $t$ be $= s$, then, on account of the similarity of the triangles $A\,M\,O$ and $ABD$, we have: $\frac{s}{x} = \frac{A\,D}{A\,B}$, and it consequently follows that this trajectory $s = \frac{x.AD}{A\,B} = \frac{c_1\,t.AD}{c_1} = A\,D\,.\,t$. In accordance with the last equation, the trajectory in the diagonal is proportional to the time $(t)$, the motion itself consequently uniform, and the diagonal $A\,D$ its velocity.

FIG. 13.

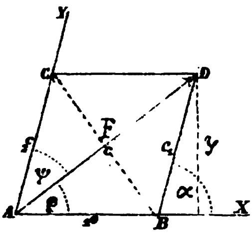
$\frac{y}{x} = \frac{c_2 \sin \alpha}{c_1 + c_2 \cos \alpha}$




*The diagonal, therefore, of a parallelogram formed by two velocities, and the angle which they make with each other, gives the direction and magnitude of the actually resulting motion.* This parallelogram is called the *parallelogram of velocities,* the simple velocities are called the *components,* and the compound velocity the *resultant.*

§ 31. By the use of trigonometrical formulæ, the direction and magnitude of the mean velocity may be ascertained by calculation. The resolution of one of the equal triangles, viz. $A\,B\,D$, of which the parallelogram of velocities $A\,B\,D\,C$ (Fig. 14) is composed, gives the mean velocity $A\,D = c$ by means of the components $A\,B = c_1$ and $A\,C = c_2$, and the angle $B\,A\,C = \alpha$ formed by their directions by the formula: $c = \sqrt{c_1{}^2 + c_2{}^2 + 2\,c_1\,c_2\,cos.\,\alpha}$, and the

FIG. 14.




angle $BAD=\phi$, included by the mean velocity, and the velocity $c_1$ is expressed by the formula $sin.\ \phi = \frac{c_2\ sin.\ a}{c}$, or $tang.\ \phi = \frac{c_2\ sin.\ a}{c_1 + c_2\ cos.\ a}$. If the velocities $c_1$ and $c_2$ are equal, and their parallelogram consequently a rhombus, then we obtain in a more simple form, in consequence of the diagonals being at right angles to each other:

$$c = 2\,c_1\,cos.\ \tfrac{1}{2}\,a \text{ and } \phi = \tfrac{1}{2}a.$$

Lastly, if the velocities enclose a right angle, then likewise we obtain more simply:

$$c = \sqrt{c_1^2 + c_2^2} \text{ and } tang.\ \phi = \frac{c_2}{c_1}.$$

*Example* 1. The water flowing from a vessel or a machine has a velocity $c_1 = 25$ ft., whilst the vessel is moved with a velocity $c_2 = 19$ feet in a direction forming an angle $a^\circ = 130^\circ$ with the direction of the flowing water. What is the direction and magnitude of the resultant, or as it is also called, the absolute velocity?

The required resulting velocity is $c = \sqrt{25^2 + 19^2 + 2\,.\,25\,.\,19\ cos.\ 130^\circ} = \sqrt{625 + 361 - 50.19\ cos.\ 50^\circ} = \sqrt{986 - 950\ cos.\ 50^\circ} = \sqrt{986 - 610{,}7} = \sqrt{375{,}3} =$ 19,37 feet.

Moreover, $sin.\ \phi = \frac{19\ sin.\ 130^\circ}{19{,}37} = 0{,}9808\ sin.\ 50^\circ = 0{,}7513$, and consequently the angle by which the resultant differs from the velocity $c_1$ $\phi = 48^\circ{,}42'$ and the angle which it makes with the direction of motion of the vessel: $a - \phi = 81^\circ{,}18'$.

2. If the former velocities were acting at right angles to each other, then $cos.\ a = cos.\ 90^\circ = 0$, thence the mean velocity $c = \sqrt{986} = 31{,}40$ feet; for its direction we should have $tang.\ \phi = \frac{19}{25} = 0.76$, and consequently its deviation from the first velocity: $\phi = 37^\circ{,}14'$.

§ 32. Any given velocity may be supposed to consist of two components, and can consequently be resolved into them, in accordance with certain conditions. If, for instance, the angles $DAB = \phi$ and $DAC = \psi$, Fig. 14, are given, and enclose the velocities required together with the mean velocity $AD = c$, then draw through the terminal point $D$ other lines which represent the degrees corresponding to $c$, parallel to the directions $AX$ and $AY$: the points of section will then cut off the required velocities $AB = c_1$ and $AC = c_2$.

Trigonometry expresses these velocities by the formulæ $c_1 =$

$\frac{c \sin. \psi}{\sin. (\phi + \psi)}, c_2 = \frac{c \sin. \phi}{\sin. (\phi + \psi)}$. In the usual practical cases, the two velocities are at right angles to each other, and then $\phi + \psi = 90^0$, $\sin. (\phi + \psi) = 1$, and it follows:

$$c_1 = c \cos. \phi \text{ and } c_2 = c \sin. \phi.$$

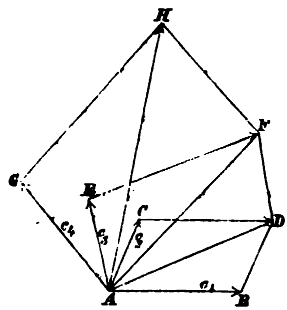
Therefore, with one component ($c_1$) and its angle of direction ($\phi$), the direction and magnitude of the other component may be estimated. Lastly, from the velocities $c$, $c_1$ and $c_2$ alone their angles of direction may be determined, as the three angles of a triangle may be computed by the three sides.

*Example.* Suppose velocity $c = 10$ feet is to be resolved into two components which deviate from its direction by the angle $\phi = 65^\circ$ and $\psi = 70^\circ$. These velocities will be:

$$c_1 = \frac{10 \sin. 70^\circ}{\sin. 135^\circ} = \frac{9{,}397}{\sin. 45^\circ} = 13{,}29 \text{ feet and } c_2 = \frac{10 \sin. 65^\circ}{\sin. 135^\circ} = \frac{9{,}063}{0{,}7071} = 12{,}81 \text{ ft.}$$

§ 33. *Composition and resolution of velocities.*—By repeated application of the parallelogram of velocities, any number of velocities may be reduced to one. By constructing the parallelogram $A B D C$, Fig. 15, the mean velocity $A D$ to $c_1$ and $c_2$ is obtained; by constructing the parallelogram $A D F E$, we get the mean velocity $A F$ to $A D$ and $A E = c_3$; and in like manner by constructing the parallelogram $A F H G$, the mean velocity $A H = c$ to $A F$ and $A G = c_4$ is obtained, and thus the mean of $c_1$, $c_2$, $c_3$ and $c_4$.

FIG. 15.



The simplest method of obtaining the mean velocity in question, is by the construction of a polygon $A B D F H$, the sides of which $A B$, $B D$, $D F$, and $F H$, are drawn parallel and equal to the given velocities $c_1$, $c_2$, $c_3$ and $c_4$; the last side $A H$ is then always the resultant velocity.

In the case also, in which the velocities are not in the same plane, the mean velocity may be ascertained by repeated application of the parallelogram of velocities. The mean velocity $A F = c$ (Fig. 16) of three velocities $A B = c_1$ $A C = c_2$ and $A E = c_3$, which are not in the same plane, is the diagonal of a parallelepipedon $B C G H$, the sides of which are equal to these velocities. The *parallelepipedon of velocities* is, therefore, also a term in general use.

FIG. 16.

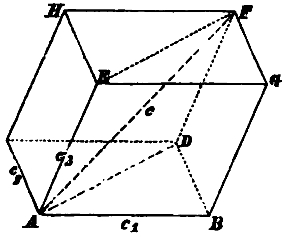
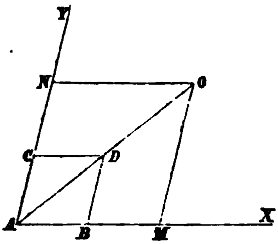



FIG. 17.




§ 34. *Composition of the accelerations.* — Two uniformly accelerated motions beginning with null velocity, produce, when combined, a uniformly accelerated motion in a straight line. If the accelerations of these motions proceeding in the directions $A\,X$ and $A\,Y$ (Fig. 17) are $p_1$ and $p_2$, then at the close of the time $t$, the spaces will be $A\,M = x = \frac{p_1\,t^2}{2}$ and $A\,N = y = \frac{p_2\,t^2}{2}$, and their relation $\frac{x}{y} = \frac{p_1\,t^2}{p_2\,t^2} = \frac{p_1}{p_2}$ is in no way dependant upon the time; and consequently the trajectory $A\,O$ of the compound motion is rectilinear. If $A\,B$ is made $= p_1$ and $B\,D = A\,C = p_2$, we obtain a parallelogram $A\,B\,D\,C$, which is similar to the parallelogram $AMON$, and from which $\frac{A\,O}{A\,D} = \frac{A\,M}{A\,B} = \frac{\frac{1}{2}\,p_1\,t^2}{p_1} = \frac{1}{2}.\,t^2$; therefore, $A\,O = \frac{1}{2}\,A\,D\,.\,t^2$. From this equation, it appears that the trajectory $A\,O$ of the compound motion is proportional to the square of the time; the motion itself, therefore, uniformly accelerated, and its acceleration is the diagonal $A\,D$ of the parallelogram constructed by the simple accelerations $p_1$ and $p_2$.

In the same manner, therefore, as velocities can be composed or resolved by the parallelogram of velocities, and according to precisely the same rules, accelerations may be united into one or broken up into several others by a parallelogram, which is called the *parallelogram of accelerations.*

§ 35. *Combination of velocity and acceleration.*—By the *combination* of a *uniform* with a *uniformly accelerated motion,* an entirely variable motion is produced when the directions of the motions do not coincide. In a certain time $t$, with the velocity $c$ in the direction $A\,Y$, (Fig. 18), the trajectory $A\,N = y = c\,t$ will be described, and in the same time with an unchangeable acceleration and a direction $A\,X$ at right angles to the former, the trajectory

$A M = x = \frac{p t^2}{2}$ will be described, and the body arrives at the terminal point $O$ of the parallelogram composed of $y = c t$ and $x = \frac{p t^2}{2}$. With the aid of these formulæ, the position of the body can be determined for any given time, but it is not always in one and the same straight line, for if from the first equation we take $t = \frac{y}{c}$, and place this value in the second, we obtain the equation $x = \frac{p y^2}{2 c^2}$. In accordance with this, the trajectories ($x$) in the direction of the second motion do not correspond with those in the first, but with the squares ($y^2$) of those in the first; and consequently the trajectory of the body is not rectilinear, but is a certain curved line, known in geometry by the name of *parabola*.

FIG. 18.

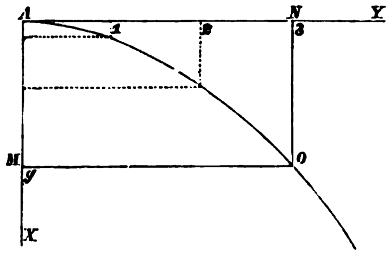


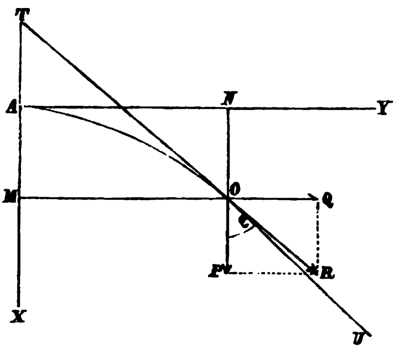

FIG. 19.


*Remark.* Let $A\ B\ C$, Fig. 19, be a cone with a circular base $A E B F$, let $D E F$ be a section of it parallel to the side $B\ C$ and at right angles to the section $A\ B\ C$, and let $O\ P\ N\ Q$ be a second section parallel to the base, and consequently also circular. Then let $E\ F$ be the line of section between the base and the second section, and $O\ N$ that between both sections; imagine then in the triangular section $A\ B\ C$, the parallel diameters $A\ B$ and $P\ Q$, and in the section $D\ E\ F$, the axis $D\ G$. Then for the half chord of the circle $M\ N = M\ O$, the equation applies $M\ N^2 = P\ M \times M\ Q$; but $M\ Q = B\ G$ and for $P\ M$ we have the proportion $P\ M : M\ D = A\ G : D\ G$; hence it follows $MN^2 = B\ G \times \frac{D\ M \times A\ G}{D\ G}$. But in like manner $G\ E^2 = B\ G \times A\ G$; if one equation is divided by the other, we obtain therefore $\frac{D\ M}{D\ G} = \frac{M\ N^2}{G\ E^2}$; the portions cut off from the axis *(abscissæ)* bear, therefore, the same proportion to each other as the squares of the corresponding perpendiculars *(ordinates)*. This law agrees completely with the law for motion

found above; this motion, therefore, takes place in a curved line $D\,N\,E$, which can be shown to be a section of the cone (Conic Section).

§ 36. *Parabolic motion.*—In order thoroughly to comprehend motion produced by the combination of velocity and acceleration, we must be able also to indicate the ***direction***, ***velocity*** and ***the space passed through*** during any length of time ($t$). The velocity parallel to $A\,Y$ is invariable and $=c$, that which is parallel to $A\,X$ is variable and $=pt$, if with these velocities, $O\,Q=c$ and $O\,P=p\,t$, the parallelogram $O\,P\,R\,Q$ is constructed, Fig. 20, we obtain in its diagonal $O\,R$ the mean, or that velocity with which the body at $O$ follows the parabolic curve $A\,O\,U$. This velocity itself is $v=\sqrt{c^2+pt^2}$. In like manner, $O\,R$ is the tangent or direction in which the body at $O$ proceeds for a single instant, and we obtain for the angle $P\,O\,R = X\,T\,O = \phi$ which it makes with the second direction (axis) $A\,X$, the formula $tang.\ \phi = \frac{O\,Q}{O\,P} = \frac{c}{p\,t}$.

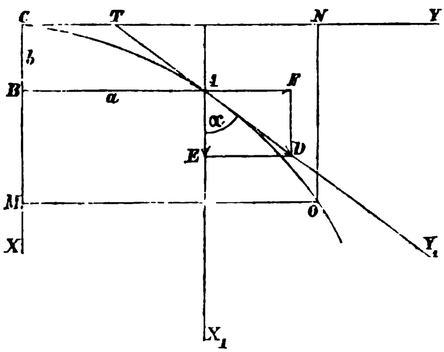
FIG. 20.




In order, lastly, to find the space passed through, or the curve $A\,O = s$ we can apply the equation ~~$\sigma = v\tau$~~ $ds = v\,dt$ (§ 19); according to which we can calculate minute portions of it, which may be considered as its elements. The higher branches of geometry supply us with a complicated formula for calculating the parabolic curve.

§ 37. As yet we have assumed that the primary directions of motion formed a right angle with each other, and we must now study more closely that case in which the direction of the acceleration makes a certain angle with that of the velocity. If the body (Fig. 21) has in the direction $A\,Y_1$ the velocity $c$, and in the direction $A\,X_1$, which makes with the first the angle $X_1\,A\,Y_1 = a$ the velocity $p$, then $A$ is no longer the vertex, and $A\,X_1$ no longer the axis, but only the direction of the axis of the parabola. The vertex $C$ is much more dependant upon the co-ordinates $A\,B = a$

FIG. 21.




and $BC=b$, the latter of which coincides with the axis, and the former is at right angles to it, beginning at the commencing point of the motion $A$. The velocity $AD=c$ is made up of the components $AF=c\ sin.\ a$ and $AE=c\ cos.\ a$. The former of these remains always the same, but the latter must be made equal to the variable velocity $pt$, supposing that the body has required the time $t$ to move from the vertex $C$ to the real commencing point $A$. We have, therefore, $c\ cos.\ a=pt$, consequently $t=\frac{c.cos.a}{p}$, and

1. $AB=a=c\ sin.\ a.\ t=\frac{c^2 sin.a cos.a}{p}=\frac{c^2 sin.2a}{2p}$, and

2. $BC=b=\frac{pt^2}{2}=\frac{c^2 cos.a^2}{2p}$.

If by these distances we have found the vertex of the parabola $C$, then, beginning from thence we can find for any required time the position $O$ of the body. Moreover we have: making $CM=x$ and $MO=y$ the general formula $x=\frac{py^2}{2c^2 sin.a^2}$, also

$$y=c\ sin.\ a\sqrt{\frac{2x}{p}}.$$

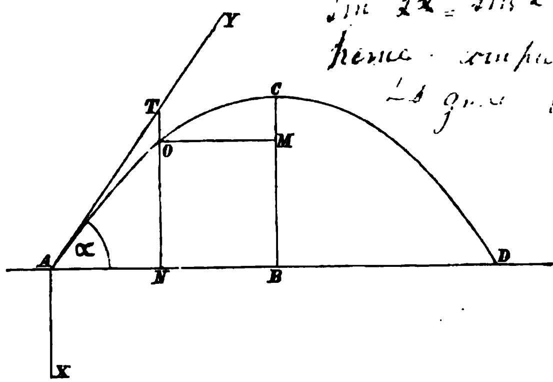
*Remarks.* The theory of parabolic motion produced by an invariable velocity and a constant acceleration, which we have just been considering, finds its application in the doctrine of *Projectiles*. The bodies projected either upwards or downwards would describe a parabolic curve as the result of their primary velocity ($c$) and the acceleration of gravity ($g=32.2$ feet), if the resistance of the air could be prevented, or the motion could take place in a vacuum. If the projectile velocity is not great, and the body is very heavy as compared with its volume, then the deviation from the parabola is small enough to be altogether neglected. The most perfect instance of the parabolic course is witnessed in columns of water flowing from vessels or from jets, &c. Bodies shot off, viz., bullets, describe curves which deviate considerably from the parabola in consequence of the great resistance of the air.

§ 38. A body projected at an angle of elevation $YAD=a$ (Fig. 22), rises to a certain point $BC$, which is called the *height*

*of projection*, and it attains the horizontal plane, from which it

FIG. 22.

sin $2x$ = sin $2(90-x)$
hence complement [illegible]
[illegible]




departed at $A$, at a distance $AD$, which is called the *range of projection.* It follows according to § 37, from the velocity $c$, the acceleration $g$, and the angle of elevation, that when $p$ is replaced by $g$ and $a^0$ by $90^0 \mp a^0$, therefore *cos.* $a$ by *sin.* $a$:

$$\text{the height of projection is } BC = b = \frac{c^2 \, sin. \, a^2}{2g}, \text{ and,}$$

$$\text{the half of the range of projection } AB = a = \frac{c^2 \, sin. 2a}{2g}.$$

On the contrary, the height corresponding to any horizontal distance $AN = AB - NB = a - y$ becomes $NO = BM = CB - CM =$

$$b - x = b - \frac{g\,y^2}{2c^2 cos. a^2} = \frac{c^2\, sin. a^2}{2g} - \frac{g\, y^2}{2c^2\, cos. a^2} = h\, sin.\, a^2 - \frac{y^2}{4\,h\, cos.\, a^2},$$

when $h$ represents the height of velocity $\frac{c^2}{2g}$.

It is evident from the formula for the range of projection, that this will be greatest when *sin.* $2\,a = 1$, therefore $2\,a = 90^0$, *i. e.* $a = 45^0$. A body ascending, therefore, at an angle of elevation of $45^0$ attains the greatest range of projection.

*Example* 1. A jet of water ascending at an angle of elevation of 66° with a velocity of 20 feet, which has therefore a height of velocity $h = 0{,}016 \,.\, 20^2 = 6{,}4$ feet, attains the height $b = h\, sin.\, a^2 = 6{,}4\, (sin.\, 66°)^2 = 5{,}34$ feet, and has a range of projection $a = 2 \,.\, 6{,}4\; sin.\, 132° = 2 \,.\, 6{,}4\; sin.\, 48° = 9{,}51$ feet. The time which each particle of water requires to perform the whole parabolic curve $A\,C\,D$ is $= \frac{2\; c.\; sin.\; a}{g} = \frac{2 \,.\, 20 \,.\, sin.\, 66°}{31{,}25} = 1{.}17$ seconds. The height, corresponding to the horizontal distance

since range $R = 2h \sin 2\alpha$ if $\alpha = 45$ ∴ $R = 2h$
$h = \frac{R}{2}$ $V = \sqrt{2gh} = \sqrt{Rg}$
∴ [illegible] range [illegible] 45 [illegible]
[illegible] angles with the same range

$AN = 3$ feet, is, as $y = \frac{9,51}{2} - 3 = 1,755$, $NO = 5,34 - \frac{1,755^2}{4.6,4\,(cos.\ 66^\circ)^2} = 5,34 - 0,73 = 4,61$ feet.

2. The jet of water flowing from a horizontal tube has at a height of $1\frac{3}{4}$ feet a range (half a range of projection) of $5\frac{1}{4}$ feet, what is the velocity of the water?

From the formula $x = \frac{g y^2}{2 c^2} = \frac{y^2}{4h}$, it follows $h = \frac{y^2}{4x}$, if $x$ in this case $= 1,75$ and $y = 5,25$, then $h = \frac{5,25^2}{4 \,.\, 1,75} = 3,937$ feet, and the velocity corresponding to this height is $c = 15,68$ feet.

*Remark.* Concerning the construction, the position of the tangents and other properties of the parabola, more minute information may be obtained in the "Engineer."

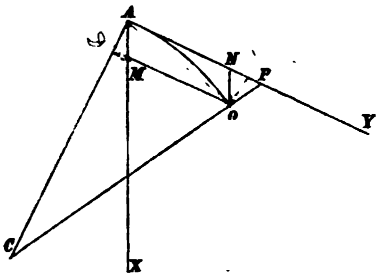
§ 39. *Curved motions in general.*—By the combination of several velocities and several invariable accelerations, a parabolic motion is likewise produced, for not only the velocities but the accelerations also may be united into a single one; the result is, therefore, the same as if there were only one velocity and one acceleration, *i. e.* only one uniform and one uniformly accelerated motion.

If the accelerations are variable, they can just as well be united into a mean as if they were constant, for it is admissible to consider them invariable within the limits of an infinitely small space of time ($\tau$); and the corresponding motions, therefore, during that space of time, as uniformly accelerated. Of course the resulting acceleration is variable, as are its components themselves. If this resulting acceleration be combined with the given velocity, it is possible to deduce a small parabolic curve according to which the motion is effected during the small portion of time. If again the velocity and mean acceleration is determined in the same manner for the next small portion of time, we are enabled to obtain a new curve belonging to another parabola; and if this be farther repeated, we at last obtain the whole course.

§ 40. Any minute portion of any curve may be considered as the arc of a circle. The circle to which this arc belongs is called the *circle of curvature,* the radius pertaining to it is the *radius of curvature.* The course of a moving body may in the same manner be composed of the arcs of circles, and thus a formula for its radius established.

Let $AM$ (Fig. 23) be a very small trajectory described with a uniformly accelerated motion $x = \frac{p\tau^2}{2}$ in the direction $AX$, and let $AN$ be a very small uniformly described trajectory $y = c\tau$, and $O$ the fourth terminating point of the parallelogram constructed

FIG. 23.




from $x$ and $y$, *i. e.* the point which a body proceeding from $A$ would occupy at the end of the short time ($\tau$). Let $A\,C$ be drawn at right angles to $A\,Y$, and let us observe from what point $C$ in this line, a small arc of a circle through $A$ and $O$ can be drawn. On account of the smallness of the arc $A\,O$, we may assume that not only $C\,A$, but also $C\,O\,P$ is at right angles to $A\,Y$; that, therefore, in the small triangle $N\,O\,P$ the angle $N\,P\,O$ is a right angle. The solution of this triangle gives us $O\,P = O\,N$ *sin.* $O\,N\,P = A\,M$ *sin.* $X\,A\,Y = \frac{p\,\tau^2}{2}$ *sin.* $\alpha$ and the tangent $A\,P = A\,N + NP = c\,\tau + \frac{p\,\tau^2}{2}$ *cos.* $\alpha = (c + \frac{p\,\tau}{2}$ *cos.* $\alpha)\,\tau$, may be made $= c\,\tau$, because $\frac{p\,\tau}{2}$ *cos.* $\alpha$, on account of the infinitely small factor $\tau$, is inappreciable with respect to $c$. But now according to the property of the circle $\overline{A\,P^2} = PO \times (PO + 2\,C\,O)$, CO being the radius, or, as $P\,O$ vanishes when compared with $2\,CO$, $\overline{A\,P^2} = PO \times 2\,CO$; we have, therefore, the desired ***radius of curvature***,

$$C\,A = C\,O = r = \frac{\overline{A\,P^2}}{2PO} = \frac{c^2\,\tau^2}{p\,\tau^2\,sin.\,\alpha} = \frac{c^2}{p\,sin.\,\alpha}.$$

By the aid of the same formula, the radii of curvature of all the elements of curves may be found, when the respective velocities ($c$) and the acceleration ($p$) are inserted, and also the angle $\alpha$ which the acceleration makes with the velocity, or with the direction of motion indicated by the line of contact.

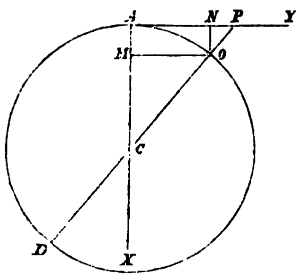
*Example.* For the parabolic path caused by the acceleration of gravity, we have $r = 0{,}031\,\frac{c^2}{sin.\,\alpha}$, and in the vertex of these curves, where $\alpha = 90°$, therefore *sin.* $\alpha = 1$, it results that $r = 0{,}031\,c^2$. With a velocity of 20 feet, it would therefore be found that $r = 12{,}4$ feet; the further, however, the body is removed from the vertex, so much the smaller $\alpha$ becomes, and so much the greater, therefore, the radius of curvature.

§ 41. Proceeding from a point $A$ (Fig. 24), where the acceleration is effected at right angles to the direction of motion $A\,Y$, if, therefore, $\alpha = 90^0$, we obtain the radius of curvature $C\,A = r = \frac{c^2}{p}$, and the velocity at the following point $O$ is composed of $c$ and

of $p\,\tau$, hence $v = \sqrt{c^2+p^2\tau^2} = c + \frac{p^2\tau^2}{2c}$, because $\tau$ is infinitely small compared with $c$. If we make $v = c + \frac{p^2}{2c}\tau.\tau$, we may then consider $\frac{p^2\,\tau}{2c}$ as the acceleration, and $\frac{p^2\,\tau}{2c}.\,\tau$ as the corresponding increase of velocity. But as $\tau$ is infinitely small, the acceleration $\frac{p^2\,\tau}{2c}$ becomes also infinitely small, and in one second of time we have an infinitely small increase of velocity, and may therefore consider the motion uniform, and consequently make $v=c$.

If with the direction of motion, the direction of acceleration also changes, and if these remain constantly at right angles to each other, then we shall always have $v=c$; the velocity of motion, therefore, remains invariably the same as it was at the commencement, namely $=c$. An acceleration such as this, which is always at right angles to the motion, or causes the body to deviate at right angles from the motional direction, is called *normal acceleration,* and we hence know that it alone never causes a change of velocity, but only a deviation from the straight direction. According to the formula above $r=\frac{c^2}{p}$ we must make the normal acceleration $p=\frac{c^2}{r}$ = *the square of the velocity divided by the respective radius of curvature.*

FIG. 24.

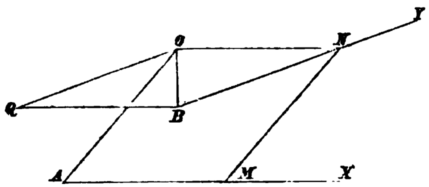



In the circle $A\,O\,D$ (Fig. 24) the radius of curvature ($r$) is the radius of the circle $C\,A = C\,O$ itself; hence when motion occurs in it, the acceleration $p=\frac{c^2}{r}$ is invariable. An invariable acceleration, therefore, which constantly causes the body to deviate at right angles from its motional direction, obliges it to revolve in a circle.

*Example.* A body which rotates in a circle of 5 feet diameter, in such a manner, that for each revolution it requires 5 seconds of time, has a velocity $c = \frac{2\,\pi\,r}{t} = \frac{2\,\pi\,.\,5}{5} = 2\,.\,\pi = 6{,}283$ feet, and a normal acceleration $p = \frac{(6{,}283)^2}{5} =$

7,896 feet; viz., in every second it will deviate from a straight line by $\frac{1}{2} p = \frac{1}{2} \times 7,896 = 3,948$ feet.

FIG. 25.




§ 42. In the *simultaneous motions of two bodies*, a constant change is taking place in their relative position, distance, &c., but with the aid of the foregoing formulæ it may be found for any given moment of time.

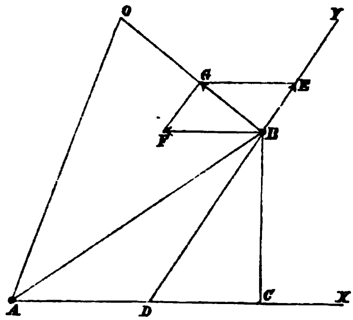
In Fig. 25, let $A$ be the point of application of the one body, $B$ that of the other; the first advances in the direction $AX$ in a certain time ($t$) to $M$, the second in the direction $BY$ in the same time to $N$; we then have in this line the relative position and distance of the bodies $A$ and $B$ at the end of this time. If we draw $AO$ parallel with $MN$, and also make $AO = MN$, then will the line $AO$ likewise give the opposite position of the bodies $A$ and $B$.

If further we draw $ON$, we obtain a parallelogram in which $ON$ is also $= AM$. If finally we make $BQ$ parallel and equal to $NO$, and draw $OQ$, we have then another parallelogram $BNOQ$, in which one side $BN$ is the absolute path ($y$) of the second body, and the other side $BQ$ the path ($x$) of the first body, described in the opposite direction. The fourth corner $O$ is the relative position of the second body, in so far as it is referred to the position of the first body, which is considered as invariable.

The relative position $O$ of a body ($B$) in motion is also found if we add to the body, besides its own proper motion ($BN$), that $AM$ of the body ($A$) to which we refer its position $BQ$, but in an inverse direction, and then resolve these motions by the parallelogram $BNOQ$ in the usual manner.

§ 43. If the motions of the bodies $A$ and $B$ are uniform, we may substitute for $AM$ and $BN$ the velocities $c$ and $c_1$, *i. e.* the spaces described in one second. We obtain, therefore, the relative velocity of the one body, when we add to the same in an opposite direction, besides its own absolute velocity, that of the body to which the first velocity is referred. The same ~~rotation~~ relation takes place with the accelerations.

FIG. 26.




*Example.* A locomotive train sets out upon the line $A\ X$, Fig. 26, from $A$ with a velocity of 35 feet; another simultaneously from $B$ upon the line $B\ Y$, which makes with the former the angle $B\ D\ X = 56°$ with a velocity of 20 feet. If now the initial distances $A\ C = 30{,}000$ feet and $C\ B = 24{,}000$ feet; how great is the distance $A\ O$ of the two trains at the end of a quarter of an hour? From the absolute velocity $B\ E = c_1 = 20$ feet of the second train, the inverse velocity $BF = c = 35$ feet of the first, and the included angle $E\ B\ F = a = 180° - B\ D\ C = 180° - 56° = 124°$, the relative velocity of the second train; $B\ G = \sqrt{c^2 + c_1^2 + 2\ c\ c_1 \cos. a} = \sqrt{35^2 + 20^2 - 2.35.20. \cos. 56°} = \sqrt{1225 + 400 - 1400 \cos. 56°} = \sqrt{1625 - 782{,}9} = \sqrt{842{,}1} = 29{,}02$ feet. For the angle $G\ B\ F = \phi$, which this makes with the first direction of motion:

$\sin. \phi = \frac{c_1 \sin. 56°}{29{,}02} = \frac{20.0{,}8290}{29{,}02}$; $Log. \sin. \phi = 0{,}75690 - 1$, hence $\phi = 34°, 50'$. Therefore in $15'$ the relative space described is $B\ O = 29{,}02.900 = 26118$ feet, the distance $A\ B = \sqrt{(30000)^2 + (24000)^2} = 38419$ feet. The angle $B\ A\ C = A\ B\ F$, whose tangent $= \frac{24000}{30000} = 0{,}8 = 38°, 40'$, therefore the angle $A\ B\ O = 38°, 40' + \phi = 38°, 40' + 34°, 50' = 73°, 30'$, and the distance of the two trains after $15'$:

$$\begin{aligned} A\ O &= \sqrt{A\ B^2 + B\ O^2 - 2\ A\ B . B\ O \cos. A\ B\ O} \\ &= \sqrt{38419^2 + 26118^2 - 2 . 38419 . 26118 \cos. 73°, 30'} \\ &= \sqrt{1588190000} = 39850 \text{ feet.} \end{aligned}$$

# SECTION II.

## MECHANICS, OR THE PHYSICAL SCIENCE OF MOTION IN GENERAL.

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## CHAPTER I.

### FUNDAMENTAL PRINCIPLES OF MECHANICS.

§ 44. *Mechanics.*—Mechanics is the science which treats of the laws of the motion of material bodies. It is an application of phoronomics to the bodies of the external world, in so far as the latter is concerned with the motion only of geometrical bodies.

Mechanics is a part of natural philosophy, or of the doctrine of laws according to which changes take place in the material world, viz. that part which considers the changes in bodies resulting from measureable motions.

§ 45. *Force.*—Force is the cause of motion or change of motion in material bodies. Every change of motion, viz., every change in the velocity of a body must be regarded as the effect of a force. For this reason we measure the force called gravity by a body falling freely, because the same incessantly changes its velocity. On the other hand, rest, or the invariability of the state of motion of a body, must not be attributed to the absence of forces; for opposite forces destroy each other and produce no effect. The gravity with which a body falls to the ground still acts though the body rest upon a table; but this action is counteracted by the solidity of the table or of the support.

§ 46. A body is in equilibrium, or the forces acting upon a body are in equilibrium, when there is no residuary effect, no motion produced or changed, or when each neutralizes the other. In a body suspended by a thread, the strength of the thread is in

equilibrium with gravity. In forces, equilibrium is destroyed, and motion arises if one of the forces be removed, or in any way counteracted; for instance, a steel spring, bent by a weight, enters into motion when the weight is taken away, because the force of the spring, called elasticity, then comes into action.

Statics is that part of mechanics which treats of the equilibrium of forces. Dynamics, on the other hand, treats of forces in so far as they produce motion.

§ 47. *Division of forces.*—According to their effects, forces are either *moving forces* or *resistances;* that is, as motion is brought about or impeded. Gravity, the elasticity of a steel spring, &c., belong to motive forces. Friction, the solidity of bodies, &c., are resisting forces or resistances, because by them motion is either diminished or destroyed, and can by no means be brought about. Moving forces are divided into accelerating and retarding; the first produces a positive, the second a negative acceleration; by the one an accelerating, by the other a retarding motion is produced. Resistances are retarding forces, but a retarding force is not always a resistance. Gravity, for example, acts upon a body projected vertically upwards to retard it; but gravity on this account is no resisting force; for by the consequent falling down of the body, it then again becomes a motive one.

There is a distinction between constant and variable forces. While constant forces always act in the same way, and therefore produce like effects in like particles of time, *i. e.* equal increments or decrements of velocity, the effects of variable forces are different at different times; while the former bring about a uniformly variable motion, to the latter corresponds a variably accelerated or a variably retarded one.

§ 48. *Pressure.—Pressure* and *traction* are the first effects of forces upon material bodies. By means of them, bodies are compressed and extended, and especially changed in their form. The pressure in traction brought about by gravity, acting vertically downwards, which the support of a heavy body, or the string to which a body is attached has to sustain, is called the weight of the body.

Pressure and traction, and weight also, are magnitudes of a particular kind, which can only virtually be compared with each other, as the action of forces serves for their measurement. The simplest, and on that account the most general, means of measuring forces is by weights.

§ 49. *Equality of Forces.*—Two weights, or two pressures, or tractions, and also the forces which correspond to these last, are equal, when one may be replaced by the other, without producing different effects. If, for example, a steel spring be bent by a weight $G$, as by another $G_1$, then are these weights, and therefore the gravities in both bodies, equal. If a loaded balance be made to vibrate as much by a weight $G$ as by another $G_1$, substituted for $G$, these two weights $G, G_1$ are equal; in this case, the arms of the balance may be equal or unequal, and the remaining load great or small.

A pressure or weight (force) is 2, 3, 4, &c., times as great as another pressure, &c., if it produces the same effect as 2, 3, 4 . . . $n$ pressures together of the second kind. If a balance, otherwise loaded at will, is brought into the same vibration by a weight ($G$) as by the addition of 2, 3, 4, equal weights ($G_1$), the weight ($G$) is 2, 3, 4, &c., times as great as the weight ($G_1$).

§ 50. *Matter.*—Matter is that by means of which bodies belonging to the external world, which in contradistinction to geometrical bodies we term material or physical, act upon our senses. Mass is the quantity of matter composing a body.

Bodies of equal volume or equal geometrical contents, have generally different weights when they consist of different kinds of matter. We cannot, therefore, infer the weight of a body from its volume until we first know the weight of a unit of volume, for instance, a cubic foot or cubic centimetre of the matter of the body.

§ 51. *Unit of Weight.*—The measurement of weights and forces consists in a comparison of them with some given invariable weight, taken as unity. The choice of this unit of weight or force is perfectly arbitrary; it is nevertheless advantageous in practice, that the weight of a volume of some universally diffused body, equivalent to that of the unit, should be chosen.

The units of weight or pressure are different in different countries; with us the unit of pressure from which all the rest are 22,:15
derived is the weight of 22,185 cubic inches of distilled water, (at a temp. 62° Fahr. taken in air, and the height of barometer at 30 inches). This weight is equal to 5760 grains, and which again is equal to one pound troy, and 7000 such grains constitute the pound avoirdupois. The gramme is the weight of a cubic centimetre of pure water in a state of maximum density (at a temperature of 4° C). The Prussian pound is also a unit referred to a weight of water. A Prussian cubic foot of distilled water in vacuo,

and at a temperature $15^0$ R. weighs 66 Prussian pounds. Now a Prussian foot $=139{,}13$ Paris lines $=0{,}3137946$ metres $=1{,}029722$ English feet: hence it follows that a Prussian pound $=467{,}711$ grammes $=1{,}031114$ pounds English.

§ 52. *Inertia.*—Inertia is that property of matter, in consequence of which it can of itself alone neither acquire nor change motion. Every material body remains at rest so long as no force acts upon it, and every material body once set into motion maintains a *uniform rectilinear motion,* so long as it is not subjected to the action of a force. Hence when a change takes place in the condition of motion of a body, when it changes its direction of motion, or when it acquires a greater or less velocity, this is not to be attributed to the body as a certain quantum of matter, but to the agency of some foreign cause or force. In as much as a development of force takes place at every change in the motion of a material body, in so far inertia may be ranked amongst forces.

If we could entirely remove the forces acting upon a mass in motion, it would move on uniformly without ceasing, but we find no where such a uniform motion, because it is not possible for us to withdraw a mass from the action of every force. When a body moves upon an horizontal table, gravity, which is then counteracted by the table, exerts upon the body no immediate action, except that from the pressure of the body against the table there arises a resistance, which we shall consider more closely in the sequel under the name of friction, which incessantly abstracts velocity from the moving body, imparts to it a retarded motion, and brings it finally to rest.

The air likewise opposes resistance to a moving body, and from this resistance, if the friction of the body were entirely put aside, a gradual diminution of velocity would ensue. But we find that the loss of velocity becomes the less, and that the motion also approximates more and more to a uniform one, the more we diminish the number and strength of these resistances; and hence we may conclude, that by the removal of all moving forces and resistances, an entirely uniform motion must take place.

§ 53. *Measure of Forces.*—The force ($P$) which accelerates an inert mass ($M$) is proportional to the acceleration ($p$), and to the mass itself ($M$): it increases in equal masses as the increment of velocity in infinitely small times, and increases by equal increments of velocity in the same ratio as the masses become greater. The *m*tuple acceleration of one and the same mass, or of equal masses

requires an $m$tuple force, and an $n$tuple mass for the same acceleration, an $n$tuple force.

As we have not yet chosen a measure of the mass, we may, therefore, at once put $P=Mp$, *i. e.* the force equal to the product of the mass and the acceleration, and at the same time, in place of the power, its effect, *i. e.* the pressure produced by it.

The correctness of this general law of motion may be readily proved by direct experiment: for example, by letting equal and differently moveable masses be impelled upon an horizontal table by means of bent springs; and it is obvious from this too, that all the consequences deduced, and all the laws developed from them for compound motions fully correspond with observation and the phenomena of nature.

§ 54. *Mass.*—All bodies fall at one and the same place of the earth, and in vacuo equally fast, viz. with an invariable acceleration $g = 9{,}81$ metres $= 32{,}2$ feet (§ 15); if, therefore, the mass of a body $= M$, and the weight measuring its gravity $= G$, we have from the last formula $G=Mg$, *i. e.*
the weight of a body is a product of its mass and the acceleration of gravity, and inversely: $M=\frac{G}{g}$, *i. e.*
the mass of a body is its weight divided by the acceleration of gravity, or the mass is that weight which a body would otherwise have if the acceleration of gravity were = to unity, as a metre, a foot, &c. At a point upon, or in the vicinity of the earth, or of any other heavenly body, where bodies do not fall with 9,81 metres = 32,2 feet, but with a velocity (after the first second) of one metre = $3\frac{1}{8}$ ft., the mass or rather its measure, is from hence immediately given by the weight of the body.

According as we express the acceleration of gravity in metres or in feet, we have, therefore, the mass

$$M=\frac{G}{9{,}81}=0{,}1019\ G, \text{ or}$$

$$M=\frac{G}{32{,}2}=0{,}031\ G.$$

The mass of a 20 lb. heavy body, $M=0{,}031\times 20=0{,}62$ lb., and inversely the weight of a mass of 20 lb. $G=32{,}2\times 20=644$ lbs.

§ 55. In so far as we assume the acceleration ($g$) of gravity as invariable, it follows that the mass of a body is exactly proportional to its weight, and that also for the masses $M$ and $M_1$, with the weights $G$ and $G_1$: $\frac{M}{M_1}=\frac{G}{G_1}$.

We hence obtain the weight as a measure of the mass of a body; the greater the mass which a body measures, the greater is its weight.

The acceleration of gravity is in fact somewhat variable, it becomes greater the nearer we approach the poles of the earth, and diminishes the more we advance towards the earth's equator; it is greatest at the poles, and least at the equator. It also diminishes the more a body is above or below the level of the sea; and attains its greatest value at the level of the sea. But since a mass, so long as nothing is added to or taken from it, is invariable, so that at all points of the earth, as well as those beyond it, at the moon, for instance, it is still the same; it hence follows that the weights also of bodies are variable and dependant upon the place of the bodies, and must be altogether proportional to the acceleration of gravity, corresponding with the place, or $\frac{G}{G_1} = \frac{g}{g_1}$.

One and the same steel spring is differently bent by one and the same weight at different places of the earth; it is least at the equator, on high mountains, and in deep mines; greatest in the vicinity of the poles, and at the level of the sea.

§ 56. *Density* is the intensity with which space is filled by matter. A body is so much the denser the more matter there is in its space. The natural measure of density is that quantity of matter (that mass) which fills a unit of volume, because matter can only be measured by weight, so that the weight of a unit of volume, a cubic metre, or cubic foot of some matter, serves as a measure of its density.

For example: the density of a cubic foot of water = 62,38 lb., and that of cast iron = 452,13 lb., because a cubic foot of water weighs 62,38 lb. = 998,08 oz. avd., and a cubic foot of cast iron weighs 452,13 lb.

From the volume $V$ of a body and its density $\gamma$, its weight $G = V\gamma$. The volume multiplied by the density gives the weight of a body.

The density of bodies is either uniform or variable, according as equal volumes of the same body are of equal or of unequal weight. The density of metals, for instance, is uniform, or they are homogeneous, because equal and very small parts of them are of the same weight: on the other hand granite is a body of variable density, because made up of parts of different densities.

*Example.*—1. If the density of lead be ~~746~~ 708 lbs., 3,2 cubic feet of lead weigh = 708 × 3,2 = 2265,5.2 lbs.—2. If the density of bar iron = 485,8 lbs.; a mass of it of

205 lbs. has a volume $V = \frac{G}{\gamma} = \frac{205}{\cancel{502}\ 485,5} = 0,4023$ cubic ft. $= \cancel{0,4083}\ 0.4217 \times 1728 = \cancel{727,48}$ 0.5554 / 27.04 cubic inches.—3. 10,4 cubic feet of deal, perfectly saturated with water, weigh 577 lbs.; the density of this wood is therefore: $\gamma = \frac{G}{V} = \frac{577}{10,4} = 55,5$ lbs.

§ 57. *Specific Gravity.*—Specific gravity or specific weight is the relation of the density of a body to that of the density of some other, generally water, taken for unity. Now the density is equal to the weight of a unit of volume: hence the specific gravity is also the relation of the weight of one body to that of another, viz. water, under the same volume.

In order not to confound the specific weight with that which belongs to a body of a certain magnitude, the last is usually called the absolute weight.

If $\gamma$ be the density of matter (of water) to which we refer the density of other matter, and $\gamma_1$ the density of any one kind of matter, whose specific gravity we will designate by $\varepsilon$, then the formula

$$\varepsilon = \frac{\gamma_1}{\gamma} \text{ and } \gamma_1 = \varepsilon \cdot \gamma.$$

holds good, and the density of a substance is equal to its specific gravity into the density of water.

The absolute weight $G$ of a mass of volume $V$ and specific gravity $\varepsilon$ is: $G = V\gamma_1 = V\varepsilon\gamma$.

*Example.*—1. The density of pure silver is 653,368 lbs. and that of water $= 62,38$ lbs., consequently the specific gravity of the former $= \frac{633,368}{62,38} = 10,474$; *i. e.* each mass of silver is 10½ times as heavy as a mass of water filling the same space.—2. The specific gravity of quicksilver $= 13,598$; its density, therefore, is $= 13,598 \times 62,38 = 848,4$ lbs.; a mass of 35 cubic inches, therefore, weighs:

$$G = 848,4 \,.\, V = \frac{848 \times 35}{1728} = 17,18 \text{ lbs.}$$

*Remark.* In these calculations the use of the French measure and weight has this advantage, that in order to effect the multiplication of $\varepsilon$ and $\gamma$, it is merely requisite to advance the decimal point; because a cubic cent. of water weighs one gramme, and a cubic metre a million, or one kilogramme. The density of quicksilver, according to the French measure and weight $= 13,598 \times 1000 = 13598$ kilog.; *i. e.* a cubic metre of quicksilver weighs 13598 kilogrammes.

§ 58. The following table contains the specific gravities of certain bodies constantly coming into application in mechanics.

| | |
|---|---|
| Mean specific gravity of dry laurel wood . . | = 0,659 |
| „ saturated with water . . | = 1,110 |

| | |
|---|---|
| Mean specific gravity of dry pine wood | = 0,453 |
| „ saturated with water | = 0,839* |
| Quicksilver | = 13,56 |
| Lead | = 11,33 |
| Copper, cast and compact | = 8,75 |
| „ forged. | = 8,97 |
| Brass | = 8,55 |
| Iron, cast, white | = 7,50 |
| „ „ grey | = 7,10 |
| „ „ medium | = 7,06 |
| „ bar iron | = 7,60 |
| Zinc, fused | = 7,05 |
| „ rolled | = 7,54 |
| Granite | 2,50 to 3,05 |
| Gneiss | 2,39 to 2,71 |
| Limestone | 2,40 to 2,86 |
| Sandstone | 1,90 to 2.70 |
| Brick | 1,40 to 2,22 |
| Masonry, with lime mortar of quarry stone: fresh | = 2,46 |
| dry | = 2,40 |
| „ „ „ of sandstone: fresh | = 2,12 |
| dry | = 2,05 |
| „ „ „ of brick: fresh | = 1,55 to 1,70 |
| dry | = 1,47 to 1,59 |
| Earth, loamy, hard stamped, fresh | = 2,06 |
| dry | = 1,93 |
| Garden earth, fresh | = 2,05 |
| dry | = 1,63 |
| Dry, poor earth | = 1,34 |

§ 59. *State of aggregation.*—Bodies appear to us, according to the different cohesion of their parts, under three principal conditions, which we term states of aggregation. They are either *solid* or *fluid,* and in the latter case, either liquid or gaseous. Solid bodies are those whose parts adhere so strongly together that a certain force is required to change the form of these bodies or to effect their division. Fluid bodies, on the other hand, are those whose parts may be displaced about each other by the smallest

* *See* "On the Absorption of Water by Wood."—*Polytechnisches Mittheilungen,* Part IV, 1844.

force. Elastic fluid bodies, whose representant is atmospheric air, are distinguished from the liquid represented by water, in as much as there is inherent in them an endeavour to dilate themselves more and more, which is not the case with water, &c.

While solid bodies have a proper form and determinate volume, liquid or aqueous bodies possess only a determinate volume without any proper form, and the elastic extensible fluid bodies have neither one nor the other.

§ 60. *Division of Forces.*—Forces are different according to their nature; we will here mention the principal:

1. *Gravity,* by means of which all bodies tend to approach towards the centre of the earth.

2. The *force of inertia,* which manifests itself when changes in the velocity of inert masses occur.

3. The *muscular force* of animated beings; the force exerted by the muscles of men and animals.

4. *Elasticity* or *spring-force,* which bodies exhibit in a change of their form or volume.

5. The *force of heat* or *caloric,* in consequence of which bodies expand or contract by a change of temperature.

6. The *magnetic force,* or the attraction and repulsion of magnets.

7. The *cohesive force,* the force by which the parts of a body are kept together, and resist separation.

8. *Adhesion,* the force with which bodies brought into close contact attract each other.

The resistances of friction, rigidity, solidity, &c., arise mainly from the force of adhesion.

§ 61. *In reference to forces we have to distinguish:*

1. Its *point of application,* that point of a body on which the force immediately acts.

2. Its *direction,* the straight line in which a force moves forward its point of application, or strives to move it forward, or to impede its motion. The *direction of a force,* like every direction of motion, has *two senses,* it can take place from left to right, or from right to left, from above to below, and from below to above. The one is termed *positive, the other negative.* As we write from left to right, and from above to below, it would be most convenient were we to call these motions positive, and those in the opposite direction, negative.

3. The *absolute magnitude* or *intensity* of a force, which,

from the above, is measured by weights, as pounds, kilogrammes, &c.

§ 62. *Action and re-action.*—The first effect which a force produces in a body, is a change of form or volume combined with extension or contraction, which begins at the point of application, and from thence diffuses itself further and further. By this inward change of the body, its inherent elasticity is called into action, puts itself into equilibrium with the force, and, therefore, is equal and opposed to the force. Action and re-action are equal and opposed to each other. This law not only prevails in reference to forces, produced by contact, but also in the so called forces of attraction and repulsion, amongst which the magnetic and gravity itself may be ranked. The stronger a magnet attracts a bar of iron, the stronger is the magnet itself attracted by the iron. The force with which the moon is attracted towards the earth (gravitation) is equal to that with which the moon reacts upon the earth. The force with which a weight presses upon its support is given back in an opposite direction; the force with which a workman draws or pushes at a machine, &c., reacts upon the workman and strives to move him in the opposite direction. When a body impinges against another, the pressures are reciprocally equal on each of the bodies.

§ 63. *Division of Mechanics.*—The whole subject of mechanics may be included under two principal divisions, according to the state of aggregation of bodies.

1. The *mechanics of solid bodies,* which is also well named, *geomechanics.*

2. The *mechanics of fluid bodies,* hydromechanics or hydraulics; the last is subdivided into:

1. Into the mechanics of water and liquid bodies especially, *hydromechanics* or *hydraulics.*

2. Into the *mechanics* of air, and *other aëriform* bodies, especially, aëromechanics, the mechanics of elastic fluids.

If we now have regard to the division of mechanics into statics and dynamics, we have the following parts:

1. *Statics of solid bodies,* or *geostatics.*
2. *Dynamics of solid bodies,* or *geodynamics.*
3. *Statics of fluids,* or *hydrostatics.*
4. *Dynamics of fluids,* or *hydrodynamics.*
5. *Statics of aëriform bodies,* or *aërostatics.*
6. *Dynamics of aëriform, aërodynamics,* or *pneumatics.*

# CHAPTER II.

## THE MECHANICS OF A MATERIAL POINT.

§ 64. A material point is a material body whose dimensions are indefinitely small in comparison with the space occupied by it. In order to simplify the representation, we will in the following consider only the motion and equilibrium of a material point. A finite body is a continuous union of infinitely many material points. If the single points or elements are all perfectly equal, *i. e.* move equally quick, in parallel straight lines, we may then apply the theory of the motion of a material point to that of the whole body, because in this case, we may assume that equal parts of the mass of the body are impelled by equal parts of the force.

§ 65. *Simple constant force.*—If ($p$) be the acceleration with which a mass ($M$) is impelled by a force, we have from § 58, the force:

$$P = Mp, \text{ and inversely, the acceleration, } p = \frac{P}{M}.$$

If further we put the mass $M = \frac{G}{g}$, where $G$ is the weight of the
weight of a unit of mass
body, and $g$ ~~the acceleration of gravity,~~ we have the force:

$$1.\ P = \frac{p}{g} G, \text{ and the acceleration:}$$

$$2.\ p = \frac{P}{G} g.$$

We find, therefore, the force ($P$) which impels a body with a certain acceleration ($p$) when we multiply the weight of the body ($G$) by the ratio $\left(\frac{p}{g}\right)$ of its acceleration, to that of gravity.

Inversely, the acceleration ($p$), with which a body is moved forward by a force ($P$) is given, when the acceleration ($g$) of gravity is multiplied by the ratio $\left(\frac{P}{G}\right)$ of the force and weight of the body.

*Example.* Let us suppose a body lying on an horizontal and perfectly smooth table, which presents no impediment to the body in its course, but counteracts the effect of gravity upon it. If this body be pressed upon by a force acting horizontally, the

body will give way to this influence, and move forward in the direction of this force. If the weight of this body be $G = 50$ lbs., and if $P = 10$ lbs. presses uninterruptedly upon it, it will enter into a uniformly accelerated motion with the acceleration $p = \frac{P}{G} \cdot g = \frac{10}{50} \times 32,2 = 6,44$ feet. On the other hand, if the acceleration with which a 42 lb. heavy body becomes accelerated by a force $(P) = 9$ feet, then will this force $P = \frac{p}{g} \cdot G = \frac{9}{32,25} \times 42 = 0,031 \times 378 = 11,7$ lbs.

§ 66. If the force which acts upon a body is constant, there arises a uniformly variable motion, and indeed a uniformly accelerated one, if the direction of the force corresponds with the initial direction of the motion; and on the other hand a uniformly retarded one, if the direction of the force is opposite to that of the initial direction of motion. If we substitute in the formulæ (§ 13 and § 14) for $p$, the value $\frac{P}{M} = \frac{P}{G} g$, we obtain the following:

I. For uniformly accelerated motions:

1. $v = c + \frac{P}{G} gt$, or $v = c + 32,2 \frac{P}{G} t$.

2. $s = ct + \frac{P}{G} \frac{gt^2}{2}$, or $s = ct + 16,1 \frac{P}{G} t^2$.

II. For uniformly retarded motions:

1. $v = c - \frac{P}{G} gt = c - 32,2 \frac{P}{G} t$.

2. $s = ct - \frac{P}{G} \frac{gt^2}{2} = ct - 16,1 \frac{P}{G} t^2$.

With the help of these formulæ all those questions may be answered which can be proposed relative to the rectilinear motions of bodies by a constant force.

*Example.*—1. A 2000 lbs. heavy carriage goes with a 4 feet velocity upon an horizontal line, offering no impediments to it, and pushed forward by an invariable force of 25 lbs. during 15 seconds, with what velocity will it proceed after the action of this force? This velocity $v = c + 32,2 \frac{P}{G} t$, but $c = 4$, $P = 25$ lbs., $G = 2000$, and $t = 15$; hence it follows, $v = 4 + 32,2 \cdot \frac{25}{2000} \cdot 15 = 10,03$ feet.—2. Under similar circumstances, a 5500 lbs. heavy carriage, which, setting out with a uniform velocity, has traversed 950 feet in 3 minutes, is so impelled forward by a force acting continuously for 30 seconds, that it afterwards passes over 1650 feet in 3 minutes; what is this force? Here the initial velocity $c = \frac{950}{3.60} = 5,277$ feet, and the terminal velocity $v = \frac{1650}{3.60} = 9,166$ feet; therefore $\frac{P}{G} gt = v - c = 3.889$, and

the force $P = \frac{3,889\,.\,G}{gt} = 0,031 \times 3,889 \times \frac{5500}{30} = 0,12056 \times \frac{550}{3} = 22,10$ lbs.—3. A sledge, weighing 1500 lbs., sliding forward with a 15 ft. velocity, loses, through friction, upon its horizontal support, its whole motion in 25 seconds; how great is this friction? Here the motion is uniformly retarded, and the terminal velocity $v = 0$; hence $c = 32,2\,\frac{Pt}{G}$, and $P = 0,031\,\frac{Gc}{t} = 0,031 \times \frac{1500 \times 15}{25} = 0,031 \times 900 = 27,9$lbs. the friction demanded.—4. Another sledge, of 1200 lbs. and 12 feet initial velocity, has to overcome by its motion a friction of 45 lbs.; what velocity has it after 8 seconds, and how great is the distance described? The terminal velocity is $v = 12 - 32,2 \times \frac{45 \times 8}{1200} = 12 - 9,868 = 2,152$ feet, and the distance described $s = \left(\frac{c+v}{2}\right) t = \left(\frac{12 + 2,132}{2}\right) \times 8 = 56,528$ feet.

§. 67. *Mechanical effect.*—The *work done,* or mechanical effect, is that effect of a force which it produces in the overcoming a ***resistance:*** as that of inertia, friction, gravity, &c. Work is performed when loads are lifted, a great velocity imparted to masses, bodies changed in their form or divided, &c. The work done, or the mechanical effect produced depends not only on the force, but also on the distance through which it is made to act or to overcome the resistance; it increases, of course, simultaneously with the force and the distance. If we lift a body slowly enough to allow of our neglecting its inertia, the labour expended is then proportional to its weight; for 1. the effect is the same whether $m$ (3) times the weight ($m\,G$) is lifted to a certain height, or whether $m$ (3) bodies of the single weight ($G$) are lifted to the same height; it is, namely, $m$ times as great as the effort necessary for the lifting of a single weight to that height; and, again, 2. the work is the same, whether one and the same weight be raised to $n$ (5) times the height ($n\,h$), or $n$ (5) times through the height, and it is of course $n$ (5) times as great as if the same weight were raised to a single height ($h$). The work again done by a slowly falling weight is proportional to the magnitude of this weight and the height from which it has descended. This proportionality also holds in every other kind of work done. In order to make a saw-cut of a given depth of double the length, there are twice as many particles to separate as from a cut of a single length; the work, therefore, is twice as great. The double length requires double the distance to be described by the force, consequently the work is proportional to the distance. Similarly will the work of a mill-set increase with the quantity of grains of a certain kind of corn, which it grinds to a certain degree. This quantity, under otherwise similar circumstances, is

proportional to the number of revolutions, or rather to the distance which the upper mill-stone, during the grinding of this quantity of corn, has gone through; consequently the work increases in proportion to the distance.

§ 68. The dependence above shewn of the work produced by a force upon the magnitude of the force and distance described by it, allows us to take that amount of work which is expended in overcoming a resistance of the magnitude of the unit of weight (as a kilogramme, pound, &c.), along a path of the magnitude of the unit of length (metre or foot), as unit of the mechanical effect or dynamical unit, and then we may put the measure of this equal to the product of the force or resistance, and the distance described in the direction of the force whilst overcoming this resistance.

If we put the amount of the resistance itself $= P$, and the distance described by the force, or rather by its point of application, in overcoming this $= s$, the labour expended is:

$$L = P s \text{ units of work.}$$

In order to define more clearly the unit of work, for which the single name, *dynam*, may be used, both factors $P$ and $s$ are generally given; and, therefore, instead of units of work, we say kilogrammetres, pounds-feet; and inversely, metrekilo. and feet-pounds according as the weight and distance are expressed in kilogrammes and metres, or in pounds and feet. These terms are usually expressed for simplicity by the abbreviations *mk* or *km*, *lb. ft.* or *ft. lb.*

*Example*.—1. In order to raise a stamper 210 lbs. 15 inches high, the mechanical effect $L = 210 \times \frac{15}{12} = 262{,}5$ ft. lbs. is necessary.—2. By a mechanical effect of 1500 ft. lbs., a sledge, which in its motion has to overcome a friction of 75 lbs., is driven forward a space $s = \frac{L}{P} = \frac{1500}{75} = 20$ feet.

§ 69. Not only in an invariable force or constant resistance is the labour a product of the force and distance, but also the labour may be expressed as a product of the distance and force, when the resistance whilst being overcome is variable, if a mean value of the continuous succession of the forces be taken as the force. The relation is here the same as that of the time, the velocity, and the space; for the last may be regarded as a product of the time by the mean value of the velocities. The same graphical representations are here also applicable. The mechanical effect produced or expended may be considered as the area of a rectangular figure, *ABCD*, Fig. 27,

FIG. 27.

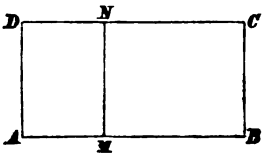
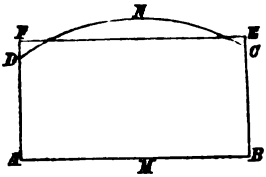



FIG. 28.

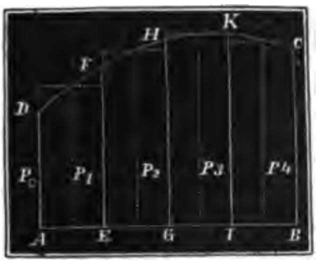



whose base $AB$ is the space described ($s$), and whose height is either the invariable force ($P$) itself, or the mean of the different values of the forces. In general, the work may be represented by the area of a figure $ABCD$, Fig. 28, which has for its base the space ($s$), and whose height above each point of the base is equal to the force corresponding with each point of the path described. If the figure $ABCD$ be transformed into a rectangular one $ABEF$ of like area, we have the height $AF = BE$ for the mean value of the force—*the mean effort*.

§ 70. Arithmetic and geometry give different methods for finding a mean value from a constant succession of magnitudes. Amongst these, Simpson's rule is that which is the most frequently applied in practice, and it combines a high degree of accuracy with great simplicity.

In every case it is necessary to divide the space $AB = s$ (Fig. 29) into $n$ (the more the better) equal parts, as $AE = EG = GI$, &c., and to measure the forces $EF = P_1$, $GH = P_2$, $IK = P_3$, &c., at the ends of these parts of the distance. If then we put the initial force $AD = P_0$ and the force at the other end $BC = P_n$, we have for the mean force:

FIG 29.




$P = (\frac{1}{2}P_0 + P_1 + P_2 + P_3 + \ldots + P_{n-1} + \frac{1}{2}P_n) \div n$, and, therefore, its work is:

$$Ps = (\tfrac{1}{2}P_0 + P_1 + P_2 + \ldots + P_{n-1} + \tfrac{1}{2}P_n)\frac{s}{n}.$$

If the number of parts ($n$) be even, viz. 2, 4, 6, 8, &c., Simpson's rule gives still more accurately the mean force:

$$P = (P_0 + 4P_1 + 2P_2 + 4P_3 + \ldots + 4P_{n-1} + P_n) \div 3n,$$

and, therefore, the corresponding work:

$$Ps = (P_0 + 4P_1 + 2P_2 + 4P_3 + \ldots + 4P_{n-1} + P_n)\frac{s}{3n}.$$

*Example.* In order to find the mechanical work which a draught horse performs in drawing a carriage over a certain way, we make use of a dynamometer, or measurer of force, which is put into communication on one side with the carriage, and on the other with the traces of the horse, and the force is observed from time to time. If the initial force $P_0$ = 110 lbs., the velocity after describing 25 feet = 122 lbs.; after 50 feet = 127 lbs.; after 75 feet = 120 lbs.; and at the end of the whole distance of 100 feet = 114 lbs.; then the mean value, according to the first formula: $P = (\frac{1}{2} \,.\, 110 + 122 + 127 + 120 + \frac{1}{2} \,.\, 114) \div 4 = 120{,}25$ lbs., and the mechanical work: $P s = 120{,}25 \times 100 = 12025$ ft. lbs.
from the second formula: $P = (110 + 4 \,.\, 122 + 2 \,.\, 127 + 4 \,.\, 120 + 114) \div 3 \times 4$

$$= \frac{1446}{12} = 120{,}5 \text{ lbs., and the mechanical work}$$

$$P s = 120{,}5 \times 100 = 12050 \text{ ft. lbs.}$$

§. 71. *Principle of the vis viva, or living forces.*—If in the formula of (§. 13) $s = \frac{v^2 - c^2}{2p}$ or $ps = \frac{v^2 - c^2}{2}$ we substitute for the acceleration $p$, its value $\frac{P}{G} g$, we thus obtain $Ps = \left(\frac{v^2 - c^2}{2g}\right) G$, or if we designate the heights due to the velocities $\frac{v^2}{2g}$ and $\frac{c^2}{2g}$ by $h$ and $h_1$:

$$P s = (h - h_1)\, G.$$

If we interpret this equation, so useful in practical mechanics, we find that the work ($P\,s$) which a mass either acquires when it passes from a lesser velocity ($c$) into a greater ($v$), or produces, when it is compelled to pass from a greater velocity into a less, is constantly equal to the product of the weight of this mass, and the difference of the heights due to the velocities $\left(\frac{v^2}{2g} - \frac{c^2}{2g}\right)$.

*Example.*—1. In order to impart to a 4000 lbs. heavy carriage, upon a perfectly smooth railroad, a velocity of 30 feet, a mechanical work $P s = \frac{v^2}{2g} G = 0{,}0155\, v^2 G$ $= 0{,}0155 \times 900 \times 4000 = 55800$ ft. lbs. is required; and just so much work will this carriage perform if a resistance be opposed to it, and it be gradually brought to rest. —2. Another carriage of 6000 lbs. goes forward with a velocity of 15 feet, which is transformed by a force acting upon it into a velocity of 24 feet, how great is the work acquired by this carriage, or done by the force? To the velocities 15 and 24 feet correspond the heights of velocity $h_1 = \frac{c}{2g} = 3{,}40$ ft., and $h = \frac{v^2}{2g}$ $= 8{,}944$ ft.; from this the mechanical work $P s = (h - h_1)\, G = 5{,}454 \times 6000 =$ 32724 ft. lbs. If now the distance be known in which this change of velocity goes on, the force may be found; and when this is known, the distance may be determined. In this last case, for example, let the distance of the carriage amount to 100 feet, and whilst describing this, the velocity passes from 15 into 24 feet; we have the force $P = (h - h_1) \frac{G}{s} = \frac{327{,}24}{100} = 327{,}24$ lbs. Were the force itself

2000 lbs., the space $s$ would be $= (h—h_1)\ \frac{G}{P} = \frac{32724}{2000} = 16,362$ feet. — 3. If a 500 lbs. sledge has entirely lost, through friction on its path, its velocity of 16 feet, after describing a space of 100 feet, then is the resistance of friction $P = \frac{h\ G}{s}$ $= 0,0155 \times 16^2 \times \frac{500}{100} = 0,0155 \times 256 \times 5 = 19,840$ lbs.

§. 72. The formula found for the work in the foregoing paragraph:

$$P\ s \quad (h—h_1)\ G$$

is not only good for constant, but also for variable forces, if instead of $P$ the mean value of the force (from §. 70) be introduced; for if the whole space ($s$) of motion be considered as consisting of equal and uniformly accelerated parts described $\left(\frac{s}{n}\right)$, then we have the amount of work for these:

$$P_1\left(\frac{s_1}{n}\right)=\frac{v_1^2—c^2}{2g}G,$$

$$P_2\left(\frac{s_2}{n}\right)=\frac{v_2^2—v_1^2}{2g}G,$$

$$P_3\left(\frac{s_3}{n}\right)=\frac{v_3^2—v_2^2}{2g}G,$$

&c., in so far as $v_1$, $v_2$, $v_3$, &c., stand for the velocities acquired at the end of these parts of space; and by the addition of all these works we have the whole work required for the transformation of the velocity $c$ into $v$:

$P\,s = (P_1 + P_2 + P_3 + \ldots)\frac{s}{n} = \frac{v^2—c^2}{2g}\ G$, because for an infinite number ($n$) of forces $(P_1 + P_2 + P_3 + \ldots) \div n$, it transforms itself into a mean force, and because the members on the right hand of the equation $\frac{v_1^2}{2g}\ G$ and $—\frac{v_1^2}{2g}\ G$, as also $\frac{v_2^2}{2g}\ G$ and $—\frac{v_2^2}{2g}\ G$, &c. are opposed to each other, so that the members $\frac{v^2}{2g}\ G$ and $\frac{c^2}{2g}\ G$, determined by the terminal velocity $v$ and the initial velocity $c$, only remain.

The formula $P\ s = \left(\frac{v^2—c^2}{2\ g}\right)\ G = (h—h_1)\ G$ is not used only for the determination of the work, but also, and at times very often, for the measurement of the terminal velocity. In the last case $h$ is put $= h_1 + \frac{Ps}{G}$ or $v = \sqrt{c^2 + 2g\frac{Ps}{G}}$. If by the constant motion of a body, the terminal velocity $v =$ the initial velocity $c$,

the work done = null, *i. e.* as much work is performed by the accelerated, as is expended by the retarded part of the motion.

*Example.*—A carriage of 2500 lbs. proceeding upon a railroad without friction, has acquired by an augmentation of its velocity, which at the commencement amounted to 10 ft., a mechanical work of 8000 lbs., its velocity after this work will be:

$$v = \sqrt{10^2 + 64,4 \cdot \frac{8000}{2500}} = \sqrt{100 + 200} = 14,53 \text{ feet.}$$

*Remark.* The product of the mass $M = \frac{G}{g}$ and the square of the velocity ($v^2$): $M v^2$ is called, without attaching to it any definite idea, the living force (*vis viva*) of the moved mass; and hereafter, the mechanical work which a moved mass acquires, may be put equal to half of the *vis viva* of the same. If a mass enters from a velocity $c$ into another $v$, the work performed is equal to half the difference of the *vis viva* at the commencement of the change of velocity. This law of the mechanical performance of bodies by means of their inertia, is called the *principle of living forces*, or the *vis viva.*

§ 78. *Composition of Forces.*—Two forces $P_1$ and $P_2$ act upon one and the same body, in the same or in an opposite direction, the effect is the same as if only one force acted upon the body, which is the sum or difference of these forces; for these forces impart to the mass $M$ the acceleration, $p_1 = \frac{P_1}{M}$ and $p_2 = \frac{P_2}{M}$, consequently from § 28, the acceleration resulting from both, is: $p = p_1 \pm p_2 = \frac{P_1 \pm P_2}{M}$, and accordingly the force corresponding to this, is: $P = M p = P_1 \pm P_2$.

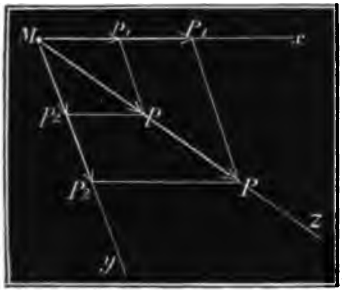
The equivalent force $P$ derived from these two is called the *resultant*; its constituents $P_1$ and $P_2$ the *components*.

*Example.*—1. A body lying flat upon the hand presses so long only upon it with its absolute weight as the hand is at rest, or is moved up and down uniformly with the body; but if the hand be raised quickly, it suffers a greater pressure; on the other hand, if it be suddenly dropped, the pressure is then less than the weight; it becomes null if the hand be drawn back with the acceleration of gravity. If the pressure on the hand = $P$, the body falls with a force $G - P$, whilst its mass $M = \frac{G}{g}$; if we put the acceleration with which the hand falls $= p$, $G - P = \frac{G}{g} p$, and therefore the pressure $P = G - \frac{p}{g} G = \left(1 - \frac{p}{g}\right) G$. If the body on the hand be raised with the acceleration $p$, $-p$ is then opposed to the acceleration $g$, therefore the pressure upon the hand $P = \left(1 + \frac{p}{g}\right) G$. According as a body ascends or descends with a 20 feet acceleration, the pressure upon the hand $= \left(1 - \frac{20}{32,2}\right) G = (1 - 0,62)\, G = 0,38$, of the weight of the body, or $= 1 + 0,62 = 1,62$.—2. If with the flat hand I throw a body of 3 lbs. 14 feet perpendicularly upwards, whilst I urge it on with the hand for the first 2 feet, the mechanical work performed is $P s = G h = 3 \times 14$

$=42$ ft. lbs., and the pressure upon the hand, $P=\frac{42}{2}=21$ lbs. Whilst the resting body presses with 3 lbs., it reacts upon the hand during the projection with 21 lbs.

§ 74. *Parallelogram of Forces.*—When a material point $M$, Fig. 30, is acted upon by two forces, $P_1, P_2$, whose directions $MX$ and $MY$ make with each other the angle $XMY=\alpha$, these lines generate the accelerations in these directions, $p_1=\frac{P_1}{M}$ and $p_2=\frac{P_2}{M}$, and from their union, there arises a mean acceleration (§ 34) in the direction $MZ$, both of which are given by the diagonal of a parallelogram formed from $p_1$, $p_2$ and the angle $\alpha$; this mean or resultant acceleration $p=\sqrt{p_1^2+p_2^2+2p_1p_2\cos.\alpha}$, and for the angle $\phi$ which its direction makes with $MX$ of the one acceleration $p_1$:

FIG. 30.



$$\sin.\phi=\frac{p_2\sin.\alpha}{p}.$$

If we substitute in these formulæ the above values of $p_1$ and $p_2$:

$$p=\sqrt{\left(\frac{P_1}{M}\right)^2+\left(\frac{P_2}{M}\right)^2+2\left(\frac{P_1}{M}\right)\left(\frac{P_2}{M}\right)\cos.\alpha} \text{ and}$$

$$\sin.\phi=\left(\frac{P_2}{M}\right)\frac{\sin.\alpha}{p}.$$

If we multiply the first equation by $M$,

$$Mp=\sqrt{P_1^2+P_2^2+2P_1P_2\cos.\alpha}, \text{ or,}$$

since $Mp$ is the force corresponding to the acceleration:

$$1.\ P=\sqrt{P_1^2+P_2^2+2P_1P_2\cos.\alpha}.$$

$$2.\ \sin.\phi=\frac{P_2\sin.\alpha}{P}.$$

*Thus, the resultant force is determined in magnitude and direction from the component forces exactly as the resultant acceleration from the component accelerations.*

If we represent the forces by straight lines, and these lines be drawn, bearing the same proportions to each other as do weights as pounds, &c., the mean force may be represented by the diagonal of the parallelogram, whose sides are formed by the lateral forces, and

one of whose angles is equal to that made by the directions of these lateral forces. The parallelogram which is constructed from the lateral forces, and whose diagonal is the mean force, is called the parallelogram of forces.

*Example.* When a body of 150 lbs. weight, resting upon a perfectly smooth table, (Fig. 31) is acted upon by two forces $P_1 =$ 30 lbs., and $P_2 = 24$ lbs., which make with each other an angle $P_1 M P_2 = \alpha + \beta =$ 105° : in what direction, and with what acceleration will the motion take place? Since $cos. (\alpha + \beta) = cos. 105° = - cos. 75°$, the mean force:

FIG. 31.


$$P = \sqrt{30^2 + 24^2 - 2 \times 30 \times 24 \, cos. \, 75°}$$
$$= \sqrt{900 + 576 - 1440 \, cos. \, 75°}$$
$$= \sqrt{1476 - 372,4} = 33,22 \text{ lbs.},$$ the acceleration corresponding with it is:

$$p = \frac{P}{M} = \frac{P g}{G} = \frac{33,2 \times 322 \, . \, 2}{150} = 7,1312 \text{ ft.}$$

The direction of motion makes with the direction of the first force an angle $\alpha$, which is determined by:

$$sin. \, \alpha = \frac{24}{33,22} sin. \, 105° = 0,7224 \, sin. \, 75°$$
$$= 0,6978, \text{ or } \alpha = 44°, 15'.$$

*Remark.* The mean force $P$ depends, from the formulæ found, only on the component forces, and not on the mass of the body upon which the forces act. For this reason we find in many works on mechanics, the correctness of the parallelogram of forces proved without regard to the mass, but with the assumption of some fundamental law. There exist many such purely statical proofs. In each of the following works we find a different one: Eytelwein's "Handbuch der Statik fester Körper," Gerstner's "Handbuch der Mechanik," Kayser's "Handbuch der Statik," Möbius' "Lehrbuch der Statik," Rühlmann's "technische Mechanik." The proof in Gerstner's Mechanics presupposes the theory of the lever; it is very simple, and is found in many old, and also in later writings, as those of Kästner, Monge, Whewell, &c. Kayser's proof is that of Poisson, in an elementary garb. Möbius' development is based upon a particular theory, that of *couples*, introduced by Poinsot. Duchayla first gave the proof in Rühlmann's Mechanics, which has also been adopted in many other works, as in Moseley's "Mechanical Principles," &c. A theory of this parallelogram, founded upon the laws of motion, is to be met with in Newton's "Principia," and is also made use of by many later writers, Venturoli, Poncelet, Burg, &c. "Elementi di Meccanica e d'Idraulica," de Venturoli; "Mécanique Industrielle," par Poncelet; "Popular Compendium of the Science of Mechanics and its application to Machines," by Burg.

§ 75. *Resolution of Forces.*—By help of the parallelogram of forces, not only two or more forces may be reduced to a single one, but also given forces under given relations may be resolved into two or more forces. If the angles $\alpha$ and $\beta$ are given, which the components $M P_1 = P_1$, and $M P_2 = P_2$, make with the

given force $M\underset{p}{R} = P$, the components may be found from the formulæ:

$$P_1 = \frac{P\ sin.\ \beta}{sin.\ (a+\beta)},\quad P_2 = \frac{P\ sin.\ a}{sin.\ (a+\beta)}.$$

If the components are at right angles to each other, $a+\beta=90^0$, and $sin.\ (a+\beta)=1$, and $P_1=P\ cos.\ a$ and $P_2=P\ sin.\ a$. If $\beta$ and $a$ be equal to one other, $P_2=P_1$, viz:

$$P_2 = \frac{P\ sin.\ a}{sin.\ 2\ a} = \frac{P}{2\ cos.\ a} = P_1.$$

*Example* 1. What is the pressure of a body $M$ upon a table $AB$, Fig. 32, whose weight $G = 70$ lbs. and upon which a force $P = 50$ lbs. acts, and whose direction is inclined to the horizon at an angle $PMP_1 = a = 40°$? The horizontal component of $P$ is $P_1 = P\ cos.\ a = 50\ cos.\ 40° = 38{,}30$ lbs., and the vertical component $P_2 = P\ sin.\ a = 50\ sin.\ 40° = 32{,}14$ lbs.; the latter strives to draw the body from the table, there remains then for the pressure: $G - P_2 = 70 - 32{,}14 = 37{,}86$ lbs.—2. If a body of 110 lbs. is so moved along an horizontal way by two forces, that it describes in the first second a space of 6,5 feet, in a direction which deviates from the two directions of force by an angle $a = 52°$ and $\beta = 77°$, the forces themselves are given as follows. The acceleration is twice the space in the first second, so that $p = 2 \times 6{,}5 = 13$ ft. Now the mean force is $P = \frac{pG}{g} = 0{,}032 \times 13 \times 110 = 44{,}33$ lbs., therefore the one component $P_1 = \frac{P\ sin.\ 77°}{sin.\ (52° + 77°)} = \frac{44{,}33\ sin.\ 77°}{sin.\ 51°} = 55{,}58$ lbs., and the other $P_2 = \frac{44{,}33\ sin.\ 52°}{sin.\ 51°} = 45{,}59$ lbs.

FIG. 32.

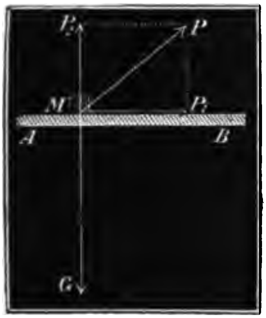
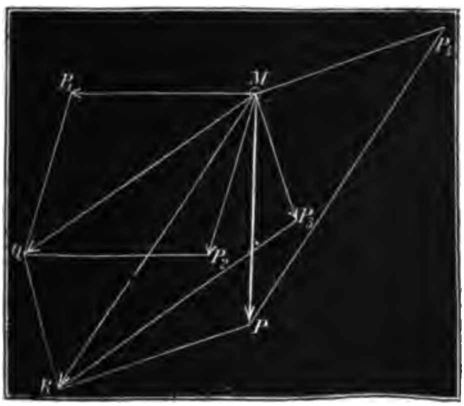



FIG. 33.




§ 76. *Forces in a Plane.*—In order to find the mean force $P$ for a system of forces $P_1$, $P_2$, $P_3$, &c., we may adopt exactly the same method (§ 33) as that followed in the composition of velocities, viz: by the repeated application of the parallelogram of forces, we may resolve them two and two and so on,

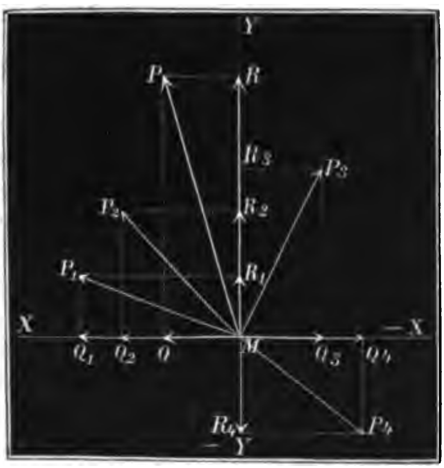
till but a single force remains. The forces $P_1$ and $P_2$, for example, give from the parallelogram $M P_1 Q P_2$, the mean force $MQ = Q$, if this be joined to $P_3$, we have from the parallelogram $MQRP_3$, $MR = R$; and this last again forms a parallelogram with $P_4$ and gives the force $MP = P$ the last, and the resultant of the four forces $P_1, P_2, P_3, P_4$.

It is not necessary, in this way of composing forces, to complete the parallelogram, and draw its diagonal. We may form a polygon $MP_1QRP$, whose sides $MP_1$, $P_1Q$, $QR$, $RP$, are parallel and equal to the given components $P_1$, $P_2$, $P_3$, $P_4$, the last side $MP$ completing the polygon will be the mean force sought, or rather its measure.

*Remark.* It is very useful to solve mechanical problems by construction also; though this method does not admit of such accuracy as that of calculation, it is free on the other hand from great errors, and may therefore serve as proof of the calculation. In Fig. 33 the forces meet each other under the given angles $P_1 M P_2 = 72°, 30'$; $P_2 M P_3 = 33°, 20'$, and $P_3 M P_4 = 92°, 40'$, and are so drawn that a pound is represented by a line of one Prussian inch. The forces $P_1 = 11{,}5$ lb., $P_2 = 10{,}8$ lbs., $P_3 = 8{,}5$ lbs., $P_4 = 12{,}2$ lbs., are therefore expressed by sides of 11,5 lines = 0827 ... inches, 10,8 lines = 0778 ... inches, 8,5 lines = 0756 ... inches, 12,2 lines = 0878 ... inches in length. A careful construction of the polygon of forces gives the magnitude of the mean force $P = 14{,}6$ lbs. and the variation of its direction $MP$ from the direction $MP_1$ of the first force = 86½°.

§ 77. The resultant $P$ is determined more simply and clearly if each of the given components $P_1$, $P_2$, $P_3$, &c., be resolved according to two axial directions $X\overline{X}$ and $Y\overline{Y}$, Fig. 34, at right angles to each other, into component forces as $Q_1$ and $R_1$, $Q_2$ and $R_2$, $Q_3$ and $R_3$, &c., the forces lying in the same direction of axis, added together, and the resultants in magnitude and direction of these two rectangular forces be then sought for. If the angles $P_1MX$, $P_2MX$, $P_3MX$, &c., which the directions of the forces $P_1$, $P_2$, $P_3$, make with the axis $XX = a_1, a_2, a_3$, &c., we have the components $Q_1 = P_1 \cos. a_1$, $R_1 = P_1 \sin. a_1$, $Q_2 = P_2 \cos. a_2$, $R_2 = P_2 \sin. a_2$, whence it follows from $Q = Q_1 + Q_2 + Q_3 + \ldots$,

FIG. 34.



1. $Q=P_1\ cos.\ a_1+P_2\ cos.\ a_2+P_3\ cos.\ a_3+\ldots$, and from $R=R_1+R_2+R_3+\ldots$,

2. $R=P_1\ sin.\ a_1+P_2\ sin.\ a_2+P_3\ sin.\ a_3+\ldots$

From the two components $Q$ and $R$ so found, the magnitude of the resultant sought, is:

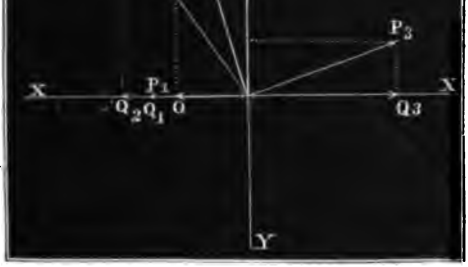
3. $P=\sqrt{Q^2+R^2}$ and the angle $PMX=\phi$, whose direction with $X\overline{X}$ is given by

4. $tang.\ \phi=\frac{R}{Q}$.

In the algebraical addition of the forces, regard must be had to the sign, for if it be different in two forces, *i. e.* if the directions of these be upon opposite sides of the point of application $M$, this addition then becomes arithmetical subtraction (§ 73). The angle $\phi$ is acute, as long as $Q$ and $R$ are positive, it is between one and two right angles, when $Q$ is negative and $R$ positive; between two and three, when $Q$ and $R$ are both negative, and lastly, between three and four, when $R$ only is negative.

*Example.* What is the magnitude and direction of the resultant of the three components $P_1$ = 30 lbs., $P_2$ = 70 lbs., $P_3$ = 50 lbs., whose directions, lying in a plane, make between them the angles $P_1 M P_2$ = 56° and $P_2 M P_3$ = 104°? If we draw the axis $X\overline{X}$ in the direction of the first force, we have $a_1 = 0$, $a_2 = 56°$, and $a_3 = 56° + 104° = 160°$; hence, 1. $Q = 30 \times cos.\ 0° + 70 \times cos. 56° + 50 \times cos. 160° = 30 + 39{,}14 - 46{,}98 = 22{,}16$ lbs.; and 2. $R = 30 \times sin.\ 0° + 70 \times sin.\ 56° + 50\ sin.\ 160° = 0 + 58{,}03 + 17{,}10 = 75{,}13$ lbs. Hence 3. $tang.\ \phi = \frac{75{,}13}{22{,}16} = 3{,}3903$; therefore the angle which the resultant makes with the positive part of the axis $M X$ or the force $P_1$ is

FIG. 35.



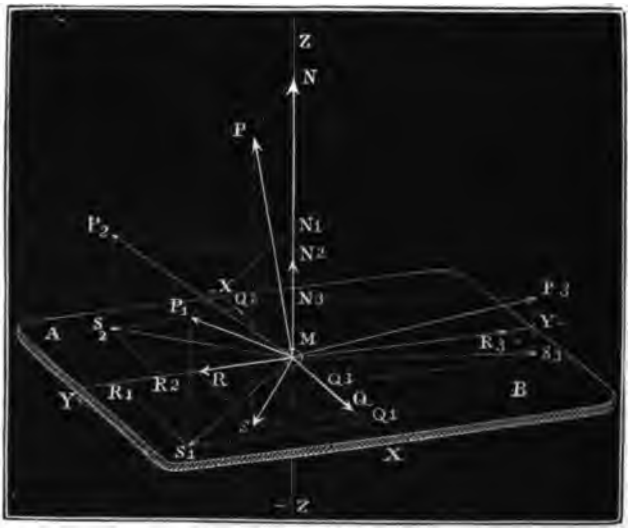

$\phi = 73°\ 34'$; lastly, the force itself $P = \sqrt{Q^2 + R^2} = \frac{Q}{cos.\ \phi} = \frac{R}{sin.\ \phi} = \frac{75{,}13}{sin.\ 73°\ 34'} = \frac{75{,}13}{0{,}9591} = 78{,}33$ lbs.

§ 78. *Forces in Space.*—If the directions of the forces do not lie in one and the same plane, we must draw through the point of appli-

cation a plane, and resolve each of the forces into two others, one lying in the plane, and the other at right angles to the plane; we must then find the resultant of the components so obtained in the plane, from the rule in the foregoing paragraph, and add together the components at right angles to the plane, and from the two rectangular components thus obtained, their resultant may be found according to the known rule (§ 74).

Fig. 36 puts the above mode of proceeding more clearly before us, let $MP_1 = P_1$, $MP_2 = P_2$, $MP_3 = P_3$ be the separate forces, $AB$ the plane (of projection) and $Z\overline{Z}$ the axis at right angles to it. From the resolution of the forces $P_1$, $P_2$, &c., the forces $S_1$, $S_2$ are given in the plane, and those of $N_1$, $N_2$, &c., in the normal to it $Z\overline{Z}$. These are again resolved according to two axes $X\overline{X}$ and $Y\overline{Y}$ into

FIG. 36.




the lateral forces $Q_1$, $Q_2$, &c., $R_1$, $R_2$, &c., and give the components $Q$ and $R$, of which the resultant $S$ consists, which joined to the sum of all the normal forces $N_1$, $N_2$, &c., gives $P$ the resultant required.

If we put $\beta_1$, $\beta_2$, for the angles at which the directions of force are inclined to the plane $AB$ or to the horizon, the forces in the plane are given, $S_1 = P_1 \cos. \beta_1$, $S_2 = P_2 \cos. \beta_2$, &c., and the normal forces, $N_1 = P_1 \sin. \beta_1$, $N_2 = P_2 \sin. \beta_2$, &c.; lastly if we designate the angles which the projections of the directions of the forces lying

in the plane $AB$, make with the axis $X\overline{X}$, by $a_1$, $a_2$, we obtain the three following forces, forming the sides of a rectangular parallelepiped.

$Q = S_1 \, cos. \, a_1 + S_2 \, cos. \, a_2 + S_3 \, cos. \, a_3$, or

1. $Q = P_1 \, cos. \, \beta_1 \, cos. \, a_1 + P_2 \, cos. \, \beta_2 \, cos. \, a_2 + \ldots$,

2. $R = P_1 \, cos. \, \beta_1 \, sin. \, a_1 + P_2 \, cos. \, \beta_2 \, sin. \, a_2 + \ldots$

3. $N = P_1 \, sin. \, \beta_1 + P_2 \, sin. \, \beta_2 + \ldots$

From these three follows the final resultant:

4. $P = \sqrt{Q^2 + R^2 + N^2}$, further

the angle of inclination to the plane of projection $PMS = \psi$, from

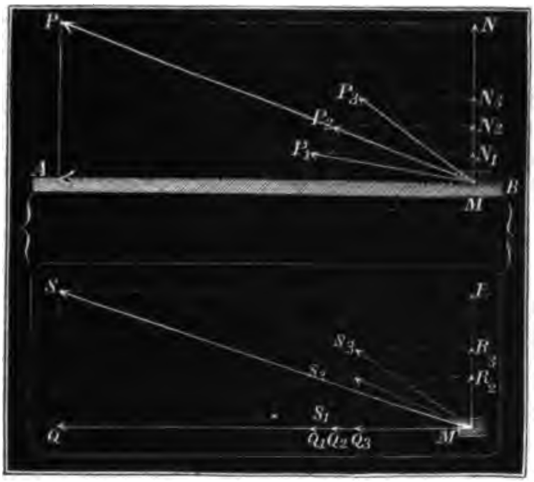
5. $tang. \, \psi = \frac{N}{S} = \frac{N}{\sqrt{Q^2 + R^2}}$, lastly

the angle $SMX = \phi$, which the projection of the resultant in the plane $AB$ makes with the first axis $X\overline{X}$, by

6. $tang. \, \phi = \frac{R}{Q}$.

*Example.* Three workmen pull at the end of three ropes, which are attached to a load $M$ lying upon an horizontal floor $AB$, Fig. 37, each with a force of 50 lbs;

FIG. 37.




the angles of inclination of these forces to the horizon are 10°, 20°, and 30°, and the horizontal angle between the first and second, and between the first and third, 20° and 35°; what is the magnitude and direction of the resultant, and how much is this less than the sum of all the forces which would result, if all three acted in the same direction? The vertical force pulling upward is:

$N = N_1 + N_2 + N_3 = 50 \times (sin. \, 10^\circ + sin. \, 20^\circ + sin. \, 30^\circ) = 50 \times 1{,}01567 = 50{,}78$ lbs.; by so much less than its own weight does the body press upon the floor.

The horizontal components are $S_1 = 50 \times cos.\ 10° = 50 \times 0{,}9849 = 49{,}24$ lbs.; $S_2 = 50 \times cos.\ 20° = 46{,}98$ lbs.; $S_3 = 50 \times cos.\ 30° = 43{,}30$ lbs. If we draw the axis $X\bar{X}$ in the direction of the first force $S_1$, we obtain the lateral force in this axis $X\bar{X}$, $Q = Q_1 + Q_2 + Q_3 = S_1\ cos.\ a_1 + S_2\ cos.\ a_2 + S_3\ cos.\ a_3 = 49{,}24 \times cos.\ 0° + 46{,}98 \times cos.\ 20° + 43{,}30 \times cos.\ 35° = 49{,}24 + 44{,}15 + 35{,}47 = 128{,}86$ lbs.; on the other hand, the lateral force in the second axis $Y\bar{Y}$: $R = R_1 + R_2 + R_3 = 49{,}24 \times sin.\ 0° + 46{,}98 \times sin.\ 20° + 43{,}30 \times sin.\ 35° = 0 + 16{,}07 + 24{,}84 = 40{,}91$ lbs.

The horizontal mean force with which the body is drawn forward is from this:

$$S = \sqrt{Q^2 + R^2} = \sqrt{(128{,}86)^2 + (40{,}91)^2} = \sqrt{18278{,}7} = 135{,}2 \text{ lbs.}$$

The angle $\phi$ which this force makes with the axis $X\bar{X}$ is determined by the $tang.\ \phi = \frac{R}{Q} = \frac{40{,}91}{128{,}86} = 0{,}3175$; $\phi = 17°, 37'$; the entire resultant is:

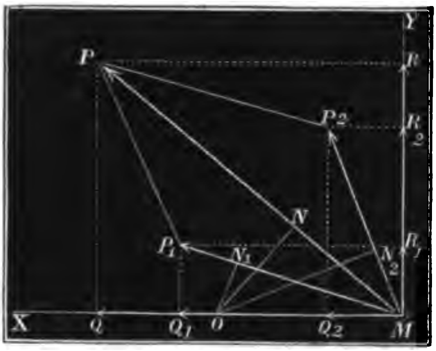
$$P = \sqrt{(135{,}2)^2 + (50{,}78)^2} = \sqrt{20856{,}6} = 144{,}42 \text{ lbs.}$$

If the forces act in the same direction, the resultant is $= 3 \times 50 = 150$ lbs., and the loss of force $= 150 - 144{,}42 = 5{,}58$ lbs.; further, because the horizontal force drawing the body forwards amounts only to 135,20 lbs., we have, with reference to the horizontal motion, the loss of force $150 - 135{,}20 = 14{,}80$ lbs.

The angle of inclination $\psi$ of the mean force to the horizon is determined by the $tang.\ \psi = \frac{N}{S} = \frac{50{,}78}{135{,}20} = 0{,}3756$, wherefore $\psi$ comes out $= 20°, 35'$.

§ 79.—From the rules found in the foregoing upon the composition of forces, two others of essential service for practical use may be deduced. In Fig. 37, let $M$ be a material point, $MP_1 = P_1$ and $MP_2 = P_2$, the forces acting upon it; lastly, let $MP = P$, the resultant of $P_1$ and $P_2$. If we draw through $M$ two axes, $MX$ and $MY$, at right angles to each other, and resolve the forces $P_1$ and $P_2$, as well as their resultant $P$, into components in the direction of these axes, viz: $P_1$ into $Q_1$ and $R_1$, $P_2$ into $Q_2$ and $R_2$, and $P$ into $Q$ and $R$, we then obtain the forces in the one axis $Q_1$, $Q_2$ and $Q$, and those in the other $R_1$, $R_2$, $R$, and $Q = Q_1 + Q_2$, and $R = R_1 + R_2$.

FIG. 37.




If now we take in the axis $MX$ any point $O$, and let fall from the same perpendiculars $ON_1$, $ON_2$ and $ON$ on the directions of the forces $P_1$, $P_2$ and $P$ we obtain rectangular triangles $MON_1$, $MON_2$, $MON$, which are similar to the triangles formed by the three forces, viz:

$$\triangle\ MON_1 \backsim \triangle\ MP_1Q_1$$
$$\triangle\ MON_2 \backsim \triangle\ MP_2Q_2$$
$$\triangle\ MON \backsim \triangle\ MPQ.$$

*Principle of virtual Velocities.*—But from these similarities $\frac{MQ_1}{MP_1}$ *i. e.* $\frac{Q_1}{P_1}=\frac{MN_1}{MO}$, also $\frac{Q_2}{P_2}=\frac{MN_2}{MO}$ and $\frac{Q}{P}=\frac{MN}{MO}$; if we put the values hence derived of $Q_1$, $Q_2$ and $Q$ into the equation $Q=Q_1+Q_2$, we then obtain

$$P.MN=P_1.MN_1+P_2.MN_2.$$

Likewise also $\frac{R_1}{P_1}=\frac{ON_1}{MO}$, $\frac{R_2}{P_2}=\frac{ON_2}{MO}$ and $\frac{R}{P}=\frac{ON}{MO}$, therefore

$$P.ON=P_1.ON_1+P_2.ON_2.$$

These equations still hold good, if $P$ the mean force be made up of three or more forces $P_1$, $P_2$, $P_3$, because generally

$$Q=Q_1+Q_2+Q_3+\ldots$$
$$R=R_1+R_2+R_3+\ldots$$

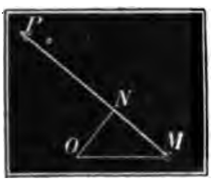
and therefore generally we may put:

1. $P.MN=P_1.MN_1+P_2.MN_2+P_3.MN_3+\ldots,$
2. $P.ON=P_1.ON_1+P_2.ON_2+P_3.ON_3+\ldots$

In both equations the mean force $P$ must correspond to the forces $P_1$, $P_2$, $P_3$, and from these equations, not only the magnitude, but also the direction of this force may be determined.

§ 80. If the point of application $M$ move in a straight line towards $O$, or if we imagine this point to have described the space $MO=s$, then the projection of this space $MN=s_1$ in the direction of the force $MP$ is called the *space of the force P*, and the product $Ps_1$ of the force and its space, *the work or labour of* the force. If we substitute in the equation (1) of the last (§) these designations, we have

FIG. 38.




$$Ps=P_1s_1+P_2s_2+P_3s_3+\ldots,$$

or the *work of the resultant is equivalent to the sum of the works of the components.*

In the summation of the mechanical effects, as in that of the forces, we must have regard to their signs. If a force ($Q_3$) of the forces $Q_1$, $Q_2$, &c., of the last § acts in an opposite direction to the rest, we must introduce it as negative, but this force $Q_3$, Fig. 39, is the component of a force $P_3$, which acting in the circumstances set forth in the former §, opposed to their proper motion $MN_3$, we are,

therefore, obliged to consider that force opposed to the motion $MN$, Fig. 40, as negative, and that one $P$, Fig. 41, acting in the direction of motion $MN$ as positive.

If the forces are variable in magnitude or direction, the formula $Ps = P_1 s_1 + P_2 s_2 + P_3 s_3 + \ldots$ is only correct for infinitely small spaces $s$, $s_1$, $s_2$, &c.

FIG. 39.

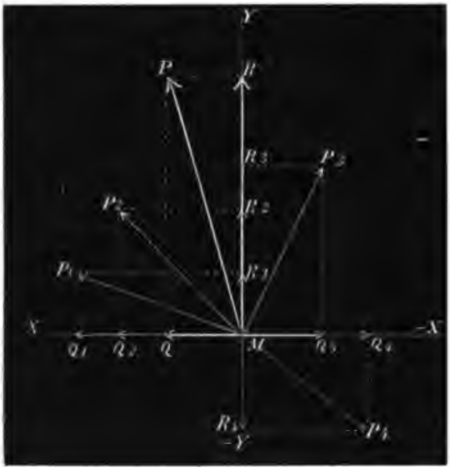


FIG. 40.

FIG. 41.

The spaces of the forces $\sigma_1$, $\sigma_2$, $\sigma_3$, corresponding to an infinitely small displacement $\sigma$ of a material point, are called their *virtual velocities*; and the law corresponding to the formula $P\sigma = P_1\sigma_1 + P_2\sigma_2 + P_3\sigma_3$, *the principle of virtual velocities.*

§ 81. *Transmission of Mechanical effect.*—From the principle of vis viva, the mechanical effect ($Ps$) in rectilinear motion, which a force ($P$) generates in changing the velocity $c$ of a mass $M$ into another $v$ is

$$Ps = \left(\frac{v^2 - c^2}{2}\right) M.$$

If $P$ be now the mean force arising from other forces, $P_1$, $P_2$, &c., acting upon the mass $M$, and the spaces which these describe be $s_1$, $s_2$, whilst the mass itself $M$ describes $s$, we then have from the foregoing:

$$Ps = P_1 s_1 + P_2 s_2 + \ldots$$

and, therefore, the following general formula:

$$P_1 s_1 + P_2 s_2 + \ldots = \left(\frac{v^2 - c^2}{2}\right) M,$$

which expresses that the sum of the mechanical effects of the single forces is equal to half the gain of vis viva of the mass taking up these forces.

If the velocity during the motion be invariable, that is $v=c$, and the motion itself be uniform, we have then $v^2-c^2=0$, consequently neither loss nor gain of vis viva, and, therefore:

$$P_1 s_1 + P_2 s_2 + P_3 s_3 + \ldots = 0,$$

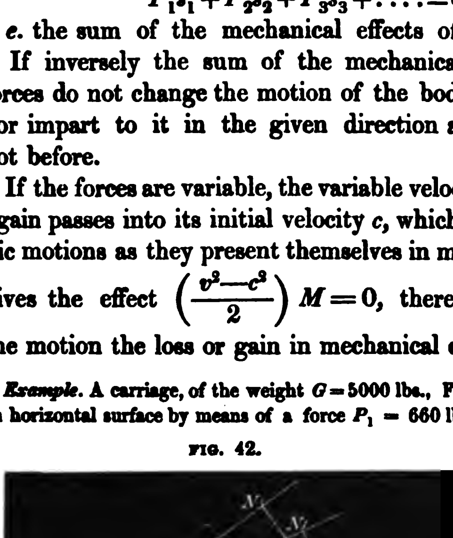
*i. e.* the sum of the mechanical effects of the single forces $= 0$.

If inversely the sum of the mechanical effects $= 0$, then the forces do not change the motion of the body in the given direction, nor impart to it in the given direction any motion which it had not before.

If the forces are variable, the variable velocity $v$ after a certain time again passes into its initial velocity $c$, which takes place in all periodic motions as they present themselves in many machines. Now $v = c$ gives the effect $\left(\frac{v^2-c^2}{2}\right) M = 0$, therefore within a period of the motion the loss or gain in mechanical effect is null.

*Example.* A carriage, of the weight $G = 5000$ lbs., Fig. 42, is moved forward upon an horizontal surface by means of a force $P_1 = 660$ lbs., ascending under an angle $\alpha = 24°$ and has during its motion two resistances to overcome; one, horizontal $P_2 = 350$ lbs., corresponding to the friction; and a resistance $P_3 = 230$ lbs., acting downwards, and inclined to the horizon at an angle $\beta = 35°$. What work will the force ($P_1$) perform, in order to convert the 2 feet initial velocity of the carriage into a velocity of 5 feet?

FIG. 42.



If we put the distance of the carriage $MO = s$, we then have for the work of the force $P_1 = P_1 . MN_1 = P_1\, s \cos. \alpha = 660 \times s \cos. 24° = 602{,}94 . s$; further, the work of the resisting force $= (-P_2) . s = -350 . s$; lastly, the work of $P_3 = (-P_3) . MN_3 = -P_3\, s \cos. \beta = -230 \times s \cos. 35° = -188{,}40 . s$. There then remains for the work of the effective force:

$$Ps = P_1\, s \cos. \alpha - P_2\, s \cos. 0 - P_3\, s \cos. \beta = (602{,}94 - 350 - 188{,}40) . s = 64{,}54 . s \text{ ft.lbs.}$$

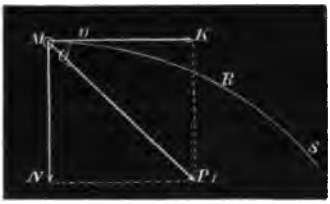
The mass, however, requires for the change of its velocity, the mechanical effect:

$$\left(\frac{v^2-c^2}{2g}\right) G = \left(\frac{5^2-2^2}{2g}\right) \times 5000 = 0{,}0155 \times (25-4) \times 5000 = 1627 \text{ ft. lbs.}$$

If now we equate both mechanical effects, we then obtain $64{,}54 . s = 1627$, consequently the distance of the carriage: $s = \frac{1627}{64{,}54} = 25{,}26$ feet; and lastly, the mechanical effect of the force $P : P_1\, s \cos. \alpha = 602{,}94 \times 25{,}26 = 15230{,}2$ ft. lbs.

§ 82. *Curvilinear Motion.*—Provided that the spaces $\sigma$, $\sigma_1$, &c. be infinitely small, we may also apply the formula last found to curved paths. Let $MOHS$, Fig. 43, be the path of a material point, and $MP_1 = P_1$ the resultant of all the forces acting upon it; if we resolve this force into two others, of which the one $MK = K$ is tangential, and the other $MN = N$ normal to the curve, we then term the one a *tangential*, and the other a *normal* force.

FIG. 43.



Whilst the material point describes the element $MO = \sigma$ of its curved path $MS$, and its velocity $c$ is transformed into $v_1$, its mass $M$ lays claim to the work $\left(\frac{v_1^2 - c^2}{2}\right) M$, but the tangential force $K$ performs at the same time the work $K\sigma$, and the normal force the work $N \cdot 0 = 0$; consequently $K\sigma = \left(\frac{v_1^2 - c^2}{2}\right) M$.

If the projection $MQ$ of the elementary space $MO$ in the direction of force be put $= \sigma_1$, then also $P_1\sigma_1 = K\sigma$; and, therefore,

$$P_1\sigma_1 = \left(\frac{v_1^2 - c^2}{2}\right) M.$$

If the whole space described by the material point $MR$ be decomposed into infinitely small parts, and each part be projected upon the direction of force at each moment, we then obtain the elementary space of the force at each moment, and the work at each moment by the multiplication of the space and force, and if we add together all these mechanical effects, we then have:

$P_1\sigma_1 + P_2\sigma_2 + P_3\sigma_3 + \ldots = \left(\frac{v_1^2 - c_1^2}{2}\right) M + \left(\frac{v_2^2 - v_1^2}{2}\right) M + \left(\frac{v_3^2 - v_2^2}{2}\right) M + \ldots = \left(\frac{v^2 - c^2}{2}\right) M = (h - h_1) M$, if $h_1$ be the height due to the initial velocity $c$, and $h$ that due to the terminal velocity $v$. Thus, *in curvilinear motion, the whole effect of the moving force is equal to half the gain of vis viva*, or equal to the product of the ~~mass~~ into the difference of the heights due to the velocities.

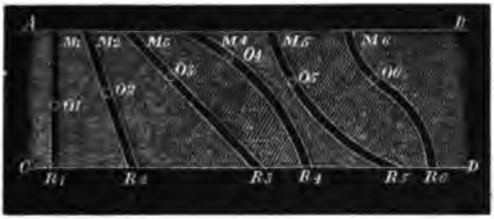
*Remark and Example.* The formula obtained which is derived from combining the principle of the vis viva with that of the virtual velocities, is especially applicable in cases when bodies are constrained by a fixed track or by suspension to describe a

determinate path. If gravity alone act upon such a body, the work which it generates in a body of the weight $G$ falling from a height corresponding to the vertical projection $M_1 R_1 = s$, is $= G s$, and therefore:

$$G s = (h - h_1) G, \text{ i.e. } s = h - h_1.$$

Which is also the space which a body describes in falling from an horizontal plane $AB$, Fig. 44, to another $CD$; the difference of the heights due to the velocity is always

FIG. 44.




equal to the perpendicular height of fall; bodies which begin to describe the paths $M_1 O_1 R_1$, $M_2 O_2 R_2$, $M_3 O_3 R_3$, &c., with equal velocity ($c$), acquire at the end of these paths, as well as at different times, equal velocities ($v$). If the initial velocity $c = 10$ feet, and the vertical height of fall $s = 20$ feet, then $h = s + h_1 = 20 + 0{,}01550 \,.\, 10^2 = 21{,}55$ feet, and the terminal velocity $v = \sqrt{2gh} = 8{,}03 \sqrt{21{,}5}$ $= 36{,}93$ feet, in whatever curved or right line the descent may take place.

$\varphi(x, y, z) = 0$ = equation of constraining curve

$v^2 = 2\int(X\,dx + Y\,dy + Z\,dz)$ $\quad Y = g$

$\therefore v^2 = \int 2g\,dy = 2gy$ $\quad \therefore y = \frac{v^2}{2g}$

# SECTION III.

## STATICS OF RIGID BODIES.

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## CHAPTER I.

### GENERAL LAWS OF THE STATICS OF RIGID BODIES.

§ 83. *Transference of the point of application.*—Although every rigid body is changed in form by the action of forces upon it, *i. e.* becomes either compressed, extended or bent, &c., it is nevertheless allowable for us to consider it for the most part as a rigid and invariable union of material points, partly because this change of form or displacement of parts is often very slight, and partly because it takes place in very short spaces of time. We shall, therefore, in the following, unless it be otherwise mentioned, regard every rigid body as a system of points, firmly connected, and we shall thereby essentially simplify the investigation.

A force $P$, Fig. 45, which acts upon a point $A$ of a rigid body $M$, is transmitted in its proper direction $\overline{XX}$ uniformly throughout the body, and an equal and opposite force $P_1$ puts itself in equilibrium

FIG. 45.

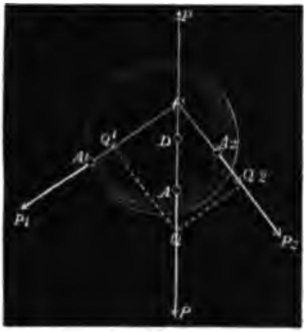
with it, then only when the point of application $A_1$ lies in the direction $\overline{XX}$ of the first force. The distance of $A$ and $A_1$ is without influence on this condition of equilibrium. The two opposite forces hold themselves in equilibrium at every distance if the two points be rigidly connected. We may, therefore, assert

*that the action of a force P*, Fig. 46, *remains the same at whatever point $A_1$, $A_2$, $A_3$, &c. of its direction it may be applied or may act directly upon the body.*

FIG. 46.

§ 84. When two forces $P_1$ and $P_2$ acting in the same plane are applied to a body at different points $A_1$ and $A_2$, their action upon the body is the same as if they had the point $C$, where the directions of the two forces intersect, for their common point of application, for from the proposition enunciated above, each of these points of application may be transferred to $C$ without thereby producing any change in their effects. If, therefore, we make $CQ_1 = A_1P_1 = P_1$ and $CQ_2 = A_2P_2 = P_2$, and then complete the parallelogram $CQ_1QQ_2$, its diagonal will give us the resultant force $CQ = P$ of $CQ_1$ and $CQ_2$, and therefore also of the forces $P_1$ and $P_2$, and whose point of application may be any other point $A$ in the direction of this diagonal.

FIG. 47.



If to the resultant force so found $AP = P$, there be put an opposite force $D\overline{P} = -P$ equally great at any point $D$ of the direction of the diagonal $C$, the two forces $P_1$ and $P_2$ will be thereby held in equilibrium; $P_1$, $P_2$ and $-P$ are therefore three forces in equilibrium.

FIG. 48.

§ 85. If there be let fall from any point $O$, Fig. 48, in the plane of the forces perpendiculars $ON_1$, $ON_2$ and $ON$ upon the directions of the component forces $P_1$ and $P_2$, and their resultant $P$, we have according to § 79.

$$P \cdot ON = P_1 \cdot ON_1 + P_2 \cdot ON_2,$$

and the distance $ON$ of the resultant force may be found from the

perpendiculars or distances $ON_1$ and $ON_2$ of the component forces, if we put;

$$ON = \frac{P_1 . ON_1 + P_2 . ON_2}{P}.$$

Whilst we find the direction and magnitude of the resultant by the application of the parallelogram of forces, its position is given with the help of the last formula, by determining its distance $ON$.

If the prolonged direction of the forces includes between them an angle $P_1\ CP_2 = \alpha$, we then have:

1. The magnitude of the resultant $P = \sqrt{P_1^2 + P_2^2 + 2P_1P_2 \cos. \alpha}$.

Further, if the resultant makes with the direction of the component $P_1$ the angle $PCP_1 = \phi$, then:

2. $\sin. \phi = \frac{P_2 \sin. \alpha}{P}$.

If the directions $CP_1$ and $CP_2$ of the given forces are distant $ON_1 = a_1$ and $ON_2 = a_2$ from an arbitrary point $O$, the distance $ON = a$ of the direction $CP$ of the resultant from this point is:

3. $a = \frac{P_1a_1 + P_2a_2}{P}$.

With the help of this distance $a$, the position of the resultant is given without regard to the point $C$, if we describe a circle from $O$ as a centre with radius $a$, and to this draw a tangent $NP$, whose direction is determined by the angle $\phi$.

*Example.* There act upon a body the forces $P_1 = 20$ lbs. and $P_2 = 34$ lbs. whose directions meet under an angle $P_1\ CP_2 = \alpha = 70°$, and are distant from a certain point $O = O\,N_1 = a_1 = 4$ feet, and $O\,N_2 = a_2 = 1$ foot; what is the magnitude, direction, and position of the resultant? The magnitude of the resultant is:

FIG. 49.

$$P = \sqrt{20^2 + 34^2 + 2 \times 20 \times 34 \cos. 70°}$$
$$= \sqrt{400 + 1156 + 1360 \times 0{,}34202}$$
$$= \sqrt{2021{,}15} = 44{,}96 \text{ lbs.};$$ further, for its direction, $\sin. \phi = \frac{34 \times \sin. 70°}{44{,}96}$,

*Log.* $\sin. \phi = 9{,}8516384$, therefore, $\phi = 45°\ 17'$, the angle which this resultant makes with the direction of $P_1$. The position finally is determined by its distance $O\,N$ from $O$, which is:

$$a = \frac{20 \times 4 + 34 \times 1}{44{,}96} = \frac{114}{44{,}96} = 2{,}536 \text{ feet.}$$

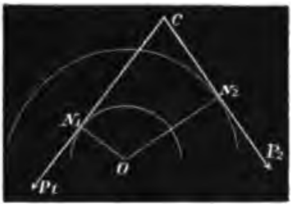
§. 86. The normal distances $ON_1 = a_1$, $ON_2 = a_2$, &c., of the

direction of the forces from an arbitrary point $O$, Fig. 50, are called the arms of the forces, because they form essential elements in the theory of the lever, to be treated of subsequently. The product $Pa$ of the force and lever arm, is called the *statical moment* of the force. But since $Pa = P_1 a_1 + P_2 a_2$; the statical moment of the resultant is equivalent to the sum of the statical moments of the components.

In the addition of the moments, regard must be had to the signs plus and minus. If the forces $P_1$ and $P_2$, Fig. 50, act about the point $O$ in like directions, and if the directions of force coincide with the direction of motion of the hands of a watch, these forces,

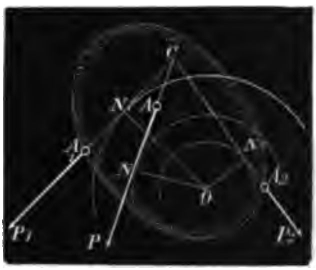
FIG. 50.


FIG. 51.



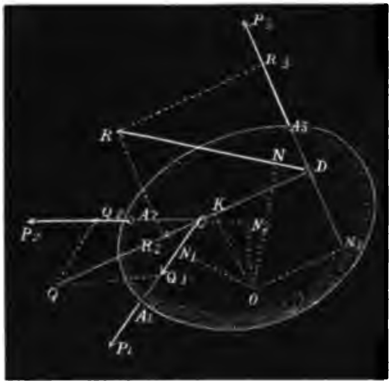

as well as their statical moments, are said to have like signs; if the one be positive, the other must be positive likewise. If, on the other hand, Fig. 51, the directions of the forces about the point $O$ be opposite to each other, then the same, as well as their statical moments, are of contrary signs; if the one be negative, the other must be positive. In the composition of forces represented in Fig. 52, $Pa = P_1 a_1 - P_2 a_2$, because $P_2$ is opposed to the force $P_1$; its statical moment is, therefore, negative.

FIG. 52.




§. 87. *Composition of forces in a plane.*—If three forces, $P_1$, $P_2$, $P_3$, Fig. 53, act upon a body at the points $A_1$, $A_2$, $A_3$, two of these forces ($P_1$, $P_2$) by the last rule must be joined, and their resultant $CQ = Q$ found, this again joined to the third force ($P_3$), and the parallelogram $DR_2RR_3$ constructed from the forces $DR_2 = CQ$ and $DR_3 = A_3P_3$. The diagonal $DR = P$ is the required resultant of $P_1$, $P_2$ and $P_3$. It is from this easy to

FIG. 53.




see how the resultant might be found if a fourth force $P_4$ were to be introduced.

In this composition of the forces, the magnitude and direction of the resultant is as accurately found as if the forces acted in one single point (§. 77); the rules of calculation (§. 77) are, therefore, applicable for finding these two first elements of the resultant; but in order to find the third, viz., the position of the resultant or its line of action, we must make use of the equation between the statical moments. Here, also, $ON_1 = a_1$, $ON_2 = a_2$, $ON_3 = a_3$, and $ON = a$, are the arms of the three components $P_1$, $P_2$, $P_3$, and of their resultant $P$, with reference to an arbitrary point $O$. So that:

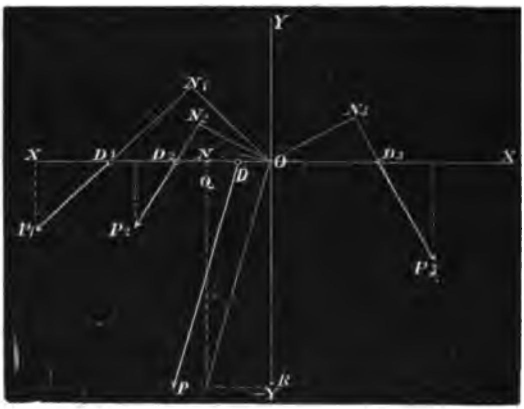
$Pa = Q \,.\, OK + P_3 a_3$, and

$Q \,.\, OK = P_1 a_1 + P_2 a_2$, provided $Q$ is the resultant of $P_1$ and $P_2$, and $OK$ the arm. If we combine these two equations we then obtain:

$Pa = P_1 a_1 + P_2 a_2 + P_3 a_3$, and also for several forces:

$Pa = P_1 a_1 + P_2 a_2 + P_3 a_3 + \ldots$, &c., *i. e.*, the (*statical*) *moment of the resultant is always equivalent to the algebraical sum of the* (statical) *moments of the components.*

FIG. 54.



§. 88. If $P_1$, $P_2$, $P_3$, Fig. 54, are the single forces of a system of forces; if, further, $\alpha_1$, $\alpha_2$, $\alpha_3$, &c., are the angles $P_1D_1X$, $P_2D_2X$, $P_3D_3X$, &c., under which an arbitrarily chosen axis $X\overline{X}$ is intersected by the directions of force, and if $a_1$, $a_2$, $a_3$, designate the arms $ON_1$, $ON_2$, $ON_3$, of these forces with regard to the point of intersection $O$ of both axes $X\overline{X}$ and $Y\overline{Y}$, we have from §§. 77 and 87:

1. The component parallel to the axis $X\overline{X}$:

$$Q = P_1 \cos. \alpha_1 + P_2 \cos. \alpha_2 + P_3 \cos. \alpha_3 \ldots,$$

2. The component parallel to the axis $Y\overline{Y}$:

$$R = P_1 \sin. \alpha_1 + P_2 \sin. \alpha_2 + P_3 \sin. \alpha_3 \ldots,$$

3. The resultant of the whole system:

$$P = \sqrt{Q^2 + R^2},$$

4. The angle which the resultant makes with the axis by

$$tang. \phi = \frac{R}{Q},$$

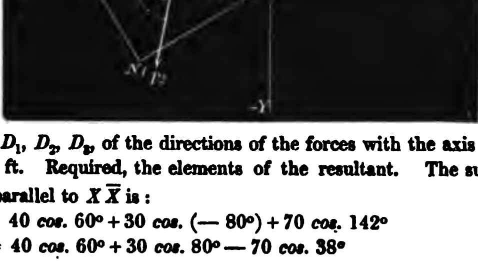
5. The arm of the resultant or the ~~diameter~~ radius of the circle to which the direction of the resultant is a tangent:

$$a = \frac{P_1 a_1 + P_2 a_2 + \ldots}{P}.$$

If this resultant be replaced by an equivalent opposite force ($-P$), then the forces $P_1$, $P_2$, $P_3$.... ($-P$) are in equilibrium.

*Example.* The forces $P_1 = 40$ lb., $P_2 = 30$ lb., $P_3 = 70$ lb., Fig. 55, intersect the axis $X\overline{X}$ at angles $\alpha_1 = 60°$, $\alpha_2 = -80°$, $\alpha_3 = 142°$, and the distances of the points

FIG. 55.



of intersection $D_1$, $D_2$, $D_3$, of the directions of the forces with the axis: $D_1 D_2 = 4$ ft., and $D_2 D_3 = 5$ ft. Required, the elements of the resultant. The sum of the component forces parallel to $X\overline{X}$ is:

$$Q = 40 \cos. 60° + 30 \cos. (-80°) + 70 \cos. 142°$$
$$= 40 \cos. 60° + 30 \cos. 80° - 70 \cos. 38°$$
$$= 20 + 5{,}209 - 55{,}161 = -29{,}952 \text{ lbs.}$$

The sum of the components parallel to $\overline{YY}$:

$$R = 40\ sin.\ 60^\circ + 30\ sin.\ (-80^\circ) + 70\ sin.\ 142^\circ$$
$$= 40\ sin.\ 60^\circ - 30\ sin.\ 80^\circ + 70\ sin.\ 38$$
$$= 34,641 - 29,544 + 48,096 = 48,193 \text{ lbs.}$$

The resultant sought is therefore:

$$P = \sqrt{Q^2+R^2} = \sqrt{29,952^2+48,193^2} = \sqrt{3219,68} = 56,742 \text{ lbs.}$$

The angle $\phi$, which it makes with the axis, is further determined by:

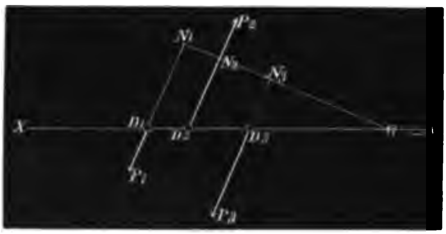
$$tang.\,\phi = \frac{R}{Q} = \frac{48,193}{-29,952} = -1,6090, \text{ it is therefore } \phi = 180^\circ - 58^\circ\ 8' = 121^\circ\ 52'.$$

The arm $ON_1$ of the force $P_1$ is $= OD_1\ sin.\ a_1 = (4+5)\ sin.\ 60^\circ = 9 \times 0.86603 = 7,794$ feet; the arm $ON_2$ of $P_2 = OD_2\ sin.\ a_2 = 5\ sin.\ 80^\circ = 4,924$ feet; lastly, the arm $ON_3$ of $P_3 = O$, when the point of application $O$ is transferred to $D_3$. The arm of the resultant is finally given by:

$$a = \frac{40 \times 7,794 - 30 \times 4,924}{56,742} = \frac{311,76 - 147,72}{56,742} = \frac{164,04}{56,742} = 2,891 \text{ feet.}$$

§. 89. *Parallel forces.*—If the forces $P_1$, $P_2$, $P_3$, &c., Fig. 56, of a rigid system are parallel, the arms $ON_1$, $ON_2$, $ON_3$ are in the same straight line; if now we draw through the point of application $O$ an arbitrary line $\overline{XX}$, the directions of the forces cut off the parts $OD_1$, $OD_2$, $OD_3$, &c., which are proportional to the arms $ON_1$, $ON_2$, $ON_3$, &c., because $\triangle OD_1N_1 \backsim \triangle OD_2N_2 \backsim \triangle OD_3N_3$. If the angle $D_1ON_1 = D_2ON_2$ be designated by $a$, &c., the arms $ON_1$, $ON_2$, &c., by $a_1$, $a_2$, &c., the abscises $OD_1$, $OD_2$, &c., by $b_1$, $b_2$, &c., we then have:

FIG. 56.




$$a_1 = b_1\ cos.\ a,\ a_2 = b_2\ cos.\ a, \text{ \&c.}$$

If, lastly, these values be substituted in the formula:

$$Pa = Pa_1 + P_2a_2 + \ldots,$$

we then obtain:

$$Pb\ cos.\ a = P_1b_1\ cos.\ a + P_2b_2\ cos.\ a + \ldots;$$

or if the common factor $cos.\ a$ be left out:

$$Pb = P_1b_1 + P_2b_2 + \ldots$$

In every system of parallel forces it is allowable to replace the arms by the distances $OD_1$, $OD_2$, cut off from any line $\overline{XX}$. Because the magnitude and direction of the resultant is the same, the forces may act at one or at different points; hence the resultant of a system of parallel forces has the same direction with the single forces, and is equivalent to their algebraical sum.

Therefore

1. $P = P_1 + P_2 + P_3 + \ldots$ and

2. $a = \dfrac{P_1 a_1 + P_2 a_2 + \ldots}{P_1 + P_2 + \ldots}$, or also :

3. $b = \dfrac{P_1 b_1 + P_2 b_2 + \ldots}{P_1 + P_2 + \ldots}$.

*Example.* The forces $P_1 = 12$ lb., $P_2 = -32$ lb., $P_3 = 25$ lb., and their directions intersect a straight line at the points $D_1$, $D_2$, and $D_3$, Fig. 56, whose distances from each other are $D_1D_2 = 21$ inches, $D_2D_3 = 30$ inches; required the resultant. The magnitude of this force is $P = 12 - 32 + 25 = 5$ lbs., its distance $D_1O$ from $D_1$, is therefore :

$$b = \frac{12 \times 0 - 32 \times 21 + 25 \times (21 + 30)}{5} = \frac{0 - 672 + 1275}{5} = 120{,}6 \text{ inches.}$$

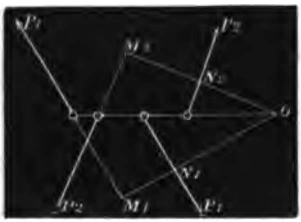
§. 90. *Couples.*—Two parallel, equal and opposite forces, $P_1$ and $-P_1$, Fig. 57, have the resultant

$$P = P_1 + (-P_1) = P_1 - P_1 = 0, \text{ with the arm}$$

$$a = \frac{P_1 a_1 + P_2 a_2}{0} = \infty.$$

FIG. 57.

FIG. 58.



For restoring equilibrium to such a couple, according to this, a single finite force $P$ acting at a finite distance, is not sufficient, but two such couples may easily hold each other in equilibrium. If $P_1$ and $-P_1$ and $-P_2$ and $P_2$, Fig. 58, are two such couples, and $OM_1 = a_1$, $ON_1 = OM_1 - M_1N_1 = a_1 - b_1$; if further, $OM_2 = a_2$ and $ON_2 = OM_2 - M_2N_2 = a_2 - b_2$ are the arms taken from a certain point $O$, we have for equilibrium :

$$P_1 a_1 - P_1 (a_1 - b_1) - P_2 a_2 + P_2 (a_2 - b_2) = 0, \text{ i. e.}$$

$$P_1 b_1 = P_2 b_2.$$

Two such couples are, therefore, in equilibrium if the product of one force, and its distance from the opposite force is as great in the one couple as in the other.

A pair of equal opposite forces is called simply a *couple*, and the product of one of the forces and its normal distance from the other force, the *moment of the couple.* From the above, two

couples acting in opposite directions are in equilibrium, if they have equal moments.

If we substitute in the formula (§. 87) for the arm $a$ of the resultant:

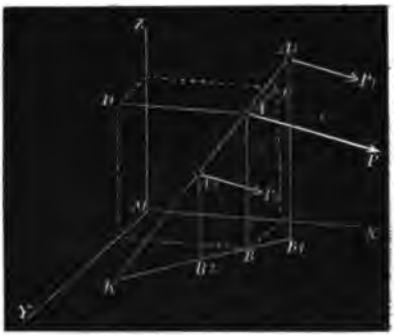
$$a = \frac{P_1 a_1 + P_2 a_2 + \dots}{P}$$

$P = 0$ without the sum of the statical moments becoming null; we obtain likewise $a = \infty$, a proof that in this case also there is no resultant, but only a couple, possible.

*Example.* If a couple consists of the forces $P_1 = 25$ lbs., and $-P_1 = -25$ lbs., and another of the forces $-P_2 = -18$ lbs., and $P_2 = 18$ lbs.; if, lastly, the normal distance of the first pair $= 3$ feet for the condition of equilibrium, the normal distance of the second must amount to $= \frac{25 \times 3}{18} = 4\frac{1}{6}$ feet.

§. 91. *Centre of parallel forces.*—If the parallel forces lie in different planes, their union may be effected in the following manner. If the straight line $A_1 A_2$, Fig. 59, which unites the points of application of two parallel forces $P_1$ and $P_2$, be prolonged to the plane $XY$ between the rectangular axes $MX$, $MY$, and if the point of intersection $K$ be taken for the initial point, we shall in this manner obtain for the point of application $A$ of the resultant $(P_1 + P_2)$ of these forces.

FIG. 59.



$$(P_1 + P_2) . KA = P_1 . KA_1 + P_2 . KA_2.$$

As now $B$, $B_1$ and $B_2$ are the projections of the points of application $A$, $A_1$, $A_2$, on the plane $XY$, we have:

$AB : A_1B_1 : A_2B_2 = KA : KA_1 : KA_2$, and therefore also

$(P_1 + P_2) AB = P_1 . A_1 B_1 + P_2 . A_2 B_2$.

If we designate by $z_1$, $z_2$, $z_3$, &c., the normal distances $A_1B_1$, $A_2B_2$, $A_3B_3$, &c., of the points of application from the principal plane $XY$, and by $w_1$ that of the point $A$ from the same plane, we have for the two forces:

$(P_1 + P_2) w_1 = P_1 z_1 + P_2 z_2$; for three or more, and generally

$(P_1 + P_2 + P_3 + \dots) w = P_1 z_1 + P_2 z_2 + P_3 z_3 \dots$ Consequently

$$1.\ w = \frac{P_1 z_1 + P_2 z_2 + \dots}{P_1 + P_2 + \dots}.$$

If we put likewise the distances $AC$ and $AD$ of the point of application of the resultant from the planes $XZ$ and $YZ = v$ and $u$, we then obtain:

$$2.\ v = \frac{P_1 y_1 + P_2 y_2 + \dots}{P_1 + P_2 + \dots} \text{ and}$$

$$3.\ u = \frac{P_1 x_1 + P_2 x_2 + \dots}{P_1 + P_2 + \dots}$$

The three distances $u$, $v$, $w$ from the principal planes, as for example, from the floor and the two side walls of a room, fully determine the point $A$, for it is the eighth terminating point of the parallelepiped, constructed from $u$, $v$, $w$, consequently, in such a system there is but one single point of application of the resultant.

As the three formulæ for $u$, $v$, $w$, do not contain the angles which the forces make with the principal planes, the point of application is independent of these forces, and also of their directions; the whole system admits, therefore, of being turned about this point without its ceasing to be the point of application, provided only that in this turning the parallelism of the forces be preserved.

In such a system of parallel forces the product of a force, and the distance of its point of application from a plane or line, is called the moment of the force with reference to this plane or line, and generally, the point of application of the resultant is called the *centre of parallel forces*. The distance of the centre of a system of parallel forces from any plane or line whatever, (the latter when the forces lie in the same plane) is obtained, when the sum of the moments is divided by the sum of the forces.

| | | | | | | sum. |
|---|---|---|---|---|---|---|
| *Example.* | If the forces are | $P_n$ | 5 | —7 | 10 | 4 8 lbs. |
| | the distances | $x_n$ | 1 | 2 | 0 | 9 ft. |
| | " " | $y_n$ | 2 | 4 | 5 | 3 " |
| | " " | $z_n$ | 8 | 3 | 7 | 10 " |
| | The moments are | $P_n x_n$ | 5 | —14 | 0 | 36 ft. lbs. |
| | " " | $P_n y_n$ | 10 | —28 | 50 | 12 " |
| | " " | $P_n z_n$ | 40 | —21 | 70 | 40 " |

Now, if the sum of the forces $= 19 - 7 - 12$ lbs., the distances of the central point of this system from the three principal planes are consequently:

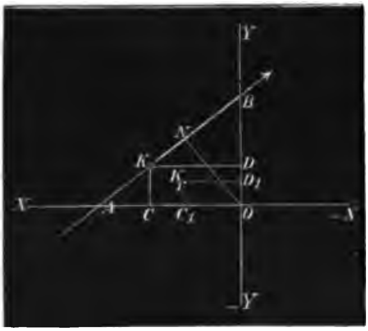
$$u = \frac{5 + 36 - 14}{12} = \frac{27}{12} = \frac{9}{4} = 2{,}25 \text{ feet};$$

$$v = \frac{10 + 50 + 12 - 28}{12} = \frac{44}{12} = \frac{11}{3} = 3{,}66\dots \text{ feet};$$

$$w = \frac{40 + 70 + 40 - 21}{12} = \frac{129}{12} = \frac{43}{3} = 10{,}75 \text{ feet.}$$

§ 92. *Forces in Space.*—If it be required to unite a system constituted of differently directed forces, a plane must be carried through the system, the different points of application transferred to this plane, and each force resolved into two component forces, the one coinciding with the plane, the other at right angles to it. If $\beta_1, \beta_2 \ldots$ are the angles under which the plane is intersected by the directions of the forces, then the normal forces are $P_1 \sin. \beta$, $P_2 \sin. \beta_2 \ldots$, and those in the plane $P_1 \cos. \beta_1$, $P_2 \cos. \beta_2$, &c. The latter from § 88, and the former from the last § 91 may be combined to a resultant. In general, the directions of both resultants will nowhere intersect each other, and accordingly a composition of these is impossible, but if the resultant of parallel forces passes through a point $K$, Fig. 60, in the direction $AB$ of the resultant of the forces in the plane (the plane of the paper) a composition is then possible. If we put the distances $OC = DK = u$, and $OD = CK = v$ for the point of application of the first resultant, on the other hand the arm $ON$ of the second $= a$, and the angle $BAO$, at which it intersects the axis $X\overline{X} = \alpha$, the condition for the possibility of a composition, is:

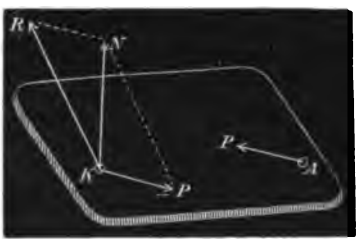
FIG. 60.



$$u \sin. \alpha + v \cos. \alpha = a.$$

If this equation is not satisfied, if, for example, the resultant of the normal forces passes through $K_1$, the reduction of the whole system of forces to a resultant is then impossible, but it readily admits of being reduced to a resultant $R$, Fig. 61, and a couple $P, - P$, if the resultant $N$ of the parallel components is resolved into the forces $-P$ and $R$, of which the one is equal, and directed parallel and opposite to the resultant $P$ of the forces in the plane.

FIG. 61.

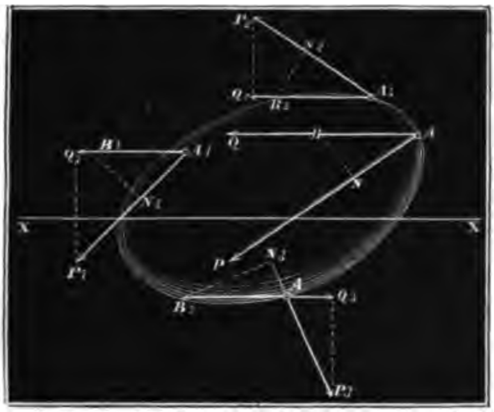


§ 93. *Principle of virtual Velocities.* — If a system of forces $P_1$, $P_2$, $P_3$, acting in a plane, Fig. 62, is progressive, *i. e.* moves forward so that all the points of application $A_1$, $A_2$, $A_3 \ldots$ pass through equal parallel

spaces $A_1 B_1$, $A_2 B_2$, $A_3 B_3$, the effect of the resultant (in the sense of § 80) is equivalent to those of the components, and in a state of equilibrium therefore $= 0$. If the projections $A_1 N_1$, $A_2 N_2$, &c., coinciding with the directions of the forces of the common spaces $A_1 B_1$, $= A_2 B_2$, &c., $= s_1, s_2$, then the mechanical effect of the resultant is:

$$Ps = P_1 s_1 + P_2 s_2 + \dots$$

FIG. 62.



This law follows from one of the formulæ of § 88, according to which the component of the resultant running parallel with the axis $XX$ is equal to the sum $Q_1 + Q_2 + \dots$ &c., of the similarly running components of the forces $P_1$, $P_2$; now from the similarity of the triangles $A_1 B_1 N_1$ and $A_1 P_1 Q_1$, there follows the proportion

$$\frac{Q_1}{P_1} = \frac{A_1 N_1}{A_1 B_1} = \frac{s_1}{AB}, \text{ and from this:}$$

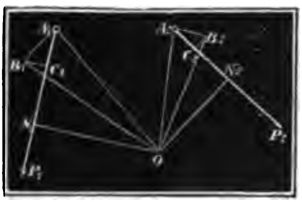
$$Q_1 = \frac{P_1 s_1}{AB},\ Q_2 = \frac{P_2 s_2}{AB}, \text{ \&c.,}$$

we may, therefore, in place of

$$Q = Q_1 + Q_2 + \dots \text{ put}$$
$$Ps = P_1 s_1 + P_2 s_2 + \dots$$

§ 94. If the system of forces $P_1$, $P_2$, &c., Fig. 63, be made to

FIG. 63.



revolve a very little about the point $O$, the law of the principle of virtual velocities ~~enumerated~~ above in 80 Enunciated and 93 holds equally good, as may be proved in the following manner. From § 86 the moment $P . ON$ of the resultant is equivalent to the sum of the moments of the components, so that:

$$Pa = P_1 a_1 + P_2 a_2 + \dots$$

The space $A_1B_1$ corresponding to a revolution through the small angle $A_1OB_1 = \phi^0$ or the arc $\phi = \frac{\phi^0}{180^0} \cdot \pi$, is perpendicular to the diameter $OA_1$, therefore, the triangle $A_1B_1C_1$, which is formed if a perpendicular line $B_1 C_1$ be let fall on the direction of the force, is similar to the triangle $OA_1N_1$ determined by the arm $ON_1 = a_1$, and accordingly

$$\frac{ON_1}{OA_1} = \frac{A_1C_1}{A_1B_1}.$$

If the virtual velocity $A_1 C_1 = \sigma_1$ and the arc $A_1 B_1 = OA_1 \cdot \phi$, we then obtain:

$$a_1 = \frac{OA_1 . \sigma_1}{OA_1 . \phi} = \frac{\sigma_1}{\phi}, \text{ also } a_2 = \frac{\sigma_2}{\phi}, \text{ \&c.}$$

If these values be substituted in the above equation for $a_1$, $a_2$, we then have

$$\frac{P\sigma}{\phi} = \frac{P_1\sigma_1}{\phi} + \frac{P_2\sigma_2}{\phi} + \ldots \text{ \&c.,}$$

or as $\phi$ is a common divisor,

$P\sigma = P_1\sigma_1 + P_2\sigma_2 + \ldots$ &c., the same as in § 80.

So that, for small revolutions the mechanical effect ($P\sigma$) of the resultant is equivalent to the sum of the mechanical effects of the components.

§ 95. The principle of virtual velocities holds likewise for arbitrarily great revolutions, if instead of the virtual velocities of the points of application, the projections $N_1D_1$, $N_2D_2$, &c., Fig. 64, of the spaces commencing at the points $N_1$, $N_2$, be introduced, and their values

FIG 64.

$$B_1C_1 = OB_1 \, sin. \, N_1OB_1 = a_1 \, sin. \, \phi,$$
$$B_2C_2 = OB_2 \, sin. \, N_2OB_2 = a_2 \, sin. \, \phi, \text{ \&c.,}$$

be substituted for $\sigma_1$, $\sigma_2$, we then obtain

$Pa \, sin. \, \phi = P_1a_1 \, sin. \, \phi + P_2a_2 \, sin. \, \phi \ldots +$, or, dividing by $sin. \, \phi$,

$$Pa = P_1a_1 + P_2a_2 + \ldots,$$

the known equation for statical moments.

This principle is correct also for finite revolutions, if the directions of the forces revolve simultaneously with the system, or if

while the point of application incessantly changes, the arm $ON_1 = OB_1$ remains invariable, then from

$$Pa = P_1 a_1 + P_2 a_2,$$

and multiplying by $\varphi$ we have

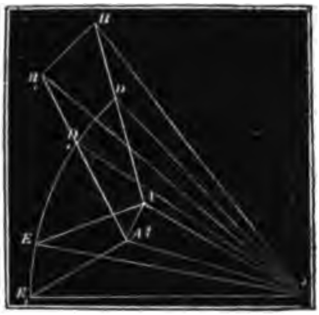
$$Pa\,\varphi = P_1 a_1 \varphi + P_2 a_2 \varphi + \ldots, \text{ i. e.}$$

$$P\sigma = P_1 \sigma_1 + P_2 \sigma_2 + \ldots,$$

if $\sigma$, $\sigma_1$, $\sigma_2$, &c., designate the circular arcs, $N_1 B_1$, $N_2 B_2$, &c., of the points $N$, $N_1$, &c.

§ 96. Every small motion or displacement of a body in a plane may be regarded as a small revolution about a moveable centre, and may be proved in the following manner. Let two points $A$ and $B$, Fig. 65, of this body (this surface or line) be advanced by a small motion to $A_1$ and $B_1$, let also $A_1B_1 = AB$. If at these points we draw perpendiculars to the small spaces described $AA_1$ and $BB_1$, they will intersect at a point $C$, from which as a centre $AA_1$ and $BB_1$ may be considered the circular arcs described. Now from the equalities $AB = A_1B_1$, $AC = A_1C$, and $BC = B_1C$, the triangles $ABC$ and $A_1B_1C$ are equal, therefore, also the $\angle B_1CA_1 = BCA$ and the $\angle ACA_1 = \angle BCB_1$. If we make $A_1D_1 = AD$, we obtain from the equality of the $\angle$s $D_1A_1C$ and $DAC$, and from that of the sides $CA_1$ and $CA$ in $CA_1D_1$ and $CAD$, again two congruent triangles in which $CD_1 = CD$, and $\angle A_1CD_1 = \angle ACD$. Consequently any arbitrary point $D$ in $AB$, by its small advancement, describes a circular arc $DD_1$. If lastly $E$ be any point without the line $AB$ and rigidly connected with it, the small space $EE_1$ may be regarded as the arc of a circle from $C$ as a centre, for if we make the $\angle E_1A_1B_1 = EAB$ and the distance $A_1E_1 = AE$, we again obtain two congruent triangles $E_1A_1C$ and $EAC$ with equal sides $CE_1$ and $CE$, and equal $\angle$s $A_1CE_1$ $ACE$, and the same may be shown for every other point rigidly connected with $AB$. We may consequently regard every small motion of a surface rigidly connected with $AB$, or of a rigid body, as a small revolution about a centre, which is given when the point of intersection $C$ is determined, in which the perpendiculars to the paths $AA_1$ and $BB_1$ of the two points of the body intersect each other.

FIG. 65.



§ 97. From § 94, for a small revolution of a system of forces, the mechanical effect of the resultant is equivalent to the algebraical sum of thos. of its components; from § 95 every small displacement may be regarded as a small revolution: hence the law of the principle of virtual velocities above enunciated is therefore applicable to every small motion of a rigid body or system of forces.

If equilibrium obtain in a system of forces, *i. e.* if the resultant be null, the sum of the mechanical effects must be also null for a small arbitrary motion. If inversely for a small motion of the body the sum of the effects be null, equilibrium does not from this necessarily follow, the sum for all possible small displacement must be = 0, if equilibrium is to take place. Since the formula expressing the law of virtual velocities only fulfils one condition of equilibrium, it is requisite for equilibrium that this law be satisfied, at least for as many motions as can be made from these conditions for example, in a system of forces in a plane, for the three motions independent of each other.

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## CHAPTER II.

### CENTRE OF GRAVITY.

§ 98. *Centre of gravity.*—The weights of the parts of a heavy body form a system of parallel forces, whose resultant is the weight of the whole, and whose centre may be determined from the three formulæ of § 91. This middle point of a body or system of bodies is called the centre of gravity, and also the centre of the mass of the body or system of bodies. If a body be turned about its centre of gravity, this point does not cease to be the central point of gravity, for if the three planes, to which the points of application of the separate weights are referred, revolve at the same time with the body, the position of the directions of force to these planes alone changes by this revolution, the distances of the points of application from these planes remain invariable. The centre of gravity is, therefore, that point of a body in which its weight acts vertically downwards, and which must be, therefore, supported, and fixed, in order that in every position the body may remain at rest.

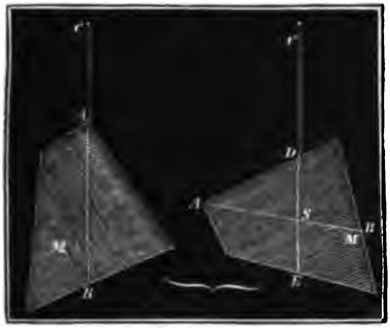
§ 99. Every straight line in which this point lies is called the line

of gravity (vertical); and every plane passing through the centre of gravity, a plane of gravity. The centre of gravity is determined by the intersection of two lines of gravity, or that of a line of gravity and a plane of gravity, or by the intersection of the planes of gravity.

Since the point of application may be displaced at will in the direction of force, without changing the action of the force, so a body is in any position in equilibrium if a point in the vertical line passing through the centre of gravity is fixed.

If a body *M*, Fig. 66, be suspended by a thread *CA*, in its prolongation *AB* we have a line of gravity, and if it be similarly suspended by a second line, we get a second line of gravity *DE*. The intersection *S* of both lines is the centre of gravity of the body. If the body be suspended upon an axis, or be brought upon a sharp edge (knife edge) into a state of equilibrium, we shall obtain in the vertical plane passing through the axis, or through the knife edge, a plane of gravity, &c. Experimental determinations of the centre of gravity, as just pointed out, are rarely applicable; we have generally to make use of geometrical rules, which will presently be given for the determination of this point with accuracy.

FIG. 66.



In many bodies, for example, in rings, the centre of gravity falls without the mass of the body. If such a body is to be fixed in its centre of gravity, it is necessary to connect a second body with the first, in such a manner that the centres of gravity of both may coincide.

§ 100. *Determination of the centre of Gravity.*—If $x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3$, &c., be the distances of the parts of a heavy body from the three planes $xz$, $yz$, $xy$, and the weights of these parts be $P_1$, $P_2$, $P_3$, &c., we then have the distances of the centre of gravity from these three planes,

$$x = \frac{P_1 x_1 + P_2 x_2 + P_3 x_3 + \dots}{P_1 + P_2 + P_3 + \dots},$$

$$y = \frac{P_1y_1 + P_2y_2 + P_3y_3 + \ldots}{P_1 + P_2 + P_3 + \ldots},$$

$$z = \frac{P_1z_1 + P_2z_2 + P_3z_3 + \ldots}{P_1 + P_2 + P_3 + \ldots}$$

If the volumes of these parts be $V_1$, $V_2$, $V_3$, &c., and their densities $\gamma_1$, $\gamma_2$, $\gamma_3$, &c., we may put therefore $V_1\gamma_1$ $V_2\gamma_2$ + c are masses

$$x = \frac{V_1\gamma_1x_1 + V_2\gamma_2x_2 + \ldots}{V_1\gamma_1 + V_2\gamma_2 + \ldots}$$

If the body be homogeneous, *i. e.* all parts of the same density $\gamma$, then:

$$x = \frac{(V_1x_1 + V_2x_2 + \ldots)\,\gamma}{(V_1 + V_2 + \ldots)\,\gamma},$$

or since the common factor $\gamma$ above and below is cancelled:

$$1.\ x = \frac{V_1x_1 + V_2x_2 + \ldots}{V_1 + V_2 + \ldots} \qquad \frac{\Sigma(V_1x_1)}{\Sigma V} = \frac{\int_{v^0}^{v'} x\,dV}{V}$$

$$2.\ y = \frac{V_1y_1 + V_2y_2 + \ldots}{V_1 + V_2 + \ldots}, \qquad = \frac{\Sigma V_1y_1}{\Sigma V} = \frac{\int_{v^0}^{v'} y\,dV}{V}$$

$$3.\ z = \frac{V_1z_1 + V_2z_2 + \ldots}{V_1 + V_2 + \ldots}. \qquad \frac{\Sigma V_1z_1}{\Sigma V} = \frac{\int_{v^0}^{v'} z\,dV}{V}$$

We may also, instead of the weights, substitute the volumes of the separate parts, and thereby make the determination of the centre of gravity a problem of pure geometry.

When bodies are a little extended in one or in two dimensions, as thin plates, fine wires, &c., they may be regarded as surfaces or lines, and their centres of gravity likewise determined with the help of the three last formulæ, if for the volumes $V_1$, $V_2$, the ~~arms~~ or lengths be substituted.

§. 101. In regular figures the centre of gravity coincides with

FIG. 67.

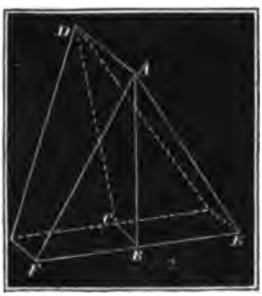



FIG. 68.

the centre of figure, as in dice, cubes, spheres, equilateral triangles, circles, &c. Symmetric figures have their centre of gravity in the plane or axis of symmetry. The plane of symmetry *ABCD* divides a body *ADFE*, Fig. 67, into two congruent halves; the proportions on both sides of this plane are equal; the moments also on the one side are equal to those on the other, and, consequently, the centre of gravity falls within this plane. Because the axis of symmetry *EF* cuts the plane surface *ABCD*, Fig. 68, into two congruent parts, here the proportions on the one side are equal to those on the other; the moments also on both sides are equal, and the centre of gravity of the whole lies in this line. Lastly, the axis of symmetry *KL* of a body *ABGH*, Fig. 69, is its line

FIG. 69.

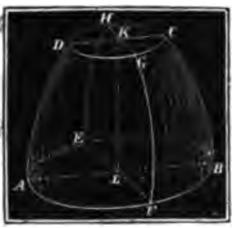
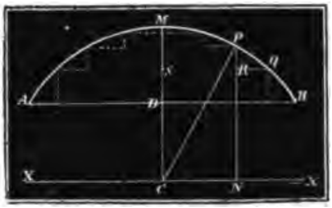



FIG. 70.




of gravity, because it arises from the intersection of two planes of symmetry, *ABCD* and *EFGH*. For this reason, the centre of gravity of a cylinder, of a cone, and of a surface of revolution, or of a rotating body formed on the potter's wheel lies in the axis of these bodies.

§. 102. *Centre of gravity of lines.*—The centre of gravity of a straight line lies in its middle.

The centre of gravity of a circular arc $AB=b$, Fig. 70, lies in wh. the diameter *CM*, ~~and~~ passes through the middle *M* of the arc, for this diameter is the axis of symmetry of this arc. But in order to find the distance $CS=x$ of the centre of gravity *S* from the middle point, the arc must be divided into many elementary parts, and statical moments of these, with reference to an axis $\overline{XX}$ passing through the centre *C* and parallel to the chord $AB=s$, be determined.

If *PQ* be a part of the arc, and *PN* be its distance from $\overline{XX}$, then the statical moment of this portion of the arc$=PQ\,.\,PN$. If now the ~~diameter~~ radius $PC=MC=r$ be drawn, and

$QR$ parallel to $AB$, we obtain the two similar $\Delta^s$ $PQR$ and $CPN$, for which:

$$PQ : QR = CP : PN,$$

and from which the statical moment of the elementary arc $PQ \,.\, PN = QR \,.\, CP = QR \,.\, r$ is determined.

Now, for the statical moments of all the remaining moments, the diameter $r$ is a common factor, and the sum of all the projections $QR$ of the elementary arcs is equal to the chord corresponding to the projection of the whole arc; it follows, therefore, that the moment of the whole arc is also = the chord ($s$) times the diameter $r$. If this moment be put equal to the arc ($b$) times the distance $x$, and therefore $bx = sr$, we then obtain:

$$\frac{x}{r} = \frac{s}{b}, \text{ and } x = \frac{sr}{b}.$$

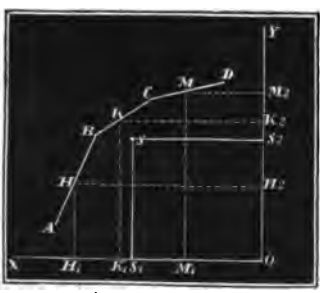
So that the distance of the centre of gravity, from the middle point to the radius, is in the ratio of the arc to the chord.

If the angle at the centre $ACB$ of the arc $b$ be $= \beta^0$, the arc corresponding to the diameter 1 is $\beta = \frac{\beta^0}{180^0} \,.\, \pi$, we have then $b = \beta r$, and $s = 2\, r\, sin.\, \frac{\beta}{2}$; whence it follows that, $x = \frac{2\, sin.\, \frac{1}{2}\beta \,.\, r}{\beta}$.

For the semicircle $\beta = \pi$ and $sin.\, \frac{\beta}{2} = 1$; therefore, $x = \frac{2}{\pi} r = 0{,}6366 \ldots r = \frac{7}{11}\, r$ nearly.

§. 103. To find the centre of gravity of a polygon or a connection of lines $ABCD$, Fig. 71, we must seek the distances of the middle points $H$, $K$, $M$, of the lines $AB = L_1$, $BC = L_2$, $CD = L_3$, &c. from two axes $OX$ and $OY$, viz: $HH_1 = y_1$, $HH_2 = x_1$, $KK_1 = y_2$, $KK_2 = x_2$, &c.; the distances of the centre of gravity sought from these axes are then:

FIG. 71.




$$SS_2 = x = \frac{L_1 x_1 + L_2 x_2 + \ldots}{L_1 + L_2 + \ldots},$$

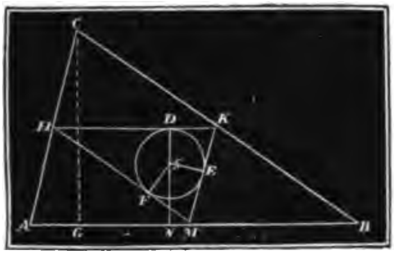
$$SS_1 = y = \frac{L_1 y_1 + L_2 + \ldots}{L_1 + L_2 + \ldots},$$

For example, the distance of the centre of gravity $S$ of a wire bent into the form of a $\triangle$ $ABC$, Fig. 72, from the base is:

$$NS = x = \frac{\frac{1}{2}ah + \frac{1}{2}bh}{a+b+c} = \frac{a+b}{a+b+c} \cdot \frac{h}{2},$$

if the sides opposite to the angles $A$, $B$, $C$ be designated by $a$, $b$, $c$, and the height $CG$ by $h$.

FIG. 72.



If the middle points $H$, $K$, $M$, of the sides of the triangle be connected with each other, and in the triangle so obtained a circle be described, its centre will coincide with the centre of gravity $S$, for the distance $SD$ from one side to $HK$ is

$$= DN - SN = \frac{h}{2} - \frac{a+b}{a+b+c} \cdot \frac{h}{2} = \frac{ch}{2(a+b+c)} = \frac{\triangle\ ABC}{a+b+c} =$$

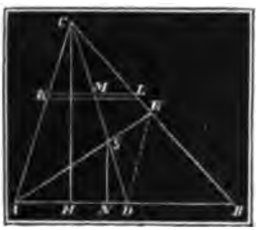
the distances $SE$ and $SF$ from the other sides.

§. 104. *Centre of gravity of plane figures.*—The centre of gravity of a parallelogram $ABCD$, Fig. 73, lies in the point of intersection of its diagonals, for all strips, such as $KL$, which are formed by drawing lines parallel to one of its diagonals $BD$, are bisected by the other diagonals $AC$; each of the diagonals, therefore, is a line of gravity.

FIG. 73.

In a $\triangle$ $ABC$, Fig. 74, every line $CD$ from one angle to the middle $D$ of the opposite side $AB$, is a line of gravity, for the same bisects all the elements $KL$ of the $\triangle$ which are given when lines parallel to $AB$ are drawn. If from a second angle $A$, a second line of gravity be drawn to the middle $E$ of the opposite side $BC$, the point of intersection of the two will give the centre of gravity of the whole $\triangle$.

FIG. 74.



Because $BD = \frac{1}{2}BA$ and $BE = \frac{1}{2}BC$, $DE$ is parallel to $AC$ and $= \frac{1}{2}AC$, and $\triangle$ $DES$ similar to the $\triangle$ $CAS$, and lastly, $CS = 2\,SD$. If further we add $SD$, it follows that $CS + SD$, or

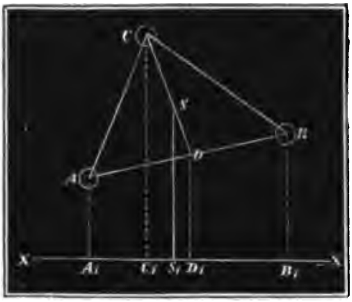
$CD = 3\ DS$, and, therefore, inversely, $DS = \frac{1}{3}\ CD$. The centre of gravity $S$ lies about $\frac{1}{3}$ of the line $CD$ from the middle point $D$ of the base, and about $\frac{2}{3}$ of the same from the angle $C$. If $CH$ and $SN$ be drawn perpendicular to the base, we have also $SN = \frac{1}{3}\ CH$; the centre of gravity $S$ is about $\frac{1}{3}$ of the height from the base of the $\Delta$.

The distance $SS_1$ of the centre of gravity of a $\Delta$ $ABC$, Fig. 75, from an axis $X\overline{X}$ is $= DD_1 + \frac{1}{3}\ (CC_1 - DD_1)$, but $DD_1 = \frac{1}{2}\ (AA_1 + BB_1)$, consequently, $x = SS_1 = \frac{1}{3}\ CC_1 + \frac{2}{3} \cdot \frac{1}{2}\ (AA_1 + BB_1)$:

$$= \frac{AA_1 + BB_1 + CC_1}{3},$$

*i. e.* the arithmetical mean of the distances of the three angular points.

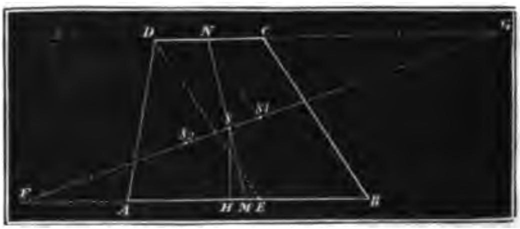
FIG. 75.



Since the distance of the centre of gravity is determined in the same manner by three equal weights at the angular points of a $\Delta$, so the centre of gravity of a plane triangle coincides with the centre of gravity of these three equal weights.

§ 105. The determination of the centre of gravity $S$ of a trapezium $ABCD$, Fig. 76, may be made in the following manner.

FIG. 76.



The straight line $MN$, which connects the middle points of the two bases $AB$ and $CD$ with each other, is a line of gravity of the trapezium; for lines drawn parallel to the bases decompose the trapezium into elementary parts, whose middle points or centres of gravity lie in $MN$. Now to determine completely the centre of gravity $S$, we have only therefore to find its distance $SH$ from a base $AB$.

Let $B$ represent the one, and $b$ the other of the parallel sides $AB$ and $CD$ of the trapezium, $h$ the height or the normal distance of these sides. Let $DE$ be now drawn parallel to the side

$BC$, we shall then obtain a parallelogram $BCDE$ of the area $bh$, and whose centre of gravity is $S_1$, and distance from $AB = \frac{h}{2}$, and a $\Delta\ ADE$ of the area $\frac{(B-b)h}{2}$ and centre of gravity $S_2$, and whose distance from $AB = \frac{h}{3}$.

The statical moment of the trapezium, about the line $AB$ is therefore

$$= bh.\ \frac{h}{2} + \frac{(B-b)\,h}{2} \cdot \frac{h}{3} = (B+2b)\frac{h^2}{6},$$

but the area of the trapezium is $= (B+b)\ \frac{h}{2}$; it follows, therefore, that the normal distance of the centre of gravity $S$ from the base is

$$HS = \frac{\frac{1}{6}\,(B+2b)\,h^2}{\frac{1}{2}\,(B+b)\,h} = \frac{B+2\,b}{B+b} \cdot \frac{h}{3}.$$

To find the centre of gravity by construction, let the two bases be prolonged, the prolongations $CG$ made $= B$ and $AF = b$, and the two extreme points obtained, $F$ and $G$, connected by a straight line: the point of intersection $S$ with the middle line $MN$ will be the centre of gravity sought; for, from $HS = \frac{B+2b}{B+b} \cdot \frac{h}{3}$, it follows that

$$MS = \frac{B+2b}{B+b} \cdot \frac{MN}{3} \text{ and } NS = \frac{2\,B+b}{B+b} \cdot \frac{MN}{3}; \text{ and}$$

$$\frac{MS}{NS} = \frac{B+2b}{2B+b} = \frac{\frac{1}{2}\,B+b}{B+\frac{1}{2}b} = \frac{MA+AF}{CG+NC} = \frac{MF}{NG},$$

which actually arises from the similarity of the triangles $MSF$ and $NSG$.

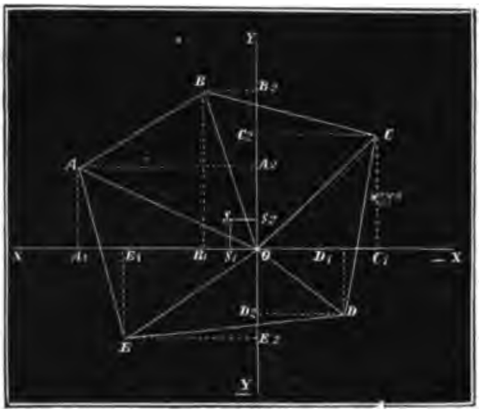
§ 106. To find the centre of gravity of any other four-sided figure $ABCD$, Fig. 77, we may decompose it by the diagonal $AC$ into two triangles, and from the foregoing, determine their centres of gravity $S_1$ and $S_2$, and thereby a line of gravity $S_1S_2$. If now the four-sided figures be decomposed into two other triangles by the diagonal $BD$, and their centres of gravity determined, we obtain another line of gravity, whose intersection with the first will give the centre of gravity of the whole figure.

FIG. 77.

We may effect this more simply if we bisect the diagonal $AC$ in $M$, apply the greater part $BE$ of the second diagonal to the less, so that $DF=BE$, join $FM$ and divide it into three equal parts; the centre of gravity lies in the first point $S$ from $M$, as may be proved in the following manner. $MS_1=\frac{1}{3}MD$ and $MS_2=\frac{1}{3}MB$, consequently $S_1 S_2$ are parallel to $BD$, but $SS_1$ times $\triangle\ ACD = SS_2$ times $\triangle\ ACB$, or $SS_1 \, . \, DE = SS_2 \, . \, BE$; therefore, $SS_1 : SS_2 = BE : DE$. Now, $BE=DF$ and $DE=BF$, consequently $SS_1 : SS_2 = DF : BF$. The straight line $MF$ intersects, therefore, the line of gravity $S_1 S_2$ in the centre of gravity of the figure.

§ 107. If it be required to find the centre of gravity $S$ of a polygon $ABCDE$, Fig. 78, we must decompose the polygon into

FIG. 78.



triangles, and determine their statical moments with reference to two rectangular axes $X\overline{X}$ and $Y\overline{Y}$.

If the co-ordinates $OA_1=x_1$, $OA_2=y_1$, $OB_1=x_2$, $OB_2=y_2$, &c. of the extremities are given, the statical moments of the triangles $ABO$, $BCO$, $COD$ &c. may be determined simply in the following manner. The area of $\triangle\ ABO$, from the remark below, $=D_1=\frac{1}{2}(x_1 y_2 - x_2 y_1)$; of the following $\triangle\, BCO = D_2 = \frac{1}{2}(x_2 y_3 - x_3 y_2)$, &c. the distance of the centre of gravity of $\triangle\, ABO$ from $Y\overline{Y}$ according to § 104 $=u_1=\frac{x_1+x_2+0}{3}=\frac{x_1+x_2}{3}$, from $X\overline{X}=v_1=\frac{y_1+y_2}{3}$; of the centre of gravity of $\triangle\ BCO=u_2=\frac{x_2+x_3}{2}$ and $v_2=\frac{y_2+y_3}{3}$ &c. If these distances are multiplied by the areas of the triangles, the

moments of these last are obtained; and if the values so obtained are substituted in the formulæ:

$$u = \frac{D_1 u_1 + D_2 u_2 + \ldots}{D_1 + D_2 + \ldots}$$

$$v = \frac{D_1 v_1 + D_2 v_2 + \ldots}{D_1 + D_2 + \ldots},$$

we have the distances $u$ and $v$ of the centre of gravity from the axes $YY$ and $XX$.

*Example.* A pentagon $ABCDE$, Fig. 78, is given by the following co-ordinates of its extremities $A$, $B$, $C$, &c.: to find the co-ordinates of its centre of gravity:

| Co-ordinates given. | | Twice the area of triangles. | Triple co-ordinates of centre of gravity. | | Six times the statical moments. | |
|---|---|---|---|---|---|---|
| $x$ | $y$ | | $3u_n$ | $3v_n$ | $6D_n u_n$ | $6D_n v_n$ |
| 24 | 11 | 24 . 21 — 7 . 11 = 427 | 31 | 32 | 13237 | 13664 |
| 7 | 21 | 7 . 15 + 21 . 16 = 441 | — 9 | 36 | —3969 | 15876 |
| —16 | 15 | 16 . 9 + 12 . 15 = 324 | —28 | 6 | —9072 | 1944 |
| —12 | — 9 | 12 . 12 + 18 . 9 = 306 | + 6 | —21 | 1836 | —6426 |
| 18 | —12 | 18 . 11 + 24 . 12 = 486 | +42 | — 1 | 20412 | — 486 |
| | | Sum: 1984 | | | 22444 | 24572 |

The distance of the centre of gravity from the axis $YY$ is:

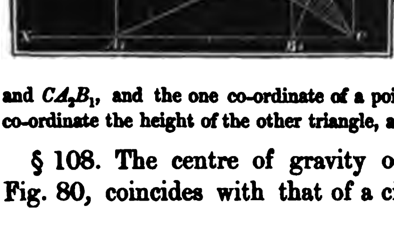
$$SS_2 = u = \frac{1}{3} \cdot \frac{22444}{1984} = 3{,}771,$$

and from the axis $XX$:

$$SS_1 = v = \frac{1}{3} \cdot \frac{24572}{1984} = 4{,}128.$$

*Remark.* If $CA_1 = x_1$, $CB_1 = x_2$, $CA_2 = y_1$, and $CB_2 = y_2$, the co-ordinates of the two angles of a triangle $ABC$, Fig. 79, whose third angle $C$ coincides with the point of application of the system of co-ordinates, we have the area of the same:

FIG. 79.



$D$ = trapezium $ABB_1A_1$ + triangle $CBB_1$ — triangle $CAA_1$

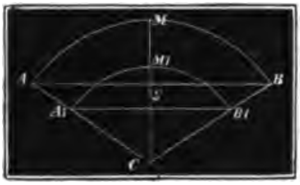
$$= \left(\frac{y_1 + y_2}{2}\right)(x_1 - x_2) + \frac{x_2 y_2}{2} - \frac{x_1 y_1}{2}$$

$$= \frac{x_1 y_2 - x_2 y_1}{2}.$$

The area of this triangle is the difference of two other triangles, $CB_2A_1$ and $CA_2B_1$, and the one co-ordinate of a point is the base of the one, and the other co-ordinate the height of the other triangle, and inversely.

§ 108. The centre of gravity of the sector of a circle $ACB$, Fig. 80, coincides with that of a circular arc $A_1B_1$ which has the

FIG. 80.

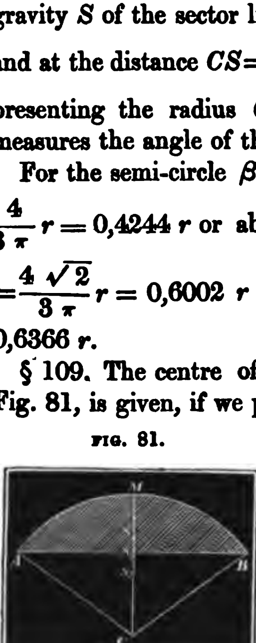


same angle with the sector, and whose radius $CA_1$ is two thirds of the radius $CA$ of the sector; for the sector may be divided by an infinity of radii into very small triangles, whose centres of gravity are distant two thirds of the radius from the centre $C$, and these form by their continuity the arc $A_1M_1B_1$. The centre of gravity $S$ of the sector lies in the radius $CM$, bisecting the surface, and at the distance $CS = x = \frac{\text{chord}}{\text{arc}} \cdot \frac{2}{3} CA = \frac{4}{3} \cdot \frac{sin. \frac{1}{2}\beta}{\beta} \cdot r$, representing the radius $CA$ of the sector, and $\beta$ the arc which measures the angle of the centre $ACB$.

For the semi-circle $\beta = \pi$, $sin. \frac{1}{2}\beta = sin. 90^0 = 1$, therefore $x = \frac{4}{3\pi} r = 0,4244\, r$ or about $\frac{14}{33} r$. For a quadrant $x = \frac{4}{3} \cdot \frac{\sqrt{\frac{1}{2}}}{\frac{1}{2}\pi} r = \frac{4\sqrt{2}}{3\pi} r = 0,6002\, r$ and for a sixth part $x = \frac{4}{3} \cdot \frac{\frac{1}{2}}{\frac{1}{3}\pi} r = \frac{2}{\pi} r = 0,6366\, r$.

§ 109. The centre of gravity of a segment of a circle $ABM$, Fig. 81, is given, if we put the moment of the sector $ACBM$ equal

FIG. 81.

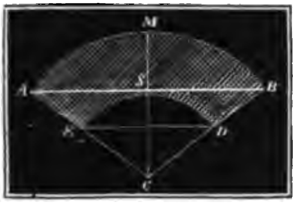


to the sum of the moments of the segment and the moment of the triangle $ACB$. If $r$ be the radius $CA$, $s$ the chord $AB$, and $A$ the area of the segment $ABM$, the moment of the sector = the sector $\times\, CS_1 = \frac{r \cdot \text{arc}}{2} \cdot \frac{\text{chord}}{\text{arc}} \cdot \frac{2}{3} r = \frac{1}{3} s r^2$, further the moment of the triangle = triangle $\times\, CS_2 = \frac{s}{2}\sqrt{r^2 - \frac{s^2}{4}} \cdot \frac{2}{3}\sqrt{r^2 - \frac{s^2}{4}} = \frac{sr^2}{3} - \frac{s^3}{12}$, and from this the moment of the segment: $A \cdot CS = Ax = \frac{1}{3} sr^2 - \left(\frac{s r^2}{3} - \frac{s^3}{12}\right) = \frac{s^3}{12}$; consequently the distance sought is $x = \frac{s^3}{12 A}$.

For the semi-circle $s = 2r$ and $A = \frac{1}{2}\pi r^2$, hence $x = \frac{8 r^3}{12 \cdot \frac{\pi r^2}{2}} = \frac{4 r}{3 \pi}$, as found above.

In like manner we may find the centre of gravity $S$ of a portion of a ring $ABDE$; Fig. 82, which is the difference of two sectors $ACB$ and $DCE$. If the radii be $CA=r$ and $CD=r_1$ and the chords $AB=s$ and $DE=s_1$, the statical moments of the sectors are: $\frac{s\,r^2}{3}$ and $\frac{s_1\,r_1^2}{3}$, therefore the statical moments of the portion of ring: $=\frac{sr^2-s_1r_1^2}{3}$, or (since $\frac{s_1}{s}=\frac{r_1}{r}$) is $=\frac{r^3-r_1^3}{3}\cdot\frac{s}{r}$. But the area $=\frac{\beta r^2}{2}-\frac{\beta r_1^2}{2}=\beta\left(\frac{r^2-r_1^2}{2}\right)$, provided that $\beta$ represents the arc corresponding to the angle at the centre $ACB$; the centre of gravity, therefore, of the portion follows from the distance $CS=x=\frac{\text{moment}}{\text{area}}=$

FIG. 82.



$$\frac{r^3-r_1^3}{r^2-r_1^2}\cdot\frac{2}{3}\cdot\frac{s}{r\beta}=\frac{2}{3}\left(\frac{r^3-r_1^3}{r^2-r_1^2}\right)\cdot\frac{\text{chord}}{\text{arc}}=\frac{4}{3}\,\frac{\sin.\frac{1}{2}\beta}{\beta}\cdot\frac{r^3-r_1^3}{r^2-r_1^2}.$$

*Example.* The radii of the surfaces of a dome are: $r=5$ ft., $r_1=3\frac{1}{2}$ ft., and the angle at the centre, $\beta^0=130^0$, then is the distance of the centre of gravity of these surfaces from their central point:

$$x=\frac{4}{3}\,\frac{\sin.\,65^0}{arc.\,130^0}\cdot\frac{5^3-3{,}5^3}{5^2-3{,}5^2}=\frac{4\,.\,0{,}9063}{3\,.\,2{,}2689}\cdot\frac{125-42{,}875}{25-12{,}25}=\frac{3{,}6252\times 82{,}125}{6{,}8067\times 12{,}75}$$
$=3{,}430$ feet.

§ 110. *Centre of gravity of curved Surfaces.*—The centre of gravity of a curved surface (envelope) of a cylinder $ABCD$, Fig. 83, lies in the middle $S$ of the axis $MN$ of this body, for all the annular elements of the cylindrical envelope which are obtained by sections drawn through the body parallel to the base are equal, and their centres of gravity lie in the axis; these centres of gravity form a uniform line of gravity. For the same reason the centre of gravity of the surfaces of a prism lies in the middle point of the straight lines connecting the centres of gravity of both the bases.

FIG. 83.

The centre of gravity of the envelope of a right cone $ABC$,

FIG. 84.

Fig. 84, lies in the axis of the cone, and is one third of this line from the base, or two thirds from the vertex; for this curved surface may be decomposed into an infinite number of small triangles by straight lines, which are called the sides of the cone whose centres of gravity form a circle *HK*, which is distant two thirds of the axis from the vertex, and whose centre of gravity or centre *S* lies in the axis *CM*.

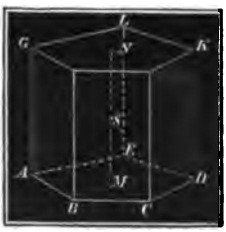
The centre of gravity of a spherical zone *ABDE*, Fig. 85, and likewise that of a spherical cup lies in the centre *S* of its height *MN*; for from the rules of geometry the zone has the same surface as a cylindrical envelope *FGHK*, whose height is equal to that of *MN* and whose radius is equal to that of the radius *CO* of the spherical zone; and this equality also exists in the annular elements, which are obtained by carrying an infinite number of planes parallel to the circular bases through the same; according to this the centre of gravity of the zone coincides with that of the cylindrical envelope.

FIG. 85.

*Remark.* The centre of gravity of the surface of an oblique cone or oblique pyramid lies at about one-third of the height from the base, but not in the straight line passing from the vertex to the centre of gravity of the base, because slices parallel to the base decompose the surface into rings, which vary in breadth at different parts of their surface.

§ 111. *Centre of gravity of Bodies.*—The centre of gravity of a prism *AK*, Fig. 86, is the centre *S* of the straight line which connects the centres of gravity *M* and *N* of both bases *AD* and *GK*, for the prism may be decomposed by sections parallel to the base into exactly congruent slices, whose centres of gravity lie in *MN*, and by their superposition make the line *MN* a uniform line of gravity.

FIG. 86.



For the same reason the centre of gravity of a cylinder lies in the middle of its axis.

The centre of gravity of a pyramid *ADF*, Fig. 87, lies in the

straight line $MF$ from the vertex $F$ to the centre of gravity $M$ of the base, for all slices as $NOPQR$, have from their similarity with the base, their centres of gravity in this line.

If the pyramid be triangular as $ABCD$, Fig. 88, each of the

FIG. 87.

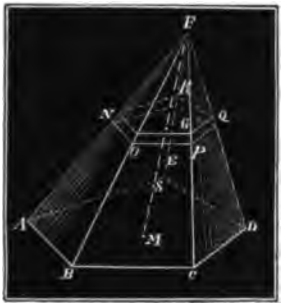
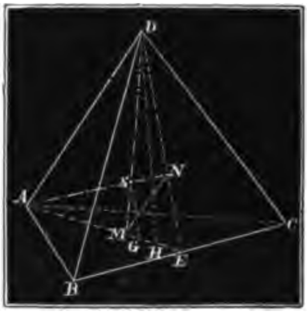


FIG. 88.



four angular points may be considered as vertices, and the opposite surfaces as bases; the centre of gravity $S$ is determined by the intersection of two straight lines drawn from $D$ and $A$ to the centres of gravity $M$ and $N$ of the opposite surfaces $ABC$ and $BCD$.

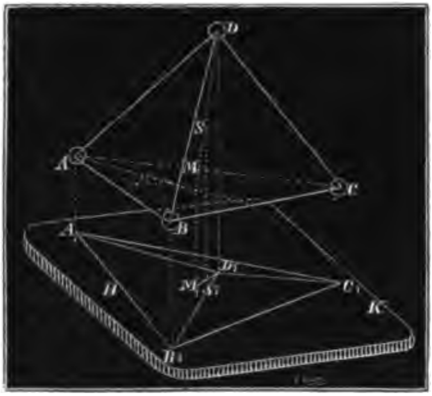
If the straight lines $EA$ and $ED$ be given, we then have from § 104 $EM = \frac{1}{3}\, EA$ and $EN = \frac{1}{3}\, ED$; therefore $MN$ is parallel to $AD$ and $= \frac{1}{3}\, AD$, and the $\triangle\ MNS$ similar to $\triangle\ DAS$. Again from this similarity we have $MS = \frac{1}{3}\, DS$, or $DS = 3\, MS$, also $MD = SD + MS = 4\, MS$, and inversely $MS = \frac{1}{4}\, MD$. Hence the centre of gravity is found to be one fourth of the line joining the centre of gravity $M$ of the base with the vertex $D$.

Further, if the heights $DH$ and $SG$ be given, and $HM$ be drawn, we then obtain the two similar $\triangle^s$ $DHM$ and $SGM$, in which from the foregoing $SG = \frac{1}{4}\, DH$. We may, therefore, say that the distance of the centre of gravity $S$ of a triangular pyramid from the base is equal to one fourth, and that from the vertex three fourths of the height of the pyramid.

As every pyramid, and also cone, is made up of an infinite number of three sided pyramids of the same height, the centre of gravity of every pyramid and cone is a fourth of the height from the base and three fourths from the vertex. We may therefore find the centre of gravity of a pyramid or cone, if a plane be drawn parallel to the base at a distance one fourth from the base, and the centre of gravity of the section or its intersection with the line joining the vertex and the centre of gravity of the base be determined.

§ 112. If the distances $AA_1$, $BB_1$, of the four angles of a a triangular pyramid $ABCD$, Fig. 89, from a plane $HK$ be known, the distance of the centre of gravity $S$ from this plane is found from the mean value

FIG. 89.



$$SS_1 = \frac{AA_1 + BB + CC_1 + DD_1}{4},$$

The distance of the centre of gravity $M$ of the base $ABC$ from this plane is (§ 104):

$$MM_1 = \frac{AA_1 + BB_1 + CC_1}{3},$$

and that of the pyramid $S$ is:

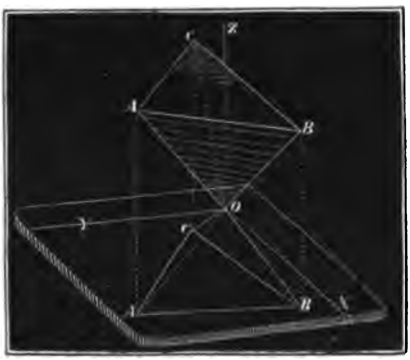
$$SS_1 = MM_1 + \tfrac{1}{4}(DD_1 - MM_1),$$

where $DD_1$ is the distance of the vertex: hence it follows by combining the two last equations, that:

$$SS_1 = \tfrac{3}{4} MM_1 + \tfrac{1}{4} DD_1 = \frac{AA_1 + BB_1 + CC_1 + DD_1}{4}.$$

The distance of the centre of gravity of four equal weights applied to the angles of a triangular pyramid is equivalent to the arithmetical mean $\frac{AA_1 + BB_1 + CC_1 + DD_1}{4}$, consequently the centre of gravity of the pyramid corresponds with that of the system of weights.

FIG. 90.



*Remark.* The determination of the volume of a triangular pyramid, from the co-ordinates of its angles is simple. If we draw planes $XY$, $XZ$, $YZ$, through the vertex $O$ of such a pyramid $ABCO$, Fig. 90, and represent the distances of the angles $ABC$ from these planes by $z_1, z_2, z_3$; $y_1, y_2, y_3$, and $x_1, x_2, x_3$, the volume of the pyramid will be

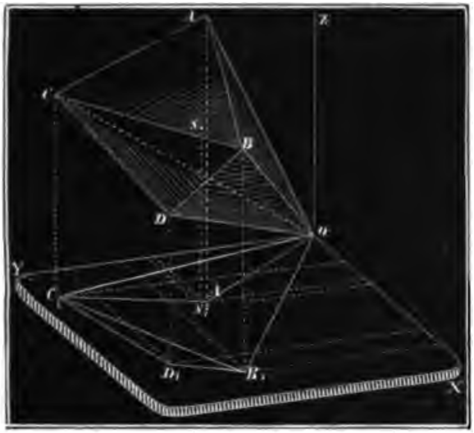
$$V = + \tfrac{1}{6}(x_1 y_2 z_3 + x_3 y_2 z_1 + x_3 y_1 z_2 - x_1 y_3 z_2 - x_2 y_1 z_3 - x_3 y_2 z_1),$$

which will be given, if the pyramid be considered as an aggregate of four ~~oblique~~ prisms.
Truncated.

The distances of the centre of gravity of these pyramids from the three planes are:

$$x = \frac{x_1 + x_2 + x_3}{4},\ y = \frac{y_1 + y_2 + y_3}{4},\ \text{and}\ z = \frac{z_1 + z_2 + z_3}{4}.$$

FIG. 91.



§ 113. Since every polyedron as $ABCDO$, Fig. 91, may be decomposed into triangular pyramids $ABCO$, $BCDO$, we may also find its centre of gravity $S$ if we calculate the volumes, and the statical moments of the single pyramids.

If the distances of the angles $A$, $B$, $C$, &c., from the co-ordinate planes passing through the common vertex $O$ of all the pyramids, are $x_1$, $x_2$, $x_3$, &c., $y_1$, $y_2$, $y_3$, &c., $z_1$, $z_2$, $z_3$, &c., the volumes of the single pyramids, are:

$$V_1 = \pm \tfrac{1}{6}\,(x_1y_2z_3 + x_2y_3z_1 + x_3y_1z_2 - x_1y_3z_2 - x_2y_1z_3 - x_3y_2z_1),$$
$$V_2 = \pm \tfrac{1}{6}\,(x_2y_3z_4 + x_3y_4z_2 + x_4y_2z_3 - x_2y_4z_3 - x_3y_2z_4 - x_4y_3z_2),$$

and the distances of their centres of gravity:

$$u_1 = \frac{x_1 + x_2 + x_3}{4},\ v_1 = \frac{y_1 + y_2 + y_3}{4},\ w_1 = \frac{z_1 + z_2 + z_3}{4},$$
$$u_2 = \frac{x_2 + x_3 + x_4}{4},\ v_2 = \frac{y_2 + y_3 + y_4}{4},\ w_2 = \frac{z_2 + z_3 + z_4}{4},\ \text{\&c.}$$

From these values the distances of the centre of gravity of the whole body may be finally calculated by the formula:

$$u = \frac{V_1u_1 + V_2u_2 + \ldots}{V_1 + V_2 + \ldots},\ v = \frac{V_1v_1 + V_2v_2 + \ldots}{V_1 + V_2 + \ldots},$$
$$w = \frac{V_1w_1 + V_2w_2 + \ldots}{V_1 + V_2 + \ldots}.$$

*Example.* A body bounded by six triangles $ADO$, Fig. 91, is determined by the following values for the co-ordinates of angles; whence the co-ordinates of its centre of gravity may be found.

| Given co-ordinates. | | | Six times the area of the triangular pyramids. | Four times the co-ordinates of centre of gravity. | | | Twenty-four times the statical moments. | | |
|---|---|---|---|---|---|---|---|---|---|
| $x$ | $y$ | $z$ | | $4u_n$ | $4v_n$ | $4w_n$ | $24\,V_n u_n$ | $24\,V_n v_n$ | $24\,V_n w_n$ |
| 20 | 23 | 41 | $6\,V_1 = \left\{\begin{matrix}20.29.28\\23.30.12\\41.45.40\end{matrix}\right\} - \left\{\begin{matrix}20.40.30\\23.28.45\\41.12.29\end{matrix}\right\} = 31072$ | 77 | 92 | 99 | 2392544 | 2858624 | 3076128 |
| 45 | 29 | 30 | | | | | | | |
| 12 | 40 | 28 | $6\,V_2 = \left\{\begin{matrix}45.35.28\\29.20.12\\30.38.40\end{matrix}\right\} - \left\{\begin{matrix}45.40.20\\29.28.38\\30.12.35\end{matrix}\right\} = 17204$ | 95 | 104 | 78 | 1634380 | 1789216 | 1341912 |
| 38 | 35 | 20 | | | | | | | |
| | | | Sum : 48276 | | | | 4026924 | 4647840 | 4418040 |

**From the results of this calculation, the distances of the centres of gravity from the three planes *YZ*, *XZ*, and *XY* follow.**

$$u = \frac{1}{4} \cdot \frac{4026924}{48276} = 20{,}853,$$

$$v = \frac{1}{4} \cdot \frac{4647840}{48276} = 24{,}069,$$

$$w = \frac{1}{4} \cdot \frac{4418040}{48276} = 22{,}879.$$

**§ 114. The centre of gravity of a truncated pyramid *ADQN*, (Fig. 88) lies in the line *MG*, which connects the centres of gravity of the two parallel bases; in order to determine the distance of this point from one of the bases, we must determine the volumes and moments of the entire pyramid *ADF* and the supplementary pyramid *NQF*. If the areas of the bases *AD* and *NQ* $= G$ and $g$, and the normal distance of both $= h$, the height of the supplementary pyramid will be given from the formulæ:**

$\dfrac{G}{g} = \dfrac{(h+x)^2}{x^2}$, or $\dfrac{h}{x} + 1 = \sqrt{\dfrac{G}{g}}$, and $x = \dfrac{h\sqrt{g}}{\sqrt{G} - \sqrt{g}}$, as also

$$h + x = \frac{h\sqrt{G}}{\sqrt{G} - \sqrt{g}}.$$

**The moment of the whole pyramid with reference to the base $G$ is now**

$\dfrac{G\,(h+x)}{3} \cdot \dfrac{h+x}{4} = \dfrac{1}{12} \cdot \dfrac{h^2\,G^2}{(\sqrt{G} - \sqrt{g})^2}$, that of the supplementary

pyramid $= \dfrac{g\,x}{3}\left(h + \dfrac{x}{4}\right) = \dfrac{1}{3}\,\dfrac{h^2\sqrt{g^3}}{\sqrt{G} - \sqrt{g}} + \dfrac{1}{12} \cdot \dfrac{h^2\,g^2}{(\sqrt{G} - \sqrt{g})^2}$; hence it follows that the moment of the truncated pyramid:

$$\frac{h^2}{12\,(\sqrt{G} - \sqrt{g})^2} \cdot (G^2 - 4\,(\sqrt{G\,g^3} - g^2) - g^2)$$

$$=\frac{h^2\ G^2-4\,g\sqrt{G\,g}+3\,g^2)}{12\,(G-2\sqrt{G\,g}+g)}=\frac{h^2}{12}\cdot(G+2\sqrt{G\,g}+3\,g).$$

Now the solid contents of the truncated pyramid are:

$$V=\frac{h}{3}\,(G+\sqrt{G\,g}+g)\,;$$

hence it follows finally that the distance of its centre of gravity $S$ from the base is

$$MS=y=\frac{h}{4}\cdot\frac{G+2\sqrt{G\,g}+3\,g}{G+\sqrt{G\,g}+g}.$$

The radii of the bases of a truncated cone are $R$ and $r$, and therefore $G=\pi\,R^2$ and $g=\pi\,r^2$, we have then for this

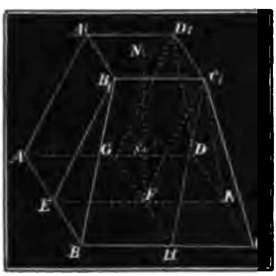
$$y=\frac{h}{4}\cdot\frac{R^2+2Rr+3\,r^2}{R^2+Rr+r^2}.$$

*Example.* The centre of gravity of a truncated cone of the height $h$ = 20 inches, and radius $R$ = 12 and $r$ = 8 inches, always lies in the line joining the centres of the two circular bases, and is distant from the greater by:

$$y=\frac{20}{4}\cdot\frac{12^2+2.12.8+3.8^2}{12^2+12.8+8^2}=\frac{5\,.\,528}{304}=\frac{2640}{304}=8{,}684 \text{ inches.}$$

§ 115. A pontoon is a body enclosed by two dissimilar rectangular bases and four trapeziums $ACC_1A_1$, Fig. 92, and may be decomposed into a parallelepiped $AFC_1A_1$, two triangular prisms $EHC_1B_1$, $GKC_1D_1$, and a quadrangular pyramid $HKC_1$; we may, therefore, with the help of these constituents, find the centre of gravity of the body.

FIG. 92.



It is easy to see that the line from the one bases to the other is the line of gravity of this body, there remains only to determine the distance of the centre of gravity from either base. If we represent the length $BC$ and breath $AB$ of one base by $l$ and $b$, and that of $A_1B_1$ and $B_1C_1$ of the other base by $l_1$ and $b_1$, and the height of the body by $h$. Then the volume of the parallelepiped $= b_1l_1h$, and its moment $b_1l_1h\,.\,\frac{h}{2}=\frac{1}{2}\,b_1l_1h^2$, further the volumes of the two triangular prisms $=(\,[b-b_1]l_1+[l-l_1]\,b_1)\,\frac{h}{2}$ and their moment $=(\,[b-b]\ l_1 + [l-l_1]\,b_1)\,\frac{h}{2}\cdot\frac{h}{3}$, lastly the volume of the pyramid$=(b-b_1)\,(l-l_1)\frac{h}{3}$

and its moment $= (b-b_1)\ (l-l_1)\ \frac{h}{3} \cdot \frac{h}{4}$. The volume of the whole body is, therefore:

$$V = (6\, b_1 l_1 + 3\, b l_1 + 3 l b_1 - 6\, b_1 l_1 + 2\, bl + 2\, b_1 l_1 - 2\, b l_1 - 2\, b_1 l) \cdot \frac{h}{6}$$

$$= (2\, bl + 2\, b_1 l_1 + b l_1 + l b_1)\ \frac{h}{6},$$ and its moment

$$Vy = (6\, b_1 l_1 + 2\, b l_1 + 2\, l b_1 - 4\, b_1 l_1 + bl + b_1 l_1 - b l_1 - l b_1) \cdot \frac{h^2}{12}$$

$$= (3\, b_1 l_1 + bl + b l_1 + b_1 l)\ \frac{h^2}{12}.$$

Hence it follows that the distance of the centre of gravity from the base $bl$ is:

$$y = \frac{bl + 3\, b_1 l_1 + b l_1 + b_1 l}{2\, bl + 2\, b_1 l_1 + b l_1 + b_1 l} \cdot \frac{h}{2}.$$

*Remark.* This formula is also applicable to bodies with elliptical bases. The axes of the one base are $a$ and $b$, and of the other $a_1$ and $b_1$; the volume of such a body, therefore, is:

$V = \frac{\pi h}{24}(2\, ab + 2\, a_1 b_1 + a b_1 + a_1 b)$, and the distance of the centre of gravity:

$$y = \frac{ab + 3\, a_1 b_1 + a b_1 + a_1 b}{2\, ab + 2\, a_1 b_1 + a b_1 + a_1 b} \cdot \frac{h}{2}.$$

*Example.* A damm, $ACC_1A_1$, Fig. 93, is of the height 20 feet, 250 feet long at the

FIG. 93.

bottom and 40 feet wide, at the top 400 feet long and 15 wide; to find the distance of its centre of gravity from the base. Here $b = 40$, $l = 250$, $b_1 = 15$, $l_1 = 400$, $h = 20$, therefore the vertical distance sought is:

$$MS = y = \frac{40 \cdot 250 + 3 \cdot 15 \cdot 400 + 40 \cdot 400 + 15 \cdot 250}{2 \cdot 40 \cdot 250 + 2 \cdot 15 \cdot 400 + 40 \cdot 400 + 15 \cdot 250} \cdot \frac{20}{2}$$

$$= \frac{4775}{5175} \cdot 10 = \frac{1910}{207} = 9{,}227 \text{ feet.}$$

§ 116. If the sector of a circle $ACD$, Fig. 94, revolves about its radius $CD$, there is generated the spherical sector $ACB$, whose

FIG. 94.

centre of gravity we wish to determine. We may represent the body as containing infinitely many and infinitely thin pyramids, whose common vertex is the centre $C$, and whose base forms the spherical surface $ADB$. The centres of gravity of all these pyramids are at $\frac{3}{4}$ of the radius of the sphere from the centre $C$; they therefore form a second spherical surface $A_1D_1B_1$ of the radius $CA_1 = \frac{3}{4} CA$. But the centre of gravity $S$ of this curved surface is the centre of gravity of the spherical sectors; because the weights of the elementary pyramids are uniformly distributed over this surface, and therefore it is uniformly heavy.

If we now put the radius $CA = CD = r$ and the height $DM$ of the outer surface $= h$, we get for the inner $CD_1 = \frac{3}{4} r$, and $M_1D_1 = \frac{3}{4} h$; consequently (§ 110) $D_1S = \frac{1}{2} M_1D_1 = \frac{3}{8} h$, and the distance of the centre of gravity of the sector from the centre:

$$CS = CD_1 - D_1S = \tfrac{3}{4} r - \tfrac{3}{8} h = \tfrac{3}{4}\left(r - \frac{h}{2}\right).$$

For the semicircle, for example, $h = r$, therefore the distance of its centre of gravity $S$ from the centre $C$ is:

$$CS = \frac{3}{4} \cdot \frac{r}{2} = \frac{3}{8} r.$$

§ 117. The centre of gravity $S$ of the segment of a sphere $ABD$, Fig. 95, is obtained when its moment is put equal to the difference of the moments of the sector $ADBC$ and that of the cone $ABC$. Again, if we put the radius of the cone $CD = r$ and the height $DM = h$, the moment of the sector $= \frac{2}{3} \pi r^2 h . \frac{3}{8} (2r - h) = \frac{1}{4} \pi r^2 h (2r - h)$ and that of the cone $= \frac{1}{3} \pi h (2r - h) . (r - h) . \frac{3}{4} (r - h) = \frac{1}{4} \pi h (2r - h) (r - h)^2$; the moment of the segment of the sphere is therefore $= \frac{1}{4} \pi h (2r - h) (r^2 - [r - h]^2) = \frac{1}{4} \pi h^2 (2r - h)^2$. The volume of the segment $= \frac{1}{3} \pi h^2 (3r - h)$; hence, the distance in question is:

FIG. 95.

$$CS = \frac{\frac{1}{4} \pi h^2 (2r - h)^2}{\frac{1}{3} \pi h^2 (3r - h)} = \frac{3}{4} \cdot \frac{(2r - h)^2}{3r - h}.$$

If, again, we put $h = r$, the segment becomes a semicircle, and as above, $CS = \frac{3}{8}\, r$.

This formula holds good for the segment of a spheroid $A_1DB_1$, which is generated by the revolution of an elliptical arc $DA_1$ about its major semi-axis $CD = r$; for both segments may be divided into thin slices by planes parallel to the base $AB$, so that the ratio of any two is constant and $= \frac{\overline{MA_1}^2}{MA^2} = \frac{CE_1^2}{CE^2} = \frac{b^2}{r^2}$, if $b$ represent the semi-axis minor of the ellipse. The volume, as well as the moment of the segment of the sphere must be multiplied by $\frac{b^2}{r^2}$, in order to give the volume and moment of the segment of the spheroid, and thereby the quotient $CS = \frac{\text{moment}}{\text{volume}}$ will remain unchanged.

§ 118. To find the centre of gravity of an irregular body $ABCD$, Fig. 96, we must decompose it into thin slices, by planes equi-distant from each other, determine the ~~solid~~ contents of each slice, their moments with reference to the first parallel plane $AB$ serving for the base, and finally connect them together by Simpson's rule.

FIG. 96.


The contents of these slices are $F_0$, $F_1$, $F_2$, $F_3$, $F_4$ and is the whole height or distance of the outermost parallel plane $= h$; the volume of the body, therefore, according to Simpson's rule (approximately) is:

$$V = (F_0 + 4\,F_1 + 2\,F_2 + 4\,F_3 + F_4)\,\frac{h}{12}.$$

If we multiply in this formula each of these volumes by their distance, we obtain the moment:

$$Vy = (0\,.\,F_0 + 1\,.\,4\,F_1 + 2\,.\,2\,F_2 + 3\,.\,4\,F_3 + 4\,F_4)\,\frac{h}{4}\cdot\frac{h}{12};$$

lastly, by dividing one expression by the other, we get the distance required:

$$MS = y = \frac{(0\,.\,F_0 + 1\,.\,4\,F_1 + 2\,.\,2\,F_2 + 3\,.\,4\,F_3 + 4\,F_4)}{F_0 + 4\,F_1 + 2\,F_2 + 4\,F_3 + F_4}\,\frac{h}{4}.$$

If the number of elementary slices $= 6$, we have:

$$y = \frac{0.F_0 + 1.4F_1 + 2.2F_2 + 3.4F_3 + 4.2F_4 + 5.4F_5 + 6.F_6}{F_0 + 4\,F_1 + 2\,F_2 + 4\,F_3 + 2\,F_4 + 4\,F_5 + F_6}\cdot\frac{h}{6}.$$

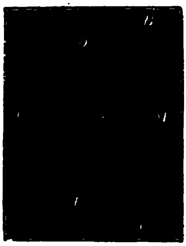
It is easy to understand how this formula may be altered when the number of slices is different from the above. This rule requires only that the number of the slices should be even, and, therefore, that of the surfaces uneven.

In most cases of application, the determination of one distance is enough, because besides this a line of gravity is known. The bodies commonly met with in practice are solids of rotation, generated in a lathe whose axis of rotation is the line of gravity.

This formula, lastly, is applicable to the determination of the centre of gravity of a surface, in which case the sections $F_0$, $F_1$, $F_2$, become lines.

*Example.*—1. For the parabolic conoid $ABC$, Fig. 97, which is generated by the revolution of a parabola $ABM$ about its axis $AM$, we obtain by making the section $DNE$, the following:

FIG. 97.

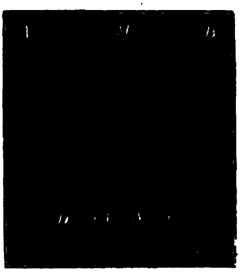


The height $AM = h$, the radius $BM = r$, $AN = NM = \frac{h}{2}$, and hence the radius $DN = r\sqrt{\frac{1}{2}}$. The area of the section through $A$ is $F_0 = 0$, of that through $N = F_1 = \pi\,\overline{DN^2} = \frac{\pi r^2}{2}$, and of that through $M = F_2 = \pi r^2$. Hence the volume of this body is:

$$V = \frac{h}{6}(0 + 4F_1 + F_2) = \frac{h}{6}(2\pi r^2 + \pi r^2) = \tfrac{1}{2}\pi r^2 h = \tfrac{1}{2} F_2 h;$$

on the other hand, the moment is $= \frac{h^2}{12}(1 . 2\pi r^2 + 2 . \pi r^2) = \frac{1}{3}\pi r^2 h^2 = \frac{1}{3} F_2 h^2$; lastly, the distance of the centre of gravity $S$ from the vertex, is:

FIG. 98.



$$AS = \frac{\frac{1}{3} F_2 h^2}{\frac{1}{2} F_2 h} = \tfrac{2}{3} h.$$

*Example* 2. A vessel $ABCD$, Fig. 98, has its mean half breadths, $r_0 = 1$ inch, $r_1 = 1{,}1$ inch, $r_2 = 0{,}9$ inch, $r_3 = 0{,}7$ inch, $r_4 = 0{,}4$ inch, with a height $MN = 2{,}5$ inch. The sections are $F_0 = 1 . \pi$, $F_1 = 1{,}21 . \pi$, $F_2 = 0{,}81 . \pi$, $F_3 = 0{,}49 . \pi$, $F_4 = 0{,}16 . \pi$; hence, the distance of the centre of gravity from the horizontal plane $AB$, is:

$$MS = \frac{0 . 1\pi + 1 . 4 . 1{,}21 . \pi + 2 . 2 . 0{,}81\pi + 3 . 4 . 0{,}49\pi + 4 \ 0{,}16 . \pi}{1\pi + 4 . 1{,}21\pi + 2 . 0{,}81\pi + 4 . 0{,}49\pi + 0{,}16\pi} \cdot \frac{2{,}5}{4}$$

$$= \frac{14{,}60}{9{,}58} . \frac{2{,}5}{4} = \frac{3650}{38{,}32} = 0{,}9502 \text{ inches.}$$

The capacity, therefore, is $= 9{,}58\,\pi . \frac{2{,}5}{12} = 6{,}270$ cubic inches.

§ 119. An interesting and sometimes very useful application of the laws of the centre of gravity is the *properties of Guldinus*, or the barocentric method. According to these the volume of a body of

revolution (or of a surface of revolution) is equal to the product of the generating surface (or generating line), and the space described by its centre of gravity during the generation of the body or surface of revolution. The correctness of this proposition may be made evident in the following manner.

*Guldinus' properties.*—If the plane area $ABC$, Fig. 99, revolve about an axis $X\overline{X}$, each element $F_1$ $F_2$, &c., of the same will describe an annulus; if the distances $F_1G_1$, $F_2G_2$, &c., of these elements from the axis of revolution $X\overline{X}_1$ be $=r_1$, $r_2$, &c., and the angle of revolution $AMA_1 = \alpha^0$, therefore the arc corresponding to the radius $1 = \alpha$, the circular paths of the elements will be $= r_1\alpha$, $r_2\alpha$, &c. The spaces described by the elements $F_1$, $F_2$, &c., may be considered as curved prisms having the bases $F_1$, $F_2$, &c., and the heights $r_1\alpha$, $r_2\alpha$, &c., and the volumes $F_1\,r_1\,\alpha$, $F_2\,r_2\,\alpha$, &c., and therefore the volume of the whole body $ABCB_1A_1C_1$ : $V = F_1r_1\alpha + F_2r_2\,\alpha \ldots = (F_1r_1 + F_2r_2 + \ldots)\,.\,\alpha$. If $MS = x$ be the distance of the centre of gravity $S$ of the generating surface from the axis of revolution, we have also $(F_1 + F_2 + \ldots)\,x = F_1r_1 + F_2r_2 + \ldots$, consequently the volume of the whole body $V = (F_1 + F_2 + \ldots)\,x\,\alpha$. But $F_1 + F_2 + \ldots$ are the contents of the whole surface $F$, and $x\,\alpha$ the circular arc $w = SS_1$, described by the centre of gravity $S$; consequently, $V = Fw$, as above enunciated. This formula holds good also for the revolution of a line, because it may be considered as a surface made up of infinitely small breadths; $F$ is namely $= Lw$: *i. e.* the surface of revolution is a product of the generating line ($L$) and the path ($w$) of its centre of gravity.

FIG. 99.

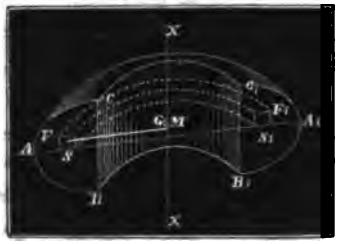
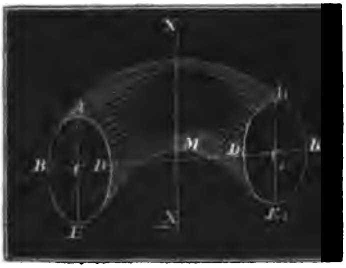


FIG. 100.



*Example.*—1. In a half-ring of an elliptical section $ABED$, Fig. 100, let the semi-axis of the section be $CA = a$ and $CB = b$, and let the distance $CM$ of the centre $C$ from the axis $X\overline{X} = r$; then the elliptical generating surface $F = \pi\,ab$, and the path of the centre of gravity ($C$) $w = \pi\,r$; hence the volume of this half-ring $V = \pi^2abr$, and that of the whole ring $= 2\pi^2 abr$. If the dimensions be, $a = 5$ inches, $b = 3$ inches, $r = 6$ inches, the volume of one-fourth of the ring $= \frac{1}{2}\,.\,\pi^2\,.\,5\,.\,3\,.\,6 = 9{,}8696\,.\,5\,.\,9 = 444{,}132$ cubic inches.

*Example.*—2. For a ring with a semicircular section $ABD$, Fig. 101, if $CA = CB = a$, represent the radius of this section, and $MC = r$ that of the hollow space or neck, the volume is

$$= V\frac{\pi a^2}{2} \cdot 2\pi\left(r + \frac{4a}{3\pi}\right) = \pi a^2\left(\pi r + \frac{4}{3}a\right).$$

*Example.*—3. To find the surface and volume of a cupola $ADB$ of the dome of a convent, Fig. 102, half the width $MA = MB = a$, and the height $MD = h$

FIG. 101.

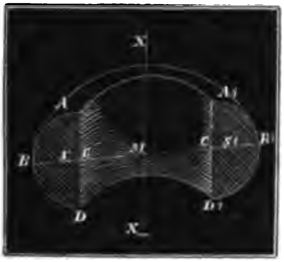


FIG. 102.

are given. From both dimensions it follows that the radius $CA = CD$ of the generating circle $= r = \frac{a^2 + h^2}{2a}$, and the angle $ACD$ subtended at the centre by $AD = \alpha^\circ$, if we put the $\sin.\alpha = \frac{h}{r}$. The centre of gravity $S$ of an arc $DAD_1 = 2\,AD$ is determined by the distance $CS = r.\ \frac{\text{chord } MD}{\text{arc } AD} = \frac{r\sin.\alpha}{\alpha}$; further, $CM = r\cos.\alpha$, consequently the distance $MS$ of the centre of gravity $S$ from the axis $MD = \frac{r\sin.\alpha}{\alpha} - r\cos.\alpha = r\left(\frac{\sin.\alpha}{\alpha} - \cos.\alpha\right)$, and the path described by the centre of gravity in the generation of the surface $ADB = 2\pi r.\left(\frac{\sin.\alpha}{\alpha} - \cos.\alpha\right)$. The generating line $DAD_1 = 2r\alpha$, and since it only is required to determine the half $ADB$, this line $= r\alpha$, and consequently we must put the whole surface $O = r\alpha . 2\pi r\left(\frac{\sin.\alpha}{\alpha} - \cos.\alpha\right) = 2\pi r^2 (\sin.\alpha - \alpha\cos.\alpha)$.

Very commonly $\alpha^\circ = 60^\circ$; therefore, $\alpha = \frac{\pi}{3}$, $\sin.\alpha = \frac{1}{2}\sqrt{3}$, and the $\cos.\alpha = \frac{1}{2}$; hence it follows that $O = \pi r^2\left(\sqrt{3} - \frac{\pi}{3}\right) = 2{,}1515 . r^2$.

For the segment $DAD_1 = A = r^2(\alpha - \frac{1}{2}\sin.2\alpha)$ the distance of the centre of gravity from the centre $C$ is $= \frac{(2 . MD)^3}{12A} = \frac{2}{3} \cdot \frac{r^3 \sin.\alpha^3}{A}$, hence the distance from the axis $MS = CS - CM = \frac{2}{3}\frac{r^3\sin.\alpha^3}{A} - r\cos.\alpha$; finally, the path of this centre of gravity described in one revolution is:

$$w = \frac{2\pi r}{A}\left(\tfrac{2}{3} r^2 \sin.\alpha^3 - A\cos.\alpha\right) = \frac{2\pi r^3}{A}\left(\tfrac{2}{3}\sin\alpha^3 - [\alpha - \tfrac{1}{2}\sin.2\alpha]\cos.\alpha\right).$$

The volume of the whole body generated by the segment $DAD_1$ is given if this path be multiplied by $A$, and the volume of the dome found by taking the half of this;

therefore, $V = \pi r^3 \left(\frac{2}{3} \sin. a^3 - [a - \frac{1}{2} \sin. 2a] \cos. a\right)$. For example, $a^\circ = 60^\circ$, $= a \frac{\pi}{3}$ $\sin. a = \frac{1}{2} \sqrt{3}$, and $\cos. a = \frac{1}{2}$; hence:

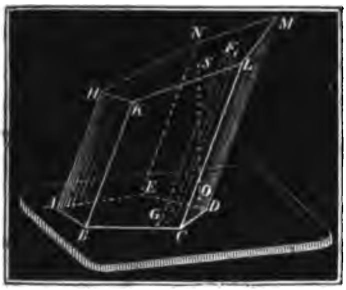
$$V = \pi r^3 \left(\frac{3}{8}\sqrt{3} - \frac{\pi}{6}\right) = 0{,}3956 \, . \, r^3.$$

*Remark.* Guldinus' properties finds its application in those bodies which arise when the generating surface so moves that in every position it remains perpendicular to the path of its centre of gravity, because we may assume every small part of a curvilinear motion to be circular. From this we may find the solid contents of the threads of screws, and sometimes also calculate the masses of earth, heaped up or removed, as in the case of canals, roads, railroads, &c.

§ 120. Another application of the doctrine of the centre of gravity, nearly allied to the last rule, is the following.

We may assume that every ~~oblique~~ prismatic body $ABCHKL$, Fig. 103, consists of an infinite number of thin prisms, similar to $F_1 G_1$. If $G_1$, $G_2$ are the bases, and $h_1$, $h_2$ the heights of these elementary prisms, we have for their solid contents $G_1 h_1$, $G_2 h_2$, &c., and the volume of the whole oblique prism $V = G_1 h_1 + G_2 h_2 + \ldots$ Now an element $F_1$ of the oblique section $HKL$ is to the element $G_1$ of the base $ABC$ as the whole oblique surface $F$ to the base $G$; therefore, $G_1 = \frac{G}{F} F_1$, $G_2 = \frac{G}{F}$, $F_2$, &c. and $V = \frac{G}{F}(F_1 h_1 + F_2 h_2 + \ldots)$.

FIG. 103.




And because $F_1 h_1 + F_2 h_2 + \ldots$ is the statical moment $Fh$ of the whole oblique section, it follows that:

$$V = \frac{G}{F} \cdot Fh = Gh, \text{ i. e.,}$$

the volume of an oblique prism is equal to the volume of a perfect prism, which stands upon the same base, and whose height is equal to the distance $SO$ of the centre of gravity $S$ of the oblique surface from the base.

In a right or oblique triangular prism, if $h_1$, $h_2$, $h_3$, be the edges of the sides, the distance of the centre of gravity of the oblique surface from the base $h = \frac{h_1 + h_2 + h_3}{3}$, hence the volume $V = G \frac{(h_1 + h_2 + h_3)}{3}$.

# CHAPTER III.

## EQUILIBRIUM OF BODIES RIGIDLY CONNECTED AND SUPPORTED.

§ 121. *Kinds of support.*—The rules developed in the first chapter of this section, on the equilibrium of a rigid system of forces, are applicable to that of rigid bodies acted upon by forces, if we consider the weight of the body as a force applied to its centre of gravity, and acting vertically downwards.

Bodies balanced by forces, are either *freely moveable, i. e.* yield to the action of forces, or they are *fixed by one or more points*, or *supported* by other bodies.

If a point of a rigid body is fixed, any other point may take up a motion whose path lies in the surface of a sphere, described from the fixed point as a centre by the distance of the other point as radius. If, on the other hand, two points of a body are fixed in every possible motion, the paths described by the remaining points are circles, which are the intersections of two spherical surfaces described from the fixed points. These circles are parallel to each other, and perpendicular to the straight line joining the two fixed points. The points of this line remain immoveable; and the body revolves about this line, which is called the axis of revolution.

The radius of the circle in which each point moves, is found by letting fall from the point a perpendicular upon the axis of revolution. The greater this is, the greater also is the circle in which the point revolves.

If three points of a body, not falling within the same line, be fixed, the body can in no sense take up motion, because the three spherical surfaces, in which a fourth point must move, intersect each other in a point only.

§ 122. *Kinds of equilibrium.*—If a body, fixed at one point, be balanced by one force or by the resultant of several forces, the direction of this force must pass through the fixed point; for a point is fixed when every force passing through it is counteracted. If this force consist merely of the weight of the body, it is then necessary that its centre of gravity should lie in the vertical line passing through the fixed point. If the centre of gravity coincide with the fixed, or the so called point of suspension, we then have indifferent equilibrium, because the body is balanced, in whatever direction it may revolve about the fixed point. If a body *AB*, Fig. 104, be fixed or sustained at a point *C*

lying above its centre of gravity $S$, it then finds itself in a condition of stable equilibrium, because, if this body be brought into any other position, the component $N$ of the weight $G$ tends to bring it back

FIG. 104.

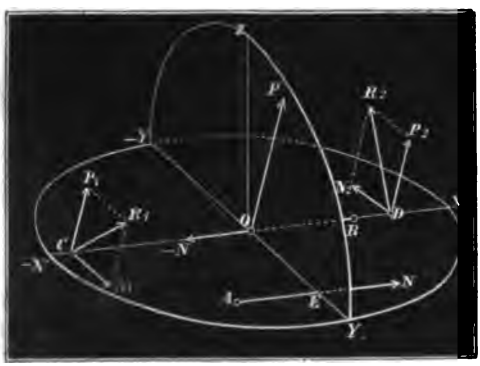
FIG. 105.

into its first position, whilst the fixed point $C$ counteracts the other component $P$. On the other hand, if a body $AB$, Fig. 105, be fixed at a point $C$ below its centre of gravity $S$, the body is then in a state of unstable equilibrium; for if the centre of gravity be drawn out of the vertical passing through $C$, the component $N$ of the weight of the body $G$ not only does not bring it back into its former position, but draws it more and more out of that position, until the centre of gravity at last comes below the fixed point.

The same reasoning will also apply to the case of a body fixed by two points, or by an axis; it will be either in indifferent, stable, or unstable equilibrium, according as the centre of gravity lies vertically above or vertically below the axis.

§ 123. *Pressure on the axis.*—If a body acted upon by forces in space be fixed by two points or by a line, relations then take place which we will investigate in the following. We may reduce, according to § 92, every system of forces to two, viz., one running parallel to the fixed axis, and the other acting in the plane normal to this line. Let $AN = N$, Fig. 106, be the first, parallel to the axis $X\bar{X}$, passing through the

FIG. 106.



fixed points $C$ and $D$ and $OP=P$, the second force acting in the plane $Y\bar{Z}Y$ at right angles to the axis $X\bar{X}$. If we introduce other forces, as $B\bar{N} = -N$, $CN_1 = N_1$, and $DN_2 = -N_1$, we change nothing in the condition of equilibrium or of motion, because these forces are entirely taken up by the axis. Now the forces $N$ and $-N$ form together a first couple, and the forces $N_1$ and $-N_1$, acting in the plane $XY$ and perpendicular to $XX$, a second couple, we may, therefore, so manage, that these shall perfectly replace each other. If $EO$ is the normal distance between the force $N$ and the axis $XX = y$, and $CD$ that of the fixed point $= x$; from § 90, we have the moments of both couples $= Ny$ and $N_1x$, and these are equivalent to each other, if $Ny = N_1x$. We may also assume inversely that the force $N$ is entirely taken up by the axis $XX$, whilst the axis has to sustain in its proper direction the pressure $N$, and the forces $N_1 = \frac{y}{x} N$ and $-N_1 = -\frac{y}{x} N$ applied perpendicularly to it at the points $C$ and $D$.

That the body may be in a state of equilibrium, it is necessary that the direction also of the resultant acting in the normal plane $YZ$ (at $O$) pass through the axis. This force $P$ may be replaced by two parallel forces $P_1$ and $P_2$ applied at the points $C$ and $D$, which may be determined, if we put $P_1 . CD = P . DO$ and $P_2 . CD = P . CO$; the axis $X\bar{X}$ will have, therefore, besides the forces $BN = -N$, $CN_1 = N_1$ and $DN_2 = -N_1$, also to react against the forces $P_1 = \frac{x_2}{x} . P$ and $P_2 = \frac{x_1}{x} . P$, which may be calculated from the distances $CD = x$, $OC = x_1$, and $OD = x_2$.

§ 124. From the results of the investigations of the former paragraph we may easily calculate the forces taken up by the axis and the fixed points $C$ and $D$. First, the axis has a pressure to sustain equivalent to the force $N$ in its own direction, which may be entirely taken up by one or other of the two fixed points. Secondly, from the forces $N_1 = \frac{y}{x} N$, $P_1 = \frac{x_2}{x} P$ and $-N_1 = -\frac{y}{x} N$ and $P_2 = \frac{x_1}{x} P$, acting in planes normal to $X\bar{X}$, and applied at the points $C$ and $D$, there arise the resultants $R_1$ and $R_2$, which must be also taken up by the fixed points $C$ and $D$.

If we put the angle $POY$, which the direction of the force $P$

makes with the plane $XY$ containing the axis $X\overline{X}$ and the direction of the force $N = a$, the angle $N_1CP_1$ is also $= a$; on the other hand, $N_2DP_2 = 180^0 - a$, and the resultant pressures are therefore given by:

$$R_1 = \sqrt{N_1^2 + P_1^2 + 2\, N_1P_1 \cos. a} \text{ and}$$
$$R_2 = \sqrt{N_1^2 + P_2^2 - 2\, N_1P_2 \cos. a}.$$

*Example.* A set of forces of a body fixed by its axis $X\overline{X}$, is resolved into a normal force $P = 36$ lb. and a parallel force $N = 20$ lb.; the distance of the last from the axis is $y = 1\frac{1}{2}$ feet, and the distance $CD = x = 4$ feet. To find the forces sustained by the axis, or by the fixed points in it, with the condition that the direction of $P$ deviate by an angle $a = 65°$ from the plane $XY$, and its point of application $O$ be distant by $CO = x_1 = 1$ foot from the fixed point $C$? The force $N = 20$ lb. imparts to the axis along its direction a thrust $N = 20$ lb.; besides, it generates also the forces $N_1 = \frac{y}{x} N = \frac{1,5}{4} \cdot 20 = 7,5$ lb. and $- N_1 = -$ 7,5 lb., against which the fixed points $C$ and $D$ react. From the force $P$ arise the forces $P_1 = \frac{x_2}{x} P = \frac{4-1}{4} \cdot 36 = 27$ lb. and $P_2 = \frac{x_1}{x} P = \frac{1}{4} \cdot 36 = 9$ lb. and by substitution of these values we have the resultant forces:

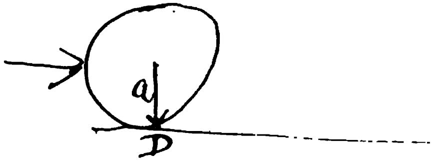
$$R_1 = \sqrt{7,5^2 + 27^2 + 2 \cdot 7,5 \cdot 27 \cdot \cos. 65°} = \sqrt{56,25 + 729 + 171,160}$$
$$= \sqrt{956,410} = 30,926 \text{ lb., and}$$
$$R_2 = \sqrt{7,5^2 + 9^2 - 2 \cdot 7,5 \cdot 9 \cdot \cos. 65°} = \sqrt{56,25 + 81 - 57,054}$$
$$= \sqrt{80,196} = 8,955 \text{ lb.}$$

§ 125. *Equilibrium of forces about an axis.*—The force $P$ is the resultant of all those component forces whose directions lie in one or more planes normal to the axis. But now in these cases, from § 86, the statical moment $Pa$ of the resultant is equivalent to the sum $P_1a_1 + P_2a_2 + \ldots$ of the statical moments of the components, and for the condition of equilibrium of the fixed body the arm $a$ of the resultant $= 0$, because this passes through the axis; hence the sum is also:

$$P_1a_1 + P_2a_2 + \ldots . = 0;$$

*i. e.* a body fixed by its axis is in a state of equilibrium, and remains also without revolving, if the sum of the moments about this axis $= 0$, or if the sum of the moments of the forces acting in one direction of revolution is equivalent to the sum of the moments of those acting in the opposite direction. By the help of this last formula we may find either a *force* or an *arm* for an element of a system of forces in equilibrium.

*Example.* The forces of rotation $P_1 = 50$ lb., and $P_2 = -35$ lb., act upon a body




I Friction causes the rotation

$\varphi$ = coefficient of friction

c = initial velocity

$\boxed{G}$ $\varphi G$ = Force of Friction

$\frac{Mk^2}{a^2}$ = Mass referred to circumference at D

$p = \frac{\text{Force}}{\text{Mass}}$ = etc in the book

II Retardation $= q = \frac{\text{Res}}{\text{Mass}} = \frac{\varphi G}{M}$

$M = \frac{G}{g}$

v = velocity of rotation at circumf.

$v_1$ = " translation

by subs for t in $v_1$ =

$c - \frac{ck^2}{a^2+k^2} = c_1 = \frac{ca^2}{a^2+k^2}$ b

p 111 $s = \frac{c+c_1}{2} t$

$\frac{k^2}{a^2}$ = cyl $= \frac{1}{2}$ ∴ $\frac{k^2+a^2}{a^2} = \frac{2+1}{2}$

$= \frac{3}{2}$ ∴ $\frac{a^2}{k^2+a^2} = \frac{2}{3}$

$$c - \cancel{\rho g}\left(\frac{K^2}{a^2+K^2}\,\frac{c}{\cancel{\rho g}}\right)$$

$$- \quad \frac{a^2c + K^2c - K^2c}{a^2+K^2}$$

$\therefore\ \beta\alpha -$

capable of turning about an axis at the arms $a_1 = 1\frac{1}{4}$ foot, and $a_2 = 2\frac{1}{2}$ feet; required, the force $P_3$ which shall act at the arm $a_3 = 4$ feet, in order to restore the balance, *i. e.* to prevent rotation about the axis? It is:

$$50 \,.\, 1{,}25 - 35 \,.\, 2{,}5 + 4\,P_3 = 0, \text{ hence}$$
$$P_3 = \frac{87{,}5 - 62{,}5}{4} = 6{,}25 \text{ lb.}$$

§ 126. *The Lever.*—A body capable of turning about a fixed axis and acted upon by forces, is called a lever. If we imagine it to be devoid of weight, it is then called a *mathematical,* but otherwise, a *material* or *physical lever.*

It is generally assumed that the forces of a lever act in a plane at right angles to the axis, and that the axis is replaced by a fixed point, called the fulcrum. The perpendiculars let fall from this point on the direction of the forces are called arms. If the directions of the forces of a lever are parallel, the arms form a single straight line, and the lever is called a straight lever. If the arms make an angle with each other, it is then called a bent lever. The straight lever acted upon by two forces, is either one-armed or two-armed, according as the points of application lie on the same or on opposite sides of the fulcrum. There is a distinction made of levers of the first, second, and third kind; the two-armed lever is termed a lever of the first kind; the one-armed, of the second or third kind, according as the weight acting vertically downwards, or the power acting vertically upwards, lies nearest to the fulcrum.

FIG. 107.


§ 127. The theory of the equilibrium of the lever has been already fully laid down; we have now, therefore, only to treat of each specially.

In the two-armed lever, *ACB,* Fig. 107, if the arm *CA* of the power *P* be designated by *a,* and the arm *CB* of the weight *Q* by *b,* from the general theory: $Pa = Qb$, *i. e.* the moment of the force is equal to the moment of the weight; or also, $P : Q = b : a$, *i. e.* the power is to the weight inversely as the arms. The pressure on the fulcrum is $R = P + Q$.

In the one-armed levers *ABC,* Fig. 108, and *BAC,* Fig. 109, the same relation takes place between the power *P* and the weight *Q,* but here the direction of the power is opposite to that of the weight, and therefore the pressure on the fulcrum is their dif-

ference, and in the first case $R = Q - P$, and in the second, $R = P - Q$.

FIG. 108

FIG. 109.

Also in the bent lever $ACB$, with the arms $CN = a$ and $CO = b$, Fig. 110, $P : Q = b : a$, here the pressure on the fulcrum is equivalent to the diagonal $R$ of the parallelogram $CP_1RQ_1$, which may be constructed from the power $P$, the weight $Q$ and the angle $P_1CQ_1 = PDQ = a$, which their directions make with each other.

FIG. 110.

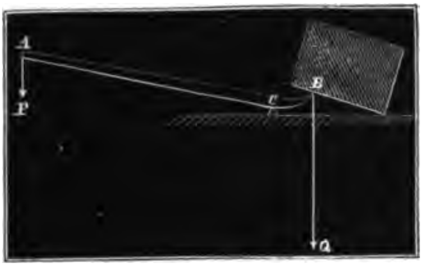
Let $G$ be the weight of the lever, and $CE = e$, Fig. 111, the distance of the fulcrum $C$ from the vertical line $SG$, passing through its centre of gravity; we shall then have to put $Pa \pm Ge = Qb$, and the plus or minus sign before $G$, according as the centre of gravity lies on the side of the power $P$, or on that of the weight $Q$.

FIG. 111.

*Remark.* The theory of the lever finds its application in many tools and machines, viz. in the different kinds of ba-

lances, crow-bars, the brakes of pumps, wheelbarrows, &c. The second part will treat fully of these.

FIG. 112.




*Example.*—1. If we press down the end $A$ of a crow-bar $ACB$, Fig. 112, with a force $P=60$ lb., and with the arm $CA$ of the power equal to 12 times that of the arm $CB$ of the weight, then will this, or rather the force exerted at $B$, be $= Q$ 12 times that of $P = 12 . 60 = 720$ lb.—2. If a load $Q$, hanging from a pole, Fig. 113, be carried by two men, which pole the one lays hold of at $A$ and the other at $B$, we may readily find out what weight each has to sustain. Let the load $Q = 120$ lb., the weight of the pole $G = 12$ lb., the distance $AB$ of both points of application $= 6$ ft. the distance $BC$ of the load from one of these points $= 2\frac{1}{2}$ feet, the distance of the centre of gravity $S$ of the pole from this same point $BS = 3\frac{1}{2}$ feet. If we take $B$ for the fulcrum, the power $P_1$ has to balance at $A$ the weights $Q$ and $G$, therefore $P_1 . BA = Q . BC + G . BS$, i. e. $6\, P_1 = 2{,}5\, .\, 120 + 3{,}5\, .\, 12 = 300 + 42 = 342$; hence, $P_1 = \frac{342}{6} = 57$ lb. On the other hand, if $A$ be considered as the fulcrum, we must put $P_2 . AB = Q . AC + G . AS$, and in numbers, $6\, P_2 = 3{,}5\, .\, 120 + 2{,}5\, .\, 12 = 420 + 30 = 450$; hence, the power $P_2$ of the second man is $P_2 = \frac{450}{6} = 75$ lb.; also, the sum of the forces $P_1 + P_2$ acting upwards, $= 57 + 75 = 132$ lb. is exactly equal to the sum of the forces acting downwards, $Q + G = 120 + 12 = 132$ lb.—3. In a heavy bent lever, $ACB$, Fig. 114, the vertically pulling force $Q = 650$ lb. and the arm $CB = 4$ ft., but the arm $CA$ of the power $P = 6$ ft., and that of the weight $CE = 1$ foot, what is the amount of the power $P$, and the pressure on the pivot $R$ required to restore the balance? $CA . P = CB . Q + CE . G$, i. e., $6\, P = 4\, .\, 650 + 1\, .\, 150 = 2750$; consequently, the power $P = \frac{2750}{6} = 458\frac{1}{3}$ lb.; the pressure on the pivot consists of the vertical force $Q + G = 650 + 150 = 800$ lb., and the horizontal power $P = 458\frac{1}{3}$ lb., and is therefore:

FIG. 113.

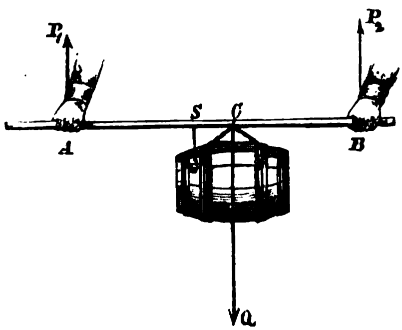
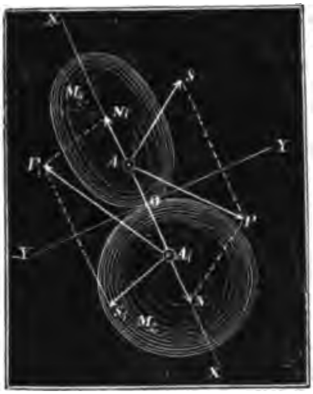



FIG. 114.

$$= R = \sqrt{(Q+G)^2+P^2}$$
$$= \sqrt{(800)^2+(458\tfrac{1}{2})^2}$$
$$= \sqrt{850070} = 922 \text{ lb.}$$

§ 128. *Pressure of bodies on one another.*—The experimental law announced in § 62, that *action and reaction are equal to each other,* is the basis of the whole mechanics of machines. It is necessary in this place to make the meaning of this still clearer. When two bodies $M_1$ and $M_2$, Fig. 115, act upon each other with the forces $P$ and $P_1$, whose directions deviate from the normal ~~common~~ $XX$ common to the two surfaces at their point of contact, a decomposition of the forces is always possible, the one component $N$ or $N_1$ which is in the direction of the normal, passes over from the one body to the other, the other component $S$ or $S_1$ remains in the body, and must be counteracted by another force or resistance, in order to maintain the bodies in equilibrium. From the principle set forth, perfect equilibrium is found to subsist between the normal components $N$ and $N_1$.

FIG. 115.



If the direction of the force $P$ deviates by the angle $NAP = a$ trom the normal $AX$ and by the angle $SAP = \beta$ from the direction of the second component $S$, we have (§ 75)

$$N = \frac{P \sin. \beta}{\sin. (a+\beta)}, \quad S = \frac{P \sin. a}{\sin. (a+\beta)}.$$

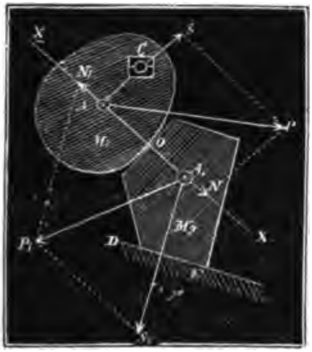
If we represent $N_1A_1P_1$ by $a_1$ and $S_1A_1P_1$ by $\beta_1$, we also have

$$N_1 = \frac{P_1 \sin. \beta_1}{\sin. (a_1+\beta_1)} \text{ and } S_1 = \frac{P_1 \sin. a_1}{\sin. (a_1+\beta_1)};$$

lastly, from the equality $N = N_1$,

$$\frac{P \sin. \beta}{\sin. (a+\beta)} = \frac{P_1 \sin. \beta_1}{\sin. (a_1+\beta_1)}.$$

FIG. 116.



*Example.* What resolution of the forces takes place if a body $M_2$, Fig. 116, sustained by a support $DE$, be pressed upon by another, capable of revolving about an axis $C$ with a force $P = 250$ lb., the angles of direction being the following:

$$PAN = \alpha = 35^\circ,$$
$$PAS = \beta = 48^\circ,$$
$$P_1A_1N_1 = \alpha_1 = 65^\circ,$$
$$P_1A_1S_1 = \beta_1 = 50^\circ.$$

From the first formula the normal pressure between the two bodies is determined by

$$N = N_1 = \frac{P\ sin.\ \beta}{sin.\ (\alpha+\beta)} = \frac{250\ sin.\ 48^\circ}{sin.\ 83^\circ} = 187{,}18 \text{ lb.};$$

from the second the pressure on the axis, or on the point $C$ is

$$S = \frac{P\ sin.\ \alpha}{sin.\ (\alpha+\beta)} = \frac{250\ sin.\ 35^\circ}{sin.\ 83^\circ} = 144{,}47 \text{ lb.; and}$$

by combining the third and fourth equation, there follows finally for the component opposed to $DE$:

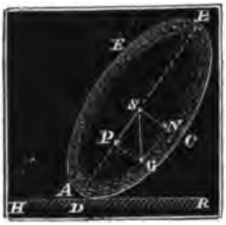
$$S_1 = \frac{N_1\ sin.\ \alpha_1}{sin.\ \beta_1} = \frac{187{,}18\ sin.\ 65^\circ}{sin.\ 50^\circ} = 221{,}46 \text{ lb.}$$

§ 129. *Stability.*—When a body pressing against a horizontal plane is acted upon by no other force than gravity, it has no tendency to move forward, because the weight acting vertically downwards is exactly sustained by this plane; nevertheless a revolution of the body is possible. If the body $ADBF$, Fig. 117, rests at a point $D$ upon the horizontal plane $HR$, it will remain at rest, if its centre of gravity $S$ be supported, *i. e.* if it lie in the vertical line passing through $D$. If a body is supported at two points on the horizontal surface of another, it is requisite for its equilibrium that the vertical line of gravity should intersect the line connecting the two points. Lastly, if a body rests at three or more points on a horizontal plane, equilibrium subsists if the vertical line containing the centre of gravity passes through the triangle or polygon which is formed by the straight lines connecting the points of support.

FIG. 117.

In bodies which are supported we must distinguish between stable and unstable equilibrium. The weight $G$ of a body $AB$, Fig. 118,

FIG. 118.




draws its centre of gravity downwards; if no resistance be opposed to this force it will cause the body to turn until its centre of gravity has attained its lowest position, and equilibrium will then be restored. We may mention that the equilibrium is stable when the centre of gravity is in its lowest possible position, Fig. 119, and unstable when in its highest, Fig. 120, and indifferent, when the centre of gravity in

FIG. 119.

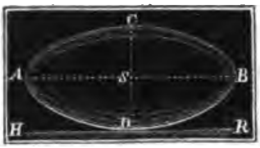



FIG. 120.


FIG. 121.


every position of the body remains at the same height Fig. 121.

*Example.*—1. The homogeneous body *ADBF*, consisting of a hemisphere and a cylinder, Fig. 117, rests upon a horizontal plane *HR*. What height $SF = h$ must its cylindrical part have, that the body may be in equilibrium? The radius of a sphere is perpendicular to the corresponding plane of contact; now the horizontal plane is such a one, consequently the radius *SD* must be perpendicular to the horizontal plane, and the centre of gravity of the body lie in it. The axis *SFL* of the body passing through the centre of the sphere is its second line of gravity; the point *S*, the intersection of the two lines, is therefore the centre of gravity of the body. Let us now put the radius of the sphere and cylinder $SA = SB = r$, and the height of the cylinder $SF = BE = h$, we then have for the volume of the hemisphere: $V_1 = \frac{2}{3}\pi r^3$, for the volume of the cylinder $V_2 = \pi r^2 h$, for the distance of the centre of gravity of the sphere $S_1 : SS_1 = \frac{3}{8} r$, and for that of the cylinder $S_2 : SS_2 = \frac{1}{2} h$. That the centre of gravity of the whole body may fall in *S*, the moment of the sphere $\frac{2}{3}\pi r^3 . \frac{3}{8} r$ must be put equal to the moment of the cylinder, $\pi r^2 h . \frac{1}{2} h$; from which we have:

FIG. 122.


$$h^2 = \frac{1}{2} r^2, \text{ i. e. } h = r \sqrt{\tfrac{1}{2}} = 0{,}7071 . r.$$

2. The pressure which each of the three legs, *A*, *B*, *C*, Fig. 122, of any loaded table has to sustain, is determined in the following manner. Let *S* be the centre of gravity of the table with its load, and *SE*, *CD*, perpendiculars

upon $AB$. If $G$ be the weight of the whole table, and $R$ the pressure on $C$, we may, considering $AB$ as the axis, put the moment of $R$ = to the moment of $G$, i. e., $R \,.\, CD = G \,.\, SE$, and we then obtain $R = \frac{SE}{CD} \,.\, G = \frac{\Delta ABS}{\Delta ABC} \,.\, G$; likewise also the pressure on $B = Q = \frac{\Delta ACS}{\Delta ABC} \,.\, G$, and that on $A = P = \frac{\Delta BCS}{\Delta ABC} \,.\, G$.

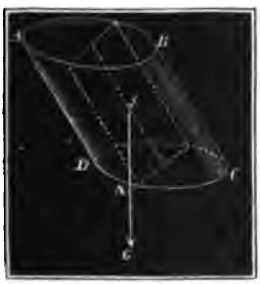
§ 130. Let us now take the case of a body having a plane base resting on a horizontal plane. Such a body possesses stability, or is in stable equilibrium when its centre of gravity is supported, *i. e.* when the vertical line containing the centre of gravity of the body passes through its base, because in this case, the tendency of the weight of the body to cause it to turn is prevented by its own rigidity. When the line of gravity passes through the edge of the base, the body is then in unstable equilibrium, and when the line passes outside the base, no equilibrium subsists. The body falls to one side and overturns. The triangular prism $ABCDE$, Fig. 123, is, according to the above, stable, because the vertical $SG$ passes through a point $N$ of the base. The parallepiped $ABCG$, Fig. 124, is in unstable equilibrium, because

FIG. 123.

FIG. 124.

$SG$ intersects a side $CD$ of the base. The cylinder $ABCD$, Fig. 125, is without stability because $SG$ no where intersects the base $CD$.

FIG. 125.

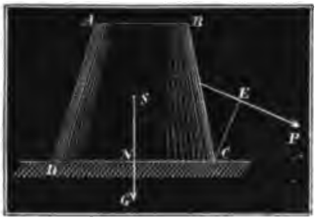


Stability is the power of a body to preserve its position by its weight alone, and to oppose resistance to any cause tending to overturn it. If we have to choose a measure of the stability of a body, we must distinguish whether this has reference to a displacement or to an actua overturning of the body

Let us now take into consideration the first only of these circumstances.

§ 131. *Formulæ of stability.*—A force, $P$, not directed vertically, tends not only to overturn a body $ABCD$, Fig. 126, but also to push it forward; let us assume in the mean time that a resistance is opposed to the pushing or pulling forwards as it may happen, and have regard only to its revolving about one of its edges $C$. If we let fall from this edge $C$ a perpendicular $CE = a$ upon the direction of the force and $CN = x$ upon the vertical line $SG$ passing through the centre of gravity, we have only to consider a bent lever $ECN$, for which $Pa = Gx$, so that $P = \frac{x}{a} G$; if the external force $P$ be greater than $\frac{x\,G}{a}$, the body revolves about the point $C$, and, therefore, loses its stability. Hence the stability depends upon the product $(Gx)$ of the weight of the body, and the shortest distance between a side of the perimeter of the base and the vertical line passing through the centre of gravity; $Gx$ may, therefore, be regarded as a measure of the stability, and for this reason is properly called the stability itself.

FIG. 126.



Hence we see that the stability increases simultaneously with the weight $G$ and the distance $x$, and may conclude that under otherwise similar circumstances a body twice or thrice as heavy does not possess more stability than one of the single weight with twice or thrice the distance or arm $x$, &c.

§ 132.—1. In a parallelepiped $ABCF$, Fig. 127, of the length $AE = l$, breadth $AB = CD = b$, and height $AD = BC = h$, the weight $G = V\gamma = bhl\gamma$, and the stability $S = G \,.\, KN = G.\frac{1}{2}CD = \frac{Gb}{2} = \frac{1}{2}\, b^2hl\gamma$, provided $\gamma$ represent the density of the mass of the parallelepiped.

2. In a body $ACFH$ consisting of two parallelepipeds, Fig. 128, the stabilities about the two edges of the base $C$ and $E$ are different from one another. Let us take the heights $BC$ and $EF = h$ and $h_1$, and the breadths $CD$ and $DE = b$ and $b_1$, the weights of the parts $G$ and $G_1 = bhl\gamma$ and $b_1h_1l\gamma$; then the arms about $C$ will be $KN_1 = \frac{1}{2}\, b$ and $KN_2 = b + \frac{1}{2}\, b_1$, and those about

FIG. 127.

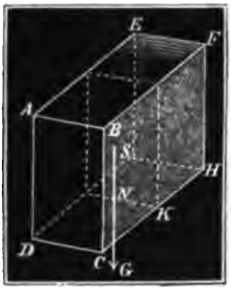



FIG. 128.

$E = b_1 + \frac{1}{2} b$ and $\frac{1}{2} b_1$. The stabilities accordingly are: first for the revolution about $C$,

$$S = \tfrac{1}{2} Gb + G_1 (b + \tfrac{1}{2} b_1), = (\tfrac{1}{2} b^2 h + b\, b_1 h_1 + \tfrac{1}{2} b_1^2 h_1)\, l\gamma,$$

secondly for that about $E$,

$$S_1 = G (b_1 + \tfrac{1}{2} b) + \tfrac{1}{2} G_1 b_1 = (\tfrac{1}{2} b_1^2 h_1 + b b_1 h + \tfrac{1}{2} b^2 h)\, l\gamma.$$

The latter stability is about $S_1 - S = (h - h_1)\, b b_1 l\gamma$ greater than the former; if we wish to increase the stability of a wall $AC$ by offsets, these must be placed on that side of the wall towards which the force of revolution (wind, water, pressure of earth, &c.) acts.

3. The following is the stability of a wall $ABCEF$, Fig. 129, battering on one side. The upper breadth $AB = b$, the height $BC = h$ and the length $CH = l$, and the batter $= n$, *i. e.* upon $AI$ = a height of 1 foot; $IL = n$ feet or inches of batter, therefore, upon $h$ feet $ED = nh$. The weight of the parallelepiped $ACF$ is $G_1 = bhl\gamma$, that of the three sided prism $ADE = G_2 = \frac{1}{2} nh \,.\, hl\gamma$, the arms for a revolution about $E$ are $= DE + \frac{1}{2} b = nh + \frac{1}{2} b$ and $\frac{2}{3} DE = \frac{2}{3} nh$, consequently for the stability we have

FIG. 129.

$$S = G_1 (nh + \tfrac{1}{2} b) + \tfrac{2}{3} G_2 nh$$
$$= (\tfrac{1}{2} b^2 + nhb + \tfrac{1}{3} n^2 h^2)\, hl\gamma.$$

A parallelepipedical wall of equal volume has the breadth $b + \frac{1}{2} nh$, hence the stability is:

$$S_1 = \tfrac{1}{2} (b + \tfrac{1}{2} nh)^2\, hl\gamma = (\tfrac{1}{2} b^2 + \tfrac{1}{2} nhb + \tfrac{1}{8} n^2 h^2)\, hl\gamma;$$

its stability is, therefore, about $S - S_1 = (b + \frac{5}{12} nh) \,.\, \frac{1}{2} nh^2 l\gamma$, less than that of the battered wall.

For a wall sloped upon the opposite side, the stability is $S_2 = (b^2 + nhb + \frac{1}{3} n^2h^2) \cdot \frac{1}{2} hl\gamma$, less also than $S$, and indeed about $S - S_2 = (b + \frac{1}{3} nh) \cdot \frac{1}{2} nh^2 l\gamma$, as well as about $S_2 - S_1 = \frac{1}{24} n^2h^3 l\gamma$ less than the stability of the parallelepipedical wall.

*Example.* What is the stability for each foot in length of a stone wall of 10 feet in height, and $1\frac{1}{4}$ feet of upper breadth with batter of 1 in 5 on the back? The specific gravity of this wall (§ 58) is taken at 2,4, its density $\gamma$ is, therefore, $= 62{,}5 \cdot 2{,}4 = 130$ lbs.; now $l = 1$, $h = 10$, $b = 1{,}25$, and $n = \frac{1}{5} = 0{,}2$; hence it follows, that the stability sought is:

$$S = (\tfrac{1}{2} \cdot [1{,}25]^2 + \ \cdot 0{,}2 \cdot 1{,}25 \cdot 10 + \tfrac{1}{3} \cdot [0{,}2]^2 \cdot 10^2)\ 10 \cdot 1 \cdot 130$$
$$= (0{,}78125 + 2{,}5 + 1{,}3333)\ 130 = 4{,}6146 \cdot 130 = 603{,}4 \text{ ft. lbs.}$$

With the same quantity of material, and under otherwise similar circumstances, the stability of a parallelepipedical wall would be:

$$S_1 = (\tfrac{1}{2} \cdot [1{,}25]^2 + \tfrac{1}{2} \cdot 0{,}2 \cdot 1{,}25 \cdot 10 + \tfrac{1}{8} \cdot 0{,}2^2 \cdot 10^2) \cdot 130$$
$$= (0{,}78125 + 1{,}25 + 0{,}5) \cdot 130 = 2{,}531 \cdot 130 = 329 \text{ ft. lbs.}$$

The same wall, with a sloping front, would have the stability:

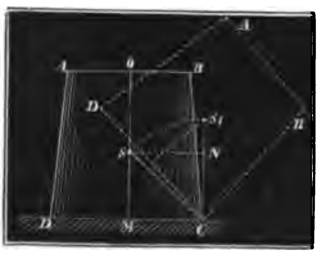
$$S_2 = (\tfrac{1}{2} \cdot [1{,}25]^2 + \tfrac{1}{2} \cdot 0{,}2 \cdot 1{,}25 \cdot 10 + \tfrac{1}{6} \cdot [0{,}2]^2 \cdot 10^2) \cdot 130$$
$$= (0{,}78125 + 1{,}25 + 0{,}666 \ldots) \cdot 130 = 2{,}6979 \cdot 130 = 350{,}7 \text{ ft. lbs.}$$

*Remark.*—It is evident from the foregoing that it allows of a saving of material to batter walls, to construct them with counterforts, to give them offsets, or to place them upon plinths, &c. The Second Part will give a further extension of this subject, when we come to treat of the pressure of earth, and of vaults, chain bridges, &c.

§ 133. *Dynamical stability.*—We may distinguish from the measure of stability treated of in the last paragraph, still another to a certain degree dynamical measure of stability, when we consider the effect which is to be expended in order to overturn a body. Now the mechanical effect of a force is equal to the product of the force and the space, but the force of a heavy body is its weight $G$, and the space equal to the vertical projection of that described by its centre of gravity, we may consequently take for the dynamical measure of the stability of a body the product $Gs$, if $s$ be the height to which the centre of gravity of the body must ascend in order to bring the body from its stable condition into an unstable one.

Let $C$ be the axis of revolution and $S$ the centre of gravity of a body $ABCD$, Fig. 130, whose dynamical stability we wish to find.

FIG. 130.



If we cause the body to revolve so that its centre of gravity comes to $S_1$, *i. e.* vertically over $C$, the body will be in unstable equilibrium, for if it only revolve a little further it will fall over. If we draw the horizontal line $SN$, this will cut off the height $Ns_1 = s$ to which the centre of gravity has ascended, from which the

stability $Gs$ is given. If now $CS = CS_1 = z$, $CM = SN = x$, and the height $CN = MS = y$, it follows that the space $S_1N = s = z - y = \sqrt{x^2 + y^2} - y$, and the stability in the last sense is

$$S = G\,(\sqrt{x^2 + y^2} - y).$$

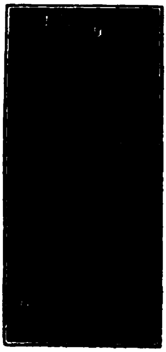
If the body is a prism with a symmetrical trapezoidal transverse section as Fig. 130 represents, and if the dimensions are the following: length $= l$, heigth $MO = h$, lower breadth $CD = b_1$, upper breadth $AB = b_2$, we then have $MS = y = \frac{b_1 + 2\,b_2}{b_1 + b_2} \cdot \frac{h}{3}$ (§ 105) and $CM = x = \frac{1}{2} b_1$, hence

$$CS = \sqrt{\left(\frac{b_1}{2}\right)^2 + \left(\frac{b_1 + 2\,b_2}{b_1 + b_2} \cdot \frac{h}{3}\right)^2},$$

and the dynamical stability or the mechanical effect required to overturn it.

$$S = G\left[\sqrt{\left(\frac{b_1}{2}\right)^2 + \left(\frac{b_1 + 2\,b_2}{b_1 + b_2} \cdot \frac{h}{3}\right)^2} - \frac{b_1 + 2 b_2}{b_1 + b_2} \cdot \frac{h}{3}\right]$$

FIG. 131.



*Example.* What is the dynamical stability or the mechanical effect necessary for the overturning of an obelisk $ABCD$, Fig. 131, of granite, if its height $h = 30$ ft., its upper length and breadth $l_1 = 1\frac{1}{2}$, and $b_1 = 1$ ft., and lower length and breadth $l_2 = 4$ ft., $b_2 = 3\frac{1}{2}$ ft.? The volume of this body is

$$(\S\ 115)\ V = (2\,b_1\,l_1 + 2\,b_2\,l_2 + b_1\,l_2 + b_2\,l_1)\,\frac{h}{6}$$

$$= (2 \,.\, \tfrac{3}{2} \,.\, 1 + 2 \,.\, 4 \,.\, \tfrac{7}{2} + 1 \,.\, 4 + \tfrac{3}{2} \,.\, \tfrac{7}{2})\,\tfrac{30}{6}$$

$= 40{,}25 \,.\, 5 = 201{,}25$ cubic feet. Now a cubic foot of granite weighs $= 3 \,.\, 62{,}5 = 187{,}5$ lb.; the whole weight of this body is: $G = 201{,}25 \,.\, 187{,}5 = 37734{,}3$ lb. The height of the centre of gravity above the base is:

$$y = \frac{b_2 l_2 + 3\,b_1 l_1 + b_2 l_1 + b_1 l_2}{2 b_2 l_2 + 2\,b_1 l_1 + b_2 l_1 + b_1 l_2} \cdot \frac{h}{2}$$

$$= \frac{4 \,.\, \tfrac{7}{2} + 3 \,.\, \tfrac{3}{2} \,.\, 1 + 1 \,.\, 4 + \tfrac{3}{2} \,.\, \tfrac{7}{2}}{40{,}25} \cdot \frac{30}{2} = \frac{27{,}75 \,.\, 15}{40{,}25} = 10{,}342 \text{ ft.}$$

Provided it be a revolution about the longer edge of the base, the horizontal distance of the centre of gravity from this edge will be: $x = \frac{1}{2}\,b_2 = \frac{1}{2} \,.\, \frac{7}{2} = \frac{7}{4}$ ft.; hence, the distance of the centre of gravity from the axis will be:

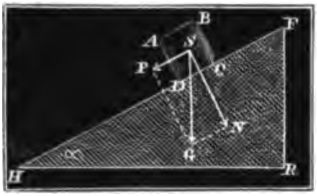
$CS = z = \sqrt{x^2 + y^2} = \sqrt{(1{,}75)^2 + (10{,}342)^2} = \sqrt{110{,}002} = 10{,}489$: and the height to which the centre of gravity must be raised to bring about an overthrow will be: $s = z - y = 10{,}489 - 10{,}342 = 0{,}147$ ft.; lastly, the corresponding mechanical effect or stability will be: $Gs = 37734.3 \,.\, 0{,}17 = 5547$ ft. lbs.

*Remark.* The factor $s = \sqrt{x^2 + y^2} - y$ gives for $y = o$, $s = x$, for $y = x$, $s = x\,(\sqrt{2} - 1) = 0{,}414\,x$, for $y = nx$, $s = (\sqrt{n^2} + 1 - n)\,x$, approximately $= (n + \frac{1}{2n} - n)\,x = \frac{x}{2n}$, also for $y = 10\,x$, $s = \frac{x}{20}$, and for $y = \infty$, $s = \frac{x}{\infty} = o$; the dynamical stability is therefore so much the greater, the lower the centre of gravity lies, and it approximates more and more to null, the higher the centre of

gravity lies above the base. Sledges, carriages, ships, &c., must on this account be so loaded, that the centre of gravity may lie as low as possible, and besides, over the middle of the base.

§ 134. *Theory of the inclined Plane.*—A body $AC$, Fig. 132, resting on an inclined plane, that is, on one inclined to the horizon, may take up two motions, it may slide down the inclined plane, and it may also revolve about one of the edges of its base and overturn. If the body is left to itself, its weight $G$ is resolved into a force $N$ normal, and to a force $P$ parallel to the base, the first is taken up by the reaction of the plane, and the last urges the body down the plane. Let the angle of inclination $FHR$ of the inclined plane to the horizon $= a$, we have therefore the angle $GSN = a$, and hence the normal pressure

FIG. 132.



$$N = G \cos. a,$$

and the force parallel to the plane:

$$P = G \sin. a.$$

If the vertical line $SG$ passes through the base $CD$ as in Fig. 132, a sliding motion only can take place, but if this line passes outside the base, as in Fig. 133, an overturn ensues, and

FIG. 133.

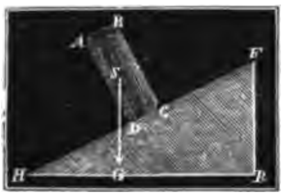


FIG. 134.

the body, therefore, is without stability. Besides, a body $AC$ resting on the inclined plane $FH$, Fig. 134, has a stability different from that of one on a horizontal plane. If $DM = x$ and $MS = y$ are the rectangular co-ordinates of the centre of gravity $S$, we have the arm of the stability $DE = DO - MN = x \cos. a - y \sin. a$, while, if the body is on a horizontal plane, it is $= x$. Since $x > x \cos. a - y \sin. a$, the stability with reference to the lower edge $D$ comes out less for the inclined than for the horizontal plane; it is null for $x \cos. a = y \sin. a$, *i. e.* for $\tang. a = \frac{x}{y}$. When a body that is stable $G\,x$ on

a horizontal plane is transferred to an inclined one, whose angle of inclination corresponds to the expression $tang.\, a = \frac{x}{y}$, it will lose its stability. On the other hand, a body may acquire on an inclined plane the stability which is wanting to it on a horizontal one. For a turning about the upper edge $C$, the arm $CE_1 = CO_1 + MN = x_1 \, cos.\, a + y \, sin.\, a$, whilst in its position on the horizontal plane it is $= x_1$. If now $x_1$ is negative, the body has no stability so long as it remains on a horizontal plane, but if it rests on an inclined one, for whose angle of inclination $tang.\, a$ is $> \frac{x_1}{y}$, the body is stable.

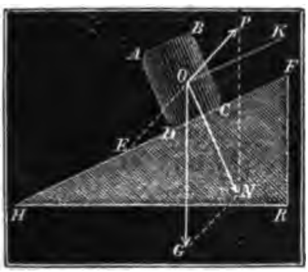
If another force besides gravity acts upon the body $ABCD$, Fig. 135, its stability continues if the direction of the resultant $N$ of the weight $G$ and the force $P$ intersects the base $CD$ of the body.

*Example.* The obelisk in the example of the preceding paragraphs has $x = \frac{7}{4}$ ft. and $y = 10,342$ ft., and will lose its stability, consequently, if transferred to an inclined plane, for whose angle of inclination:

$$tang.\, a = \frac{7}{4 \, . \, 10,342} = \frac{7000}{41368} = 0,16922, \text{ and inclination } a = 9^\circ \, 36'.$$

§ 135. As the inclined plane only counteracts that pressure which is directed perpendicularly against it, the force $P$ which is necessary to prevent a body supported upon an inclined plane from overturning, is determined by the condition that the resultant $N$ of $P$ and $G$, Fig. 135, must be at right angles to the inclined plane. From the theory of the parallelogram of forces we have $\frac{P}{G} = \frac{sin.\, ONP}{sin.\, PON}$, now the $\angle \, PNO = \angle \, GON = FHR = a$, and $\angle \, PON = POK + KON = \beta + 90^\circ$, in so far as we represent by $\beta$ the $\angle \, PEF = POK$, by which the direction of the force deviates from the inclined plane; hence we have

FIG. 135.




$$\frac{P}{G} = \frac{sin.\, a}{sin.\, (90 + \beta)}, \; i.\, e.\, \frac{P}{G} = \frac{sin.\, a}{cos.\, \beta},$$

therefore the force which maintains the body on the plane is:

$$P = \frac{G \, sin.\, a}{cos.\, \beta}.$$

For the normal pressure $N$

$\frac{N}{G} = \frac{sin.\ OGN}{sin.\ ONG}$, but the $\angle\ OGN = 90^0 - (a+\beta)$ and

$ONG = PON = 90 + \beta$, hence it follows

$$\frac{N}{G} = \frac{sin.\ [90^0 - (a+\beta)]}{sin.\ (90^0 - \beta)} = \frac{cos.\ (a+\beta)}{cos.\ \beta}$$

and for the normal pressure against the plane

$$N = \frac{G\ cos.\ (a+\beta)}{cos.\ \beta}.$$

If the force $P$ is parallel to the plane, $\beta = 0$ and $cos.\ \beta = 1$, since $P = G\ sin.\ a$ and $N = G\ cos.\ a$.

If $P$ acts vertically $a + \beta = 90^0$, hence

$$cos.\ \beta = sin.\ a,\ cos.\ (a+\beta) = 0 \text{ and}$$

$P = G$ and $N = 0$, the inclined plane has then no control over the body.

Lastly, if the force acts horizontally, $\beta = -a$, and $cos.\ \beta = cos.\ a$, hence

$$P = \frac{G\ sin.\ a}{cos.\ a} = G\ tang.\ a; \text{ and } N = \frac{G\ cos.\ 0}{cos.\ a} = \frac{G}{cos.\ a}.$$

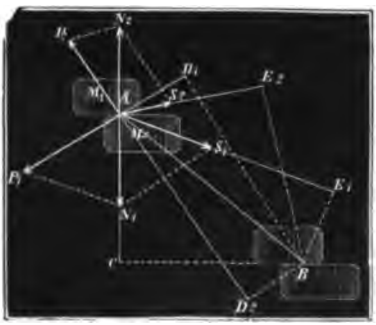
*Example.* To maintain a body of 500 lb. upon an inclined plane of 50° inclination to the horizon, a force is applied whose direction makes an angle of 75° with the horizon, what is the magnitude of this force, and the pressure of the body against the plane? The force is:

$P = \frac{500\ sin.\ 50^\circ}{cos.\ (75-50)} = \frac{500\ .\ sin.\ 50}{cos.\ 25} = 422{,}6$ lb.; and the pressure on the plane:

$N = \frac{500\ .\ cos.\ 75^\circ}{cos.\ 25} = 142{,}8$ lb.

§ 136. *Principle of virtual velocities.*—If we combine the principle of the equality of action and re-action set forth in § 128, with that of virtual velocities (§ 80 and 93), the following law transpires. If two bodies $M_1$ and $M_2$, Fig. 136, hold each other in equilibrium, *then for a finite rectilinear or infinitely small curvilinear motion of the point of contact or pressure A, the sum of the mechanical effects of the forces of the one body is equivalent to the sum of the mechanical effects of those of the other.* If $P_1$ and $S_1$ be the forces of the one body, and $P_2$ and $S_2$ those of the other, then, for a displacement of the

FIG. 136.



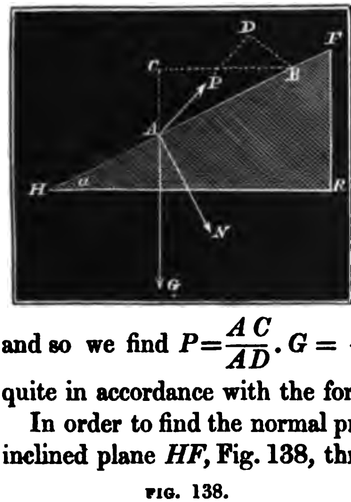
virtual velocities
point of contact from $A$ to $B$, the respective ~~distances~~ described are $AD_1$, $AE_1$, $AD_2$ and $AE_2$, and according to the above law:

$$P_1 . AD_1 + S . AE_1 = P_2 . AD_2 + S_2 . AE_2.$$

The correctness of this proposition may be proved in the following manner. As the normal pressures $N_1$ and $N_2$ are equal, there is also equilibrium between their mechanical effects, $N_1 . AC$ and $N_2 . AC$, with this difference, that the mechanical effect of the one force is positive and that of the other negative. Now from what has preceded we have the mechanical effect $N_1 . AC$ of the resultant $N_1$ equivalent to the sum $P_1 AD_1 + S_1 . AE_1$ of the mechanical effects of its components $P_1$ and $S_1$, and likewise $N_2 . AC = P_2 . AD_2 + S_2 . AE_2$; hence also $P_1 . AD_1 + S_1 . AE_1 = P_2 . AD_2 + S_2 . AE_2$.

The application of the principle of virtual velocities thus made more general possesses great advantagein statical investigations, as by it the evolution of algebraical expressions becomes much simplified. If, for example, we move a body $A$ up an inclined plane $FH$, Fig. 137, a distance $AB$, the ~~corresponding~~ path of the weight $G$, $= AC = AB \sin. ABC = AB . \sin. FHR = AB . \sin. a$. On the other hand, the path of the force $P$ is $AD = AB . \cos. BAD = AB . \cos. \beta$, and lastly, that of the normal force $N = o$; now the mechanical effect of $N$ is equivalent to that of $G +$ that of $P$, hence we have to put

FIG. 137.



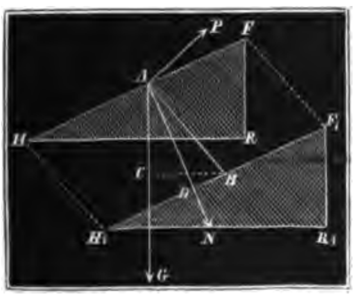
$$N . o = - G . AC + P . AD,$$

and so we find $P = \frac{AC}{AD} . G = \frac{G \sin. a}{\cos. \beta}$,

quite in accordance with the former paragraph.

In order to find the normal pressure $N$, we must move forward the inclined plane $HF$, Fig. 138, through a space $AB$ at right angles to the direction of the force $AP$, to determine the corresponding paths of the forces, and again put the mechanical effect of $N$ equivalent to that of $G +$ the mechanical effect of $P$. The path of $N$ is $AD = AB \cos. BAD = AB \cos. \beta$, that of $G$ is $AC = AB \cos. BAC = AB \cos. (a + \beta)$ and that of $P = o$, hence the mechanical effect

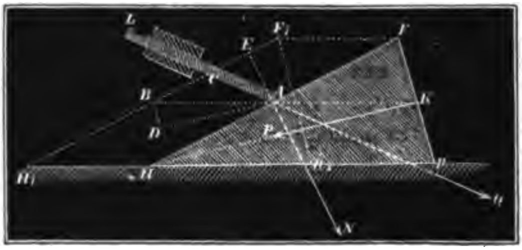
FIG. 138.



$N . AD = G . AC + P . o$ and $N = \frac{G . AC}{AD} = G . \frac{cos. (\alpha + \beta)}{cos. \beta}$, just as was found in the former paragraph.

§ 137. *Theory of the wedge.*—After this the theory of the wedge comes out very simply. The wedge is a moveable inclined plane formed by a triangular prism *FHR*, Fig. 139, generally the

FIG. 139.



force *KP* is $= P$, and at right angles to the back *FR* of the wedge, and holds in equilibrium another force or load $AQ = Q$, which presses against its lateral surface *FH*. If $FHR = \alpha$ be the angle measuring the sharpness of its edge, and further, the angle by which the direction of the force *KP* or *AD* deviates from the surface *FH*, therefore $FHK = HAD$, $= \delta$, and lastly the angle *LAH*, the deviation of the direction of *Q* from this same surface, $= \beta$, then the paths will be given which are described by the advance of the wedge from the position *FHR* into that of $F_1 H_1 R_1$, in the following manner. The path of the wedge is $AB = FF_1 = HH_1$ and that of the force is $= AD = AB\ cos.\ BAD = AB\ cos.\ (BAH - DAH) = AB\ cos.\ (\alpha - \delta)$; further, the path of the bar *AL* or load is $AC = \frac{AB\ sin.\ ABC}{sin.\ ACB} = \frac{AB\ sin.\ \alpha}{sin.\ HAC} = \frac{AB\ sin.\ \alpha}{sin.\ \beta}$, and the simultaneous path of the normal pressure *N* between the wedge and the foot of the bar $= AE = AB\ sin.\ \alpha$.

By the advance of the wedge a distance *AB*, the normal pressure *N* produces the mechanical effect $N . AE = N . AB\ sin.\ \alpha$, the force, however, develops the mechanical effect $P . AD = P . AB\ cos.\ (\alpha - \delta)$ and the resistance the mechanical effect, $Q . AC = Q . AB \frac{sin.\ \alpha}{sin.\ \beta}$, hence $N . AB\ sin.\ \alpha = P . AB\ cos.\ (\alpha - \delta)$ *i. e.* $N\ sin.\ \alpha = P\ cos.\ (\alpha - \delta)$, as also $N . AB\ sin.\ \alpha = Q . AB \frac{sin.\ \alpha}{sin.\ \beta}$, *i. e.* $N\ sin.\ \alpha = Q \frac{sin.\ \alpha}{sin.\ \beta}$, and

from these equations the equation between the power and resistance sought is given :

$$P\ cos.\ (\alpha-\delta) = \frac{Q\ sin.\ \alpha}{sin.\ \beta}, \text{ or}$$

$$P = \frac{Q\ sin.\ \alpha}{sin.\ \beta\ cos.\ (\alpha-\delta)},$$

which may likewise be obtained by the decomposition of the forces.

If the direction of the force is parallel to the base or lateral surface $HR$, $\delta = \alpha$, hence $P = \frac{Q\ sin.\ \alpha}{sin.\ \beta}$, and if, further, the direction of the load is perpendicular to the side $FH$, $\beta = 90^0$, and $P$ follows $= Q\ sin.\ \alpha$.

*Example.* The edge $FHR$ of a wedge $= \alpha = 25°$, the force is directed parallel to the base $HR$, therefore, $\delta = \alpha$, and the weight $Q$ acts at right angles to the side $FH$, therefore $\beta = 90°$, in what proportions are the power and weight to each other? $P$ is $= Q\ sin.\ \alpha$, therefore $\frac{P}{Q} = sin.\ 25° = 0,4226$. For a weight $Q$ of 130 lbs. the power $P$ comes out $= 130 . 0,4226 = 54,938$ lbs. In order to drive forward the weight or bar 1 foot, the wedge must pass over the space $AB = \frac{AC}{sin.\ \alpha}$ $= \frac{1}{0,4226}$ 2,3662 feet.

*Remark.* The theories of the inclined plane and the wedge will be more fully developed in the fifth chapter, where the effect of friction is taken into account.

## CHAPTER IV.

### EQUILIBRIUM IN FUNICULAR MACHINES.

§ 138. *Funicular machines.*—We have hitherto assumed that bodies, on which forces act, do not change their form in consequence of this action, we will now take up the equilibrium of such bodies as suffer a change in their form by the smallest forces. The former are called solid or rigid, the latter flexible bodies. In truth there is no body perfectly flexible; many of them, however, such as strings, ropes, cords, &c., and in some respects chains also, require so small a force to bend them that they may in many cases be regarded as perfectly flexible. Such bodies, which are moreover inextensible, will be the subject of the following investigations.

We understand by a funicular machine, a cord or a connection of cords (the word cord taken in its general sense) which becomes stretched by forces, and in this chapter we will consider the theory of the equilibrium of these machines.

That point of a funicular machine to which the force is applied, and where the cord forms an angle with the direction of the force, is called a knot or node. This may be either fixed or moveable. Tension is the force which a stretched cord transmits in the direction of its axis. The tensions at the ends of a straight cord or portion of a cord are equal and opposite § 83; also a straight cord cannot transmit other forces than the tension acting in the direction of its axis, because it must otherwise bend, and, therefore, cannot remain straight.

§ 139. *Knots or nodes.*—Equilibrium obtains in a funicular machine, when there is equilibrium at each of its nodes. Hence we must next find what are the relations of equilibrium at any one node.

Equilibrium takes place at a node $K$, which a portion of a cord $AKB$, Fig. 140, forms, when the resultant $KS$ of the tensions of the cord $KS_1 = S_1$ and $KS_2 = S_2$ are equal and opposite to the force $P$ applied at the node $K$, for the tensions $S_1$ and $S_2$ produce the same effects as equal and opposite forces, and three forces hold each other in equilibrium, if one of them is equal to and acts opposite to the resultant of the other two (§ 75). The resultant $R$ of the force $P$ and the first tension $S_1$ is equal and opposite to the second tension $S_2$, &c. In every case, this equation may be used to find out two of the quantities to be determined, viz. the tension of the cord and its direction. Let, for example, the force be $P$, the tension $S_1$ and the $\angle$ between the two $AKP = 180° - AKS = 180° - a$, we have for the other tension

FIG. 140.


$$S_2 = \sqrt{P^2 + S_1^2 - 2\, PS_1 \cos. a}$$

and for its direction or deviation from $KS$, $BKS = \beta$, and

$$\sin. \beta = \frac{S_1 \sin. a}{S_2}.$$

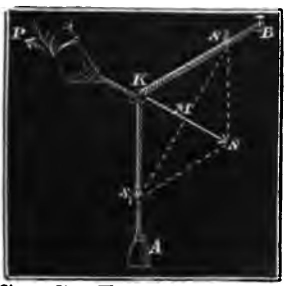
*Example.* If the cord $AKB$, Fig. 140, is fixed at the extremity $B$, and at the extremity $A$ stretched by a weight $G$ = 135 lbs. and the middle $K$ by a force $P$ = 109 lbs. which pulls upwards under an angle of 25°; required the direction and tension of the portion of cord $KB$. The magnitude of the tension is:

$$S_2 = \sqrt{109^2 + 135^2 - 2 . 109 . 135 \; cos. \; (90^\circ - 25^\circ)}$$
$$= \sqrt{11881 + 18225 - 29430 \; . \; cos. \; 65^\circ} = \sqrt{17668,3} = 132,92 \text{ lbs.}$$

For the angle $\beta$, $sin. \; \beta = \frac{S_1 \; sin. \; \alpha}{S_2} = \frac{135 \; . \; sin. \; 65^\circ}{132,92}$, $Log. \; sin. \; \beta = 0,964017 - 1$, hence $\beta = 67^\circ \; 0'$, and the inclination of the portion of the cord to the horizon $= \alpha + \beta - 90^\circ = 65^\circ + 67^\circ - 90^\circ = 42^\circ$.

§ 140. If the node $K$ is a running or moveable one, or the force $P$ acts by means of a ring running along the cord $AKB$, Fig. 141, the resultant $S$ of the tensions $S_1$ and $S_2$ is equal and opposite to the force $P$ at the ring; besides this, the tensions are equal, for if the cord be drawn a certain space $s$ through the ring, each of the tensions $S_1$ and $S_2$ will pass over the space $s$, and the force $P$ over a space $= 0$; consequently, provided there is perfect flexibility, the mechanical effect $P . 0 = S_1 . s - S_1 . s$, *i. e.* $S_1 s = S_2 s$ and $S_1 = S_2$. From this equality of the tensions there follows the equality of the angles $AKS$ and $BKS$, by which the resultant $S$ deviates from the directions of the cords, if we put these angles $= \alpha$, the resolution of the rhomb $KS_1SS_2$, gives

FIG. 141.



$$S = P = 2 \; S_1 \; cos. \; \alpha \text{ and inversely}$$
$$S_1 = S_2 = \frac{P}{2 \; cos. \; \alpha}.$$

$A$ and $B$ are the fixed points of a cord $AKB$ of given length $(2 \; a)$ with a moveable node $K$, the place of this node may be found by constructing an ellipse, whose foci are $A$ and $B$, and whose major axis is equal to the length of the cord $2 \; a$, and if a tangent is drawn to this curve at right angles to the given direction of the force, the resulting point of contact is the place of the node, because the normal to the ellipse $KS$ makes equal angles with the radii vectores $KA$

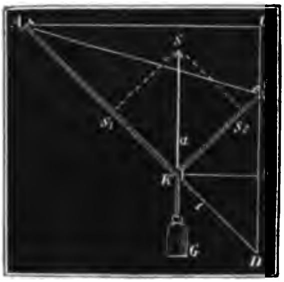
FIG. 142.

and $KB$, as does the resultant $S$ with the tensions of the cord $S_1$ and $S_2$.

If $AD$ be drawn parallel to the given direction of the force, and $BD$ be made equal to the given length of the cord, $AD$ bisected at $M$ and the perpendicular $MK$ be raised, the place of the node $K$ may likewise be obtained without the construction of an ellipse, for since the $\angle\ AKM = \angle\ DKM$ and $AK = DK$, it follows that $\angle\ AKS$ also $= \angle\ BKS$ and $AK + KB = DK + KB = DB$.

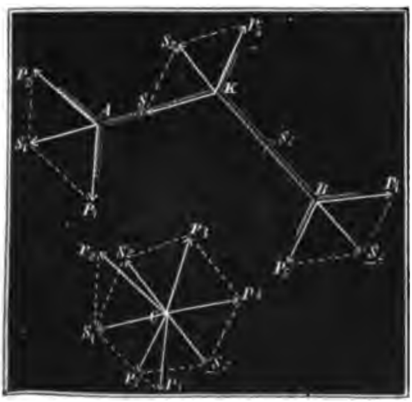
*Example.* Between the points $A$ and $B$, Fig. 143, a rope of 9 feet in length is stretched by a weight $G$ of 170 lbs. suspended to it by a ring; the horizontal distance $AC$ of the two points is $6\frac{1}{2}$ ft. and the vertical distance $BC = 2$ ft.; to find the position of the node, the tensions and directions of the rope. From the length $AD = 9$ ft. as hypothenuse and the horizontal line $AC = 6\frac{1}{2}$ ft.; it follows that the vertical $CD = \sqrt{9^2 - 6{,}5^2} = \sqrt{81 - 42{,}25} = \sqrt{38{,}75} = 6{,}225$ feet; and from this the base $BD$ of the equilateral triangle $BDK, = CD - CB = 6{,}225 - 2 = 4{,}225$ ft. The similarity of the triangles $DKM$ and $DAC$ gives $DK = BK = \frac{DM}{DC} \cdot DA = \frac{4{,}225 \cdot 9}{2 \cdot 6{,}225} = 3{,}054$ ft.; hence it follows, that $AK = 9 - 3{,}054 = 5{,}946$ feet; and for the angle $\alpha$, by which the sides of the rope are inclined to the vertical: $cos.\ \alpha = \frac{BM}{BK} = \frac{2{,}1125}{3{,}054} = 0{,}6917$; hence, $\alpha = 46^\circ\ 14'$; and lastly, the tension of the rope $S_1 = S_2 = \frac{G}{2\ cos.\ \alpha} = \frac{170}{2 \cdot 0{,}6917} = 122{,}9$ lbs.

FIG. 143.



§ 141. *Funicular polygon.*—The relations of equilibrium in the funicular polygon, *i. e.* in a stretched cord which is acted upon by forces applied to different points, are in accordance with those of the equilibrium of forces, which are applied to one point. Let $AKB$, Fig. 144, be a cord stretched by the forces $P_1$, $P_2$, $P_3$, $P_4$, $P_5$: let $P_1$ and $P_2$ act at $A$, $P_3$ at $K$, and $P_4$ and $P_5$ at $B$. Let us put the tension of the portion $AK = S_1$ and that of $BK = S_2$, we shall then obtain $S_1$ for the resultant of $P_1$ and $P_2$

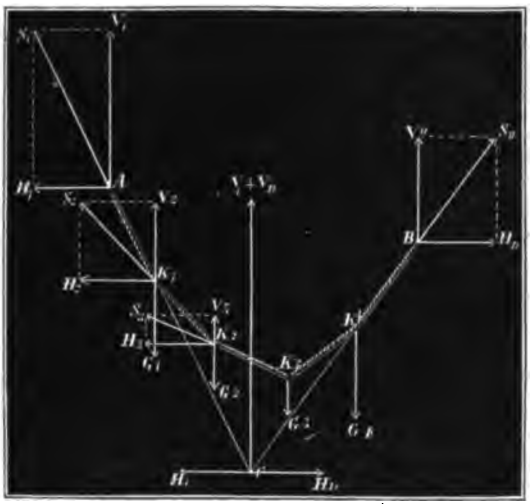
FIG. 144.



applied to $A$, and if we carry the point of application $A$ of this tension from $A$ to $K$, we shall again get $S_2$ for the resultant of $S_1$ and $P_3$, or of $P_1$, $P_2$, $P_3$; lastly, if we transport the point of application of $S_2$ from $K$ to $B$, we shall then obtain in $S_2$, $P_4$ and $P_5$, or since $S_2$ is the resultant of $P_1, P_2, P_3$, also in $P_1$, $P_2$, $P_3$, $P_4$, $P_5$ a set of forces balancing each other. We may accordingly assert that, *when certain forces $P_1$, $P_2$, $P_3$, &c., hold a funicular polygon in equilibrium, they will hold each other in equilibrium also, if applied at a single point $C$, their direction and magnitude remaining invariable.*

If the cord $AK_1K_2 \ldots B$, Fig. 145, be stretched at the points

FIG. 145.



or nodes, $K_1$, $K_2$ by weights $G_1$, $G_2 \ldots$ and the extremities $A$ and $B$ by the vertical forces $V_1$ and $V_n$, and the horizontal forces $H_1$ and $H^n$, the sum of the vertical forces will be : $V_1 + V_n—(G_1 + G_2 + G_3 + \ldots)$ and of the horizontal forces: $H_1—H_n$. The condition of equilibrium requires that both sums $= 0$; therefore

1. $V_1 + V_n = G_1 + G_2 + G_3 + \ldots$ and
2. $H_1 = H_n$; *i. e.*

*In a funicular polygon stretched by weights, the sum of the vertical forces or vertical tensions at the extremities or points of suspension is equivalent to the sum of the suspended weights, and the horizontal tension at the one extremity is equal and oppositely directed to the horizontal tension at the other extremity.*

If the directions of the tensions $S_1$ and $S_n$ at the cords $A$ and $B$ be prolonged to their intersection $C$, and the points of application of these tensions be transferred to this point, we shall then have the single force $P = V_1 + V_n$, because the horizontal forces $H_1$ and $H_n$ counteract each other. Since this force holds in equilibrium the sum $G_1 + G_2 + G_3 + \ldots$ of the suspended weights, the point of application or centre of gravity of these weights must therefore lie in the direction of the same, *i. e.* in the vertical line passing through the point $C$.

§ 142. From the tension $S_1$ of the first portion $AK_1$ whose angle of inclination $S_1AH_1 = a_1$, the vertical tension follows; $V_1 = S_1 \sin. a_1$, and the horizontal $H_1 = S_1 \cos. a_1$. If now we transfer the point of application of these forces from $A$ to the first node $K_1$, the weight $G_1$ acting vertically downwards meets these tensions, and now for the following portion $K_1K_2$, the vertical tension $V_2 = V_1 - G_1 = S_1 \sin. a_1 - G_1$, for which the horizontal tension $H_2 = H_1 = H$ remains unchanged. Both forces united give the tension of the axis of the second portion $S_2 = \sqrt{V_2^2 + H^2}$ and its inclination $a_2$ by the formula

$$tang.\, a_2 = \frac{V_2}{H} = \frac{S_1 \sin. a_1 - G_1}{S_1 \cos. a_1}, \text{ i. e.}$$

$$tang.\, a_2 = tang.\, a_1 - \frac{G_1}{H}.$$

If the point of application of the forces $V_2$ and $H_2$ is transferred from $K_1$ to $K_2$, we obtain in the weight $G_2$ meeting them another new vertical force, and therefore the vertical force of the third portion of the cord

$$V_3 = V_2 - G_2 = V_1 - (G_1 + G_2) = S_1 \sin. a_1 - (G_1 + G_2),$$

whilst the horizontal force $H_3$ remains $= H$. The whole tension of the third portion is

$S_3 = \sqrt{V_3^2 + H^2}$, and for its angle of inclination $a_3$, we have

$$tang.\, a_3 = \frac{V_3}{H} = \frac{S_1 \sin. a_1 - (G_1 + G_2)}{S_1 \cos. a_1}, \text{ i. e.}$$

$$tang.\, a_3 = tang.\, a_1 - \frac{G_1 + G_2}{H}.$$

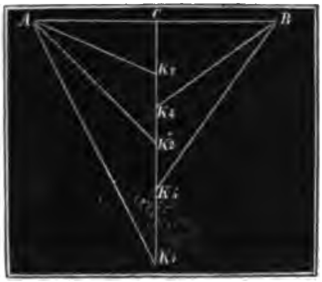
For the angle of inclination of the fourth portion of the cord,

$$tang.\, a_4 = tang.\, a_1 - \frac{G_1 + G_2 + G_3}{H}, \text{ \&c.}$$

Besides, the tensions $S_1$, $S_2$, $S_3$, &c., as well as the angles of inclination $a_1$, $a_2$, $a_3$, &c. of the separate portions of the cord,

may easily be represented geometrically. If we make the horizontal line $CA=CB$, Fig. 146, = the horizontal tension $H$ and the vertical $CK_1$ = vertical tension $V_1$ at the point of suspension $A$, the hypothenuse $AK_1$ gives the whole tension $S_1$ and the $\angle\, CAK_1$, also its inclination to the horizon : if now further we apply the weights $G_1$, $G_2$, $G_3$, &c. as parts $K_1K_2$, $K_2K_3$, &c. of $CK$, and draw the transversal lines $AK_2$, $AK_3$, &c., we shall have in them the tensions of the successive portions of the cord, and in the angles $K_2AC$, $K_3AC$, &c. the angles of inclination $a_2$, $a_3$, &c. of these portions.

FIG. 146.



§ 143. From the investigations of the preceding paragraph, the law for the equilibrium of cords stretched by weights, comes out thus:

1. *The horizontal tension is at all points of the cord one and the same, viz:*

$$H=S_1 \cos. a_1 = S_n \cos. a_n.$$

2. *The vertical tension at any one point is equal to the vertical tension at the other extremity above it, less the sum of the intermediate suspended weights, therefore*

$$V_m = V_1 - (G_1 + G_2 + \dots G_{m-1}).$$

If the angle $a_1$ be known and the horizontal tension $H$, the vertical tension at the extremity $A$ is known; $V_1 = H \, . \, tang. \, a_1$, and accordingly that at the extremity $B$; $V_n = (G_1 + G_2 + \dots + G_n) - V_1$.

If, on the other hand, the angles of inclination $a_1$ and $a_n$ at both points of suspension $A$ and $B$ are known, the horizontal and vertical tensions are given at the same time, viz:

$$\frac{V_n}{V_1} = \frac{tang. \, a_n}{tang. \, a_1}, \text{ and, therefore,}$$

$$V_n = \frac{V_1 \, tang. \, a_n}{tang. \, a_1}.$$

Since $V_1 + V_n = G_1 + G_2 + \dots$, *i. e.*

$$\left(\frac{tang. \, a_1 + tang. \, a_n}{tang. \, a_1}\right) V_1 = G_1 + G_2 \dots, \text{ it follows that:}$$

$$V_1 = \frac{(G_1 + G_2 + \dots) \, tang. \, a_1}{tang. \, a_1 + tang. \, a_n},$$

$$V_n = \frac{(G_1 + G_2 + \ldots) \; tang. \, a_n}{tang. \, a_1 + tang. \, a_n}, \text{ and from this:}$$

$$H = V_1 \; cotg. \, a_1 = V_n \; cotg. \, a_n.$$

If both sides have the same inclination $a_n = a_1$, then $V_1 = V_n = \frac{G_1 + G_2 + \ldots + G_n}{2}$, and the one extremity $A$ supports as much as the other $B$.

For the rest, these laws hold good also for the funicular polygon, especially when stretched by forces, if the directions of the forces are substituted for the verticals.

*Example.* The funicular polygon $AK_1K_2K_3B$, Fig. 147, is stretched by three weights $G_1 = 20$, $G_2 = 30$, and $G_3 = 16$ lbs., as well as by the horizontal force $H_1 = 25$ lbs.; required to find the tensions of the axis and the angles of inclination of the sides, in the hypothesis that the ends of the string have the same inclination. Here the vertical tensions are equal, viz. $V_1 = V_4 = \frac{G_1 + G_2 + G_3}{2} = \frac{20 + 30 + 16}{2} = 33$ lbs. The vertical tension of the second portion of the string is $V_2 = V_1 - G_1 = 33 - 20 = 13$ lbs., that of the third $V_3 = V_4 - G_2$ or $(G_1 + G_2 - V_1) = 33 - 16 = 17$ lbs.; the angles of inclination $a_1$ and $a_4$ of the ends are determined by $tang. \, a_1 = tang. \, a_4 = \frac{V_1}{H} = \frac{33}{25} = 1{,}32$, that of the second and third portions by the $tang. \, a_2 = tang. \, a_1 - \frac{G_1}{H} = 1{,}32 - \frac{20}{25} = 0{,}52$, and $tang. \, a_3 = tang. \, a_4 - \frac{G_3}{H} = 1{,}32 - \frac{16}{25} = 0{,}68$; hence $a_1 = a_4 = 52° \, 51'$; $a_2 = 27° \, 28'$, $a_3 = 34° \, 13'$; lastly, the tensions of the axis are $S_1 = S_4 = \sqrt{V_1^2 + H^2} = \sqrt{33^2 + 25^2} = \sqrt{1714} = 41{,}40$ lbs., $S_2 = \sqrt{V_2^2 + H^2} = \sqrt{13^2 + 25^2} = \sqrt{794} = 28{,}18$ lbs., and $S_3 = \sqrt{V_3^2 + H^2} = \sqrt{17^2 + 25^2} = 30{,}23$ lbs.

FIG. 147.

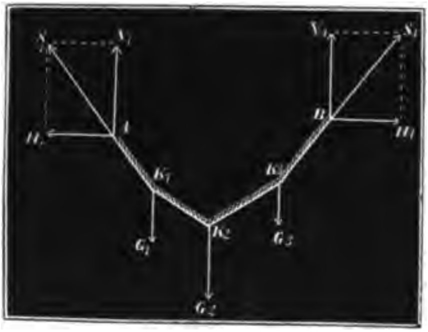
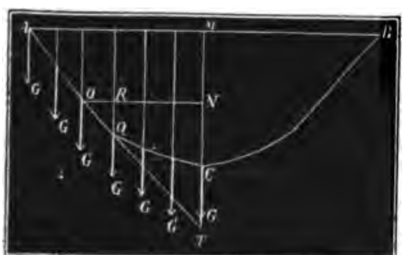


FIG. 148.



§ 144. *The Parabola as catenary.*—Let us suppose that the string $ACB$, Fig. 148, is stretched by equal weights $G_1$, $G_2$, &c., suspended at equal horizontal distances from each other. Let us represent by $b_1$ the horizontal distance $AM$ between

the point of suspension $A$ and the lowest $C$, but the vertical distance $CM$ by $a$. Let us put further for another point $O$ of the polygon, the corresponding co-ordinates $ON=y$ and $CN=x$. If now the vertical tension of $A$ be $=V$, that of $O$ will be $=\frac{y}{b}.V$, and hence for the angle of inclination to the horizon, $NOT=ROQ=\phi$ of the portion of the string $OQ$, we shall have $tang.\ \phi=\frac{y}{b}.\frac{V}{H}$, where $H$ is the constant of the horizontal tension.

Hence $QR=OR.tang.\ \phi=OR.\frac{y}{b}.\frac{V}{H}$ is the vertical distance of two adjacent angles of the funicular polygon. If we substitute for $y$ $OR$, $2\ OR$, $3\ OR$, &c., the last equation will give the corresponding vertical distances of the first, second and third angles, &c., reckoned from below upwards; then if we add together all these values, whose amount may be $=m$, we shall obtain the height $CN$ of the point $O$ vertically above the lowest point $C$, viz:

$$x=CN=\frac{V}{H}.\frac{OR}{b}(OR+2\ OR+3\ OR+\ldots+m.OR)$$

$$\frac{V}{H}.\frac{\overline{OR}^2}{b}(1+2+3+\ldots+m)=\frac{V}{H}.\frac{m(m+1)}{1.2}.\frac{OR^2}{b},$$

in accordance with the theory of arithmetical series.

Lastly, if $OR$ be put $=\frac{y}{m}$, we shall have:

$$x=\frac{V}{H}.\frac{m(m+1)}{2m^2}.\frac{y^2}{b}.$$

If the number of weights be very great, $m+1$ may be taken $=m$, whence we shall have:

$$x=\frac{V}{H}.\frac{y^2}{2b}.$$

For $x=a$, $y=b$, hence also:

$a=\frac{V}{H}.\frac{b}{2}$, and more simply:

$\frac{x}{a}=\frac{y^2}{b^2}$, which is the equation to a parabola.

If, therefore, a string devoid of weight be stretched by infinitely many weights applied at equal horizontal distances, the funicular polygon will pass into a parabola.

For the angle of inclination $\phi$ we have besides:

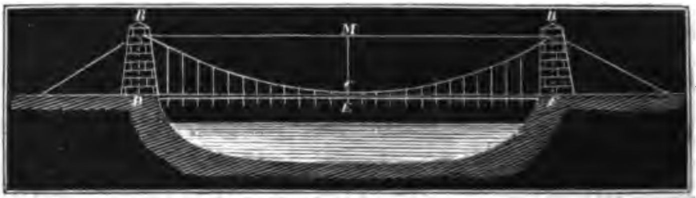
$$tang.\,\phi=\frac{y}{b}\cdot\frac{2a}{b}=2y\cdot\frac{a}{b^2}=2y\cdot\frac{x}{y^2}=\frac{2x}{y},\text{ as also}$$

$$tang.\,a=\frac{2a}{b}.$$

Therefore the tangent $OT$ cuts the axis of the abscissæ, so that $CT=CN=x$.

If the chains and rods of a chain bridge, Fig. 149, were without

FIG. 149.



weight, or light enough in respect to the weight of the loaded bridge $DEF$, which only is to be taken into consideration, then the chain $ACB$ would form a parabola.

*Example.* The whole load of a chain-bridge in Fig. 149, = 320000 lbs.; the span $AB = 2b = 150$ feet, and the height of the arch $CM = a = 15$ feet; to find the tensions and other relations of the chains. The inclination of the ends of the chain to the horizon is determined by the formula, $tang.\,a = \frac{2a}{b}=\frac{30}{75}=\frac{2}{5}=0{,}4$, therefore $a=21°\ 48'$. The vertical tension at each point of suspension is $V_1 = \frac{1}{2}$ the weight = 160000 lbs.; the horizontal, $H = V_1\,cotg.\,a = 160000\cdot\frac{1}{0{,}4}=400000$ lbs.; lastly, the whole tension at one end:

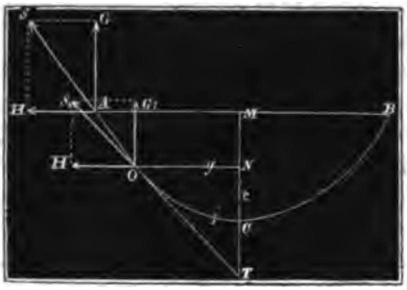
$$S = \sqrt{V^2+H^2} = V\sqrt{1+cotg.\,a^2} = 160000\,.\sqrt{1+\left(\frac{1}{0{,}4}\right)^2}$$
$$= 160000\sqrt{\frac{29}{4}} = 80000\sqrt{29} = 430813 \text{ lbs.}$$

§ 145. *Catenary.*—When a perfectly flexible and inextensible string suspended from two points, or a chain consisting of short links, is stretched by its own weight, its axis forms a curved line, to which the name of catenary has been given. The imperfectly elastic and extensible cords, ropes, bands, chains, &c., met with in practice, give curved lines which approximate to the catenary only, but may usually be treated as such. From the foregoing, the horizontal tension of the catenary is equally great at all points, on the other hand, the vertical tension is equivalent to the vertical tension of the points of suspension lying above it, less the weight of the portions of the chain above.

Since the tension at the vertex, where the catenary is horizontal, is null; the vertical tension, therefore, at the point of suspension is equivalent to the weight of the chain from that point to the vertex, and the vertical tension at each place also equivalent to the weight of the portion of the rope or chain lying below it.

If equal lengths of the chain be equally heavy we have then the common catenary, which only we will now consider. If a portion of the rope, or chain one foot in length, weighs $\gamma$, and if the arc corresponding to the co-ordinates $CM = a$ and $MA = b$, Fig. 150,

FIG. 150.



$AOC = l$, we then have the weight of the portion of the chain $AOC = l\,\gamma$; if, on the other hand, the length of the arc ($l$) corresponding to the co-ordinates ($CN = x$ and $NO = y$) $= s$, we have the weight of this arc $= s\,\gamma$. If we put the length of a similar portion, whose weight $= H, = c$, (the horizontal tension) we have further $H = c\,\gamma$, and, therefore, for the angles of inclination $\alpha$ and $\phi$ at the points $A$ and $O$:

$$tang.\ \alpha = tang.\ SAH = \frac{G}{H} = \frac{l\gamma}{c\gamma} = \frac{l}{c} \text{ and}$$

$$tang.\ \phi = tang.\ NOT = \frac{s\,\gamma}{c\,\gamma} = \frac{s}{c}.$$

FIG. 151.

§ 146. If we make the horizontal line $CH$, Fig. 151, = the length $c$ of the portion of chain measuring the horizontal tension, and $CG$ = the length $l$ of the arc of the chain on one side, we have, in accordance with § 142, in the hypothenuse $GH$, the measure and direction of the funicular tension at the point $A$, for

$$tang.\ CHG = \frac{CG}{CH} = \frac{l}{c} \text{ and}$$

$$\overline{GH} = \sqrt{\overline{CG}^2 + \overline{CH}^2} = \sqrt{l^2 + c^2},$$

$$\text{or } S = \sqrt{G^2 + H^2} = \sqrt{l^2 + c^2}\,.\,\gamma = \overline{GH}\,.\,\gamma.$$

If now we divide $CG$ into equal parts and draw from $H$ to the points 1, 2, 3, &c., straight lines, these will give the measure and directions of the tensions of those points of the catenary which we obtain when we divide the length of the catenary arc $AC$ into as many equal parts. So, for example, the line $H3$ gives the measure and direction of the tension or the tangents at the point (3) to the arc $AC$, because in this point the vertical tension $= C3 \,.\, \gamma$, whilst the horizontal tension remains the same $= c \,.\, \gamma$, therefore for this point $tang.\, \phi = \dfrac{C3 \,.\, \gamma}{c\gamma} = \dfrac{C3}{CH}$, which the figure actually gives.

This peculiarity of the catenary is of use in constructing this curve mechanically, with an approximation to correctness. After the given length $CG$ of the catenary arc for construction has been divided into very many equal parts, the line $CH = c$ measuring the horizontal tension is applied to it, and the transversal lines $H1$, $H2$, $H3$, &c., drawn; if a part $C1$ of the arc be placed upon $CH$, and through the point of division obtained (1) a parallel to $H1$ be drawn, which cuts off from it a part (12); and likewise through the point (2) another line parallel to $H2$ be drawn, and which cuts off from it a point (23) equal to a part of the arc, and again through this (3) another, parallel to $H3$, and (34) be made equal to another part of the arc, and we proceed in this manner, we shall obtain a polygon ($C\,1\,2\,3\,4\ldots$); as we have taken these sides very small, we may consider it as a curve and easily find the curve to it, if we connect the middle points of the small sides ($C1$), (12), (23), by a trace or line.

For practical purposes, a finely linked chain suspended against a perpendicular wall enables us to determine accurately enough a catenary answering certain conditions, as those of given length and height, or of given width or length of the arc.

§ 147. In many cases, and also in applications to architecture and to machines, the horizontal tension of the catenary is very great, and the height of the arc small in comparison with the width. Under this supposition, an equation to this curve is obtained in the following manner.

Let $s$ be the length, $x = CM$ the absciss, and $y = AM$ the ordinate of a very compressed arc $AC$, Fig. 152. If we make $AK = CK$, we may consider this arc as a circular one described from $K$ as a centre. Since from the known equation of the circle $y^2 = x(2r - x)$, it follows that the radius $CK$ of the circle, $r = \dfrac{y^2}{2x} + \dfrac{x}{2}$,

FIG. 152.

or more simply, if we neglect $\frac{x}{2}$ as small in comparison with $\frac{y^2}{2x}$, $r = \frac{y^2}{2x}$. For the angle $AKC = \phi^0$, subtended at the centre by $AB$, $sin. \phi = \frac{AM}{AK} = \frac{y}{r} = \frac{2x}{y}$, and the arc $\phi = sin. \phi + \frac{1}{6} sin. \phi^3 + \frac{3}{40} sin. \phi^5 + \dots$; if we have regard only to the two first ~~members~~ terms, it therefore follows that:

$$\phi = \frac{2x}{y} + \frac{1}{6} \cdot \left(\frac{2x}{y}\right)^3 = \frac{2x}{y} + \frac{4}{3} \cdot \left(\frac{x}{y}\right)^3.$$

Now the arc $AC = s = r\,\phi = \frac{y^2}{2x} \cdot \phi$; hence:

$$s = y + \frac{2}{3} \cdot \frac{x^2}{y} = y\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right]. \quad (b)$$

But inversely, $y = \frac{s}{1 + \frac{2}{3}\left(\frac{x}{y}\right)^2}$, which may be put:

$$y = s\left[1 - \frac{2}{3}\left(\frac{x}{y}\right)^2\right], \text{ and on the other hand:}$$

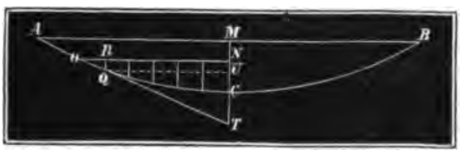
$$x = \sqrt{\frac{3}{2} y\,(s - y)}.$$ deduced from (b)

*Example.* The width of a very compressed arc, whose law for the rest is not known, is $2b = 3,5$ feet, and the height $a = 0,25$ feet; its length, therefore, is:

$$2l = 3,5\left[1 + \frac{2}{3} \cdot \left(\frac{0,25}{1,75}\right)^2\right] = 3,5\,(1 + \frac{2}{3} \cdot 0,143^2) = 3,5 + 3,5 \cdot 0,0136 = 3,548 \text{ ft.}$$

§ 148. We will now apply the formula $s = y\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right]$

FIG. 153.



for the length of a compressed arc to a strongly stretched catenary $ACB$, Fig. 153, while we put the vertical tension at a point

$O, = V = s\gamma = y\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right] . \gamma$, and therefore for the angle made by the tangent $TON = \phi$, $tang. \phi = \frac{s}{c} = \frac{y}{c}\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right]$.

If we divide the ordinate $y$ into $m$ equal parts, we find the portion $RQ = NU$ of the absciss $x$ corresponding to one such part $OR$, when we put $RQ = OR \,.\, tang. \phi = OR \,.\, \frac{y}{c}\left[1 + \frac{1}{3}\left(\frac{x}{y}\right)^2\right]$.

Since $x$ is small in comparison with $y$, $RQ$ is approximately $= OR \,.\, \frac{y}{c}$. If now we put $OR = \frac{y}{m}$ and successively for $y$: $\frac{y}{m}, \frac{2y}{m}, \frac{3y}{m}$, &c., we obtain by degrees the several parts of $x$, whose sum therefore is $x = \frac{y^2}{c\,m^2}(1 + 2 + 3 + \ldots + m) = \frac{y^2}{c\,m^2} \cdot \frac{m\ (m+1)}{2}$ (§ 144) $= \frac{y^2}{2c}$, and which corresponds with the equation to the parabola.

But if we wish to attain greater accuracy, we must put $QR = OR \,.\, \frac{y}{c}\left[1 + \frac{2}{3}\left(\frac{x}{y}\right)^2\right]$, substitute for $x$ its value last found $\frac{y^2}{2c}$, and we shall then obtain:

$$QR = OR \,.\, \frac{y}{c}\left(1 + \frac{1}{6} \cdot \frac{y^2}{c^2}\right) = \frac{OR}{c}\left(y + \frac{1}{6} \cdot \frac{y^3}{c^2}\right).$$

Let us again successively put $y = \frac{y}{m}, \frac{2y}{m}, \frac{3y}{m}$, &c., and for $OR$ likewise $\frac{y}{m}$, we shall then find the several values of $x$, and the sum itself:

$$x = \frac{y}{cm}\left[\frac{y}{m}(1+2+3+\ldots+m) + \frac{1}{6\,c^2} \cdot \left(\frac{y}{m}\right)^3 (1^3+2^3+3^3+\ldots+m^3)\right].$$

Now for a very great number of ~~members~~, the sum of the natural numbers from 1 to $m = \frac{m^2}{2}$, and the sum of their cubes $= \frac{m^4}{4}$, accordingly:

$$x = \frac{y}{c}\left(\frac{y}{2} + \frac{1}{6\,c^2} \cdot \frac{y^3}{4}\right) \text{ i. e.}$$

1. $x=\frac{y^2}{2c}+\frac{y^4}{24c^3}=\frac{y^2}{2c}\left[1+\frac{1}{12}\cdot\left(\frac{y}{c}\right)^2\right]$, the equation of a strongly stretched catenary.

By inversion it follows that $y^2=2cx-\frac{y^4}{12c^2}=2cx-\frac{4c^2x^2}{12c^2}$ $=2cx-\frac{x^2}{3}$, therefore:

2. $y=\sqrt{2cx-\frac{x^2}{3}}$, or approximately $=\sqrt{2cx}\left(1-\frac{x}{12c}\right)$.

The measure of the horizontal tension is further given:

$$c=\frac{y^2}{2x}+\frac{y^4}{2x\,.\,12c^2}=\frac{y^2}{2x}+\frac{y^4}{24x}:\frac{4x^2}{y^4},\ i.\ e.$$

3. $c=\frac{y^2}{2x}+\frac{x}{6}$.

The angle of the tangent $\varphi$ is determined by:

$$tang.\ \varphi=\frac{y}{c}\left[1+\frac{2}{3}\left(\frac{x}{y}\right)^2\right]=\frac{y\left[1+\frac{2}{3}\left(\frac{x}{y}\right)^2\right]}{\frac{y^2}{2x}\left[1+\frac{1}{3}\left(\frac{x}{y}\right)^2\right]}$$

$$=\frac{2x}{y}\left[1+\frac{2}{3}\left(\frac{x}{y}\right)^2\right]\div\left[1+\frac{1}{3}\left(\frac{x}{y}\right)^2\right],\ i.\ e.$$

4. $tang.\ \varphi=\frac{2x}{y}\left[1-\frac{1}{3}\left(\frac{x}{y}\right)^2\right]$.

Lastly, we must here place the formula of rectification found in the former paragraph:

5. $s=y\left[1+\frac{2}{3}\left(\frac{x}{y}\right)^2\right]=y\left[1+\frac{1}{6}\left(\frac{y}{c}\right)^2\right]$.

*Example.*—1. For a span $2b=16$ feet and height of arc $a=2\frac{1}{2}$ feet, the length $2l$ is $=16\left[1+\frac{2}{3}\left(\frac{2,5}{8}\right)^2\right]=16+16\ .\ 0,065=17,04$ feet, the length of the portion of chain which measures the horizontal tension: $c=\frac{b^2}{2a}+\frac{a}{6}=\frac{64}{5}+\frac{5}{12}=12,8+0,417$ $=13,217$ feet; the tangent of the angle of suspension: $tang.\ \alpha=\frac{2a}{b}\left[1+\frac{1}{3}\left(\frac{a}{b}\right)^2\right]$ $=\frac{5}{8}\left[1+\frac{1}{3}\left(\frac{5}{16}\right)^2\right]=\frac{5\,.\,1,03255}{8}=0,6453\ldots$, the angle of suspension, therefore, $\alpha=32°\ 50'$.—2. A chain of 10 feet length and $9\frac{1}{2}$ span, has the height of its arc

$$a=\sqrt{\frac{3}{2}(l-b)\,b}=\sqrt{\frac{3}{2}\,\frac{(10-9\frac{1}{2})}{2}\,\frac{9\frac{1}{2}}{2}}=\sqrt{\frac{3}{2}\cdot\frac{19}{16}}=\sqrt{\frac{57}{32}}$$

$=\sqrt{1,7812}=1,335$ feet, and the measure of the horizontal tension:

$$c=\frac{b^2}{2a}+\frac{a}{6}=\frac{4,75^2}{2\,.\,1,335}+\frac{1,335}{6}=8,673\text{ feet.}$$

3. If a 30 feet long and 8 lb. heavy line be stretched horizontally by a force of 20 lbs., the vertical tension $V = \frac{1}{2} G = 4$ lbs., the horizontal force $H = \sqrt{S^2 - V^2}$ $= \sqrt{20^2 - 4^2} = \sqrt{384} = 19{,}596$ lbs., the tangent of the angle of suspension: $tang.\ \phi = \frac{V}{H} = \frac{4}{19{,}596} = 0{,}20412$, the angle $\phi$ itself $= 11° 32'$; the measure of the horizontal tension $c = \frac{H}{\gamma} = H : \frac{8}{30} = \frac{30}{8} H = 73{,}485$ feet, the span $2\,b$ $= 2\,l \left[1 - \frac{1}{6} \cdot \left(\frac{l}{c}\right)^2\right] = 30 \cdot \left[1 - \frac{1}{6} \cdot \left(\frac{15}{73{,}48}\right)^2\right] = 30 - 0{,}208 = 29{,}792$ ft., and the height of the arc $a = \sqrt{\frac{3}{2} b\,(l - b)} = \sqrt{\frac{3}{2}\,\frac{29{,}792 \cdot 0{,}208}{2 \cdot 2}}$ $= \sqrt{29{,}792 \cdot 0{,}078} = 1{,}524$ feet.

§ 149. The higher calculus gives the following general formulæ for the catenary, and which hold good for all tensions.

1. $s = \sqrt{2\,c\,x + x^2}$, and inversely, $x = \sqrt{c^2 + s^2} - c$ and $c = \frac{s^2 - x^2}{2x}$.

2. $s = \frac{c}{2}\left(e^{\frac{y}{c}} - e^{-\frac{y}{c}}\right)$, inversely $y = c\,L\,n\left(\frac{s + \sqrt{c^2 + s^2}}{c}\right)$, where $e$ is the base 2,71828 of the natural system of logarithms, and $L\,n$ the logarithm $=$ 2,30258 times the common logarithm.

3. $y = c\,L\,n\left(\frac{c + x + \sqrt{2\,c\,x + x^2}}{c}\right)$, inversely $x = \frac{c}{2}\left(e^{\frac{y}{c}} + e^{-\frac{y}{c}}\right) - c$,

4. $y = \frac{s^2 - x^2}{2\,x}\,L\,n\left(\frac{s + x}{s - x}\right)$.

The use of these formulæ is very troublesome, especially in complicated problems, where a direct solution is generally not possible.

*Example.* The two co-ordinates of a catenary are $x = 2$ feet, and $y = 3$ feet; required the horizontal tension $c$ of this curve? Approximately from No. 3 of the former paragraphs $c = \frac{y^2}{2x} + \frac{x}{6} = \frac{9}{4} + \frac{2}{6} = 2{,}58$. From No. 3 of the present paragraphs $y$ is exactly $= c\,Ln\left(\frac{c + x + \sqrt{2\,c\,x + x^2}}{c}\right)$, i. e. $3 = c\,Ln\left(\frac{c + 2 + \sqrt{4\,c + 4}}{c}\right)$. If $c$ be here put $= 2{,}58$, we then have the error $f = 3 - 2{,}58\,Ln\left(\frac{4{,}58 + 2\sqrt{3{,}58}}{2{,}58}\right)$ $= 3 - 2{,}58\,Ln\left(\frac{8{,}3642}{2{,}58}\right) = 3 - 3{,}035 = -0{,}035$; but if $c$ be put $= 2{,}53$, we then have the error $f_1 = 3 - 2{,}53\,Ln\left(\frac{4{,}53 + 2\sqrt{3{,}53}}{2{,}53}\right) = 3 - 2{,}53\,Ln\left(\frac{8{,}2876}{2{,}53}\right)$ $= 3 - 3{,}002 = -0{,}002$. In order now to find the true value of $c$; if according to a known rule we put

$\frac{c-2,58}{c-2,53}=\frac{f}{f_1}=\frac{0,035}{0,002}=17,5$; in this manner it will follow that:
$16,5 \,.\, c = 17,5 \,.\, 2,53 - 2,58 = 41,69$; therefore:

$$c = \frac{41,69}{16,5} = 2,527 \text{ feet.}$$

*Remark.* Practical applications of the catenary will be given when, in the Second Part, we come to treat of the construction of vaults, chain-bridges, &c.

§ 150. *The Pulley.*—Ropes, cords, &c., are the usual means by which forces are transmitted over the wheel and axle. We will here develop what is most general in the theories of these two arrangements, without, however, taking into account friction and rigidity.

A pulley is a circular disc, *ABC*, Fig. 154 and Fig. 155, turning

FIG. 154.

FIG. 155.

about an axis on whose circumference lies a cord or string, and whose extremities are stretched by the forces $P$ and $Q$. In a fixed pulley, the block in which the axis or pivot reposes is immoveable; in a free pulley, on the other hand, it is moveable.

In the condition of equilibrium of a pulley, the forces $P$ and $Q$ at the extremities of the string are equal; for every pulley is a bent lever, the arms of which are equal in length, which we may obtain if we let fall perpendiculars $CA$ and $CB$ from the axis $C$ on the directions of the forces, or of the strings $DP$ and $DQ$. It is clear that the forces $P$ and $Q$ in any revolution about $C$ describe the same space, viz. $r\phi$, if $r$ be the radius $CA = CB$ and $\varphi^0$ the angle of revolution; and that from this we may infer the equality between $P$ and $Q$. From the forces $P$ and $Q$ there arises the resultant $CR = R$, which is taken up by the block and is dependant on the angle $ADB = a$, which the directions of the string include;

and moreover it gives as the diagonal of the rhomb $CP_1\ RQ_1$ constructed from $P$ and $a$ : $R = 2\ P\ cos.\ \frac{a}{2}$.

§ 151. In the fixed pulley, Fig. 154, the force $Q$ consists of the weight to be overcome or raised at one extremity of the string; here, therefore, the force is equal to the weight, and the application of this pulley effects nothing but a change of direction. In the moveable pulley, Fig. 155, on the other hand, the weight on the hook $R$ acts at the extremity of the block, whilst the one extremity of the string is fastened to a fixed object; here, therefore, the force $P$ is to be put $= \frac{R}{2\ cos.\ \frac{a}{2}}$. If we represent the chord $AMB$, which corresponds to the arc over which the string passes, by $a$, the radius $CA = CB$, as before $= r$, then $a = 2\ AM = 2\ .\ CA\ cos.\ CAM = 2\ CA\ cos.\ ADM = 2\ r\ cos.\ \frac{a}{2}$, hence $\frac{r}{a}$ may be put $= \frac{1}{2\ cos.\ \frac{a}{2}}$, and likewise $\frac{P}{R} = \frac{r}{a}$. From this, therefore, *the power in the ~~fixed~~* moveable *pulley is to the weight as the radius of the pulley to the chord of the arc over which the string passes.*

FIG. 156.

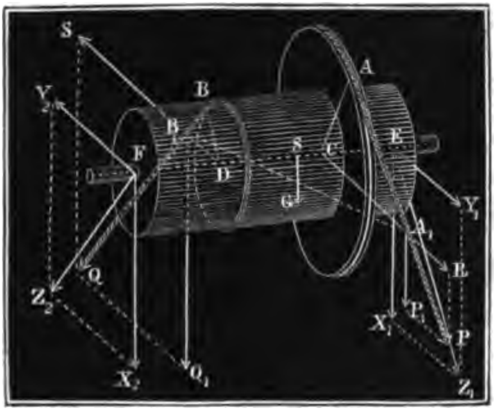
If $a = 2\ r$, the string passes over a semicircle, Fig. 156, the force then is at a minimum; viz. $P = \frac{1}{2} R$; if $a = r$, that is $60^0$ of the part of the pulley over which the string passes, we have $P = R$; the smaller, therefore, $a$ becomes, the greater is $P$, and for $a$ infinitely small, the force $P$ becomes infinitely great. An inverse proportion takes place in the spaces; if $s$ is the space of $P$, which corresponds to a space $R = h$, we have then $Ps = Rh$, therefore, $\frac{s}{h} = \frac{a}{r}$.

The moveable pulley is thus a means of modifying force; for example, a given weight may by this means be raised by a smaller force, but in proportion as there is gain in force, there is loss in space.

*Remark.* We shall treat of the composition of pulleys and systems of pulleys, as well as of the resistances arising from friction and rigidity, more fully in a subsequent Part.

§ 152. *The wheel and axle.*—The wheel and axle is a rigid connection of two fixed pullies or wheels, capable of revolving about a common axis $ABFE$, Fig. 157. The smaller of these wheels is called the axle, the greater one the wheel. The round extremities $E$ and $F$, on which this arrangement rests are called gudgeons. The axis of revolution of the wheel and axle is either horizontal, or vertical, or inclined. Here we shall only speak of the wheel and axle which revolves about a horizontal axis. We shall also here suppose, that the forces $P$ and $Q$, or the power $P$ and the weight $Q$ act at the extremities of a perfectly flexible string, which passes round the circumference of the wheel and axle. The questions to be answered are, in what relations the powers and weights are to each other, and what pressures the gudgeons $E$ and $F$ have to sustain?

FIG. 157.




Let us imagine a plane, a horizontal plane passed through the axis $CD$ and the points of application $A$ and $B$ of the power $P$, and the weight $Q$ transferred to this plane, and therefore $P$ and $Q$ applied at $A_1$ and $B_1$. If the angles $AA_1C$ and $BB_1D$, which both forces make with the horizon $= a$ and $\beta$, these forces may be replaced by the horizontal forces $R = P\ cos.\ a$, $S = Q\ cos.\ \beta$, and by the vertical forces $P_1 = P\ sin.\ a$, $Q_1 = Q\ sin.\ \beta$. The horizontal forces are directed towards the axis, and being applied at $C$ and $D$ become perfectly counteracted by the axis. The vertical forces $P_1$ and $Q_1$, on the other hand, tend to turn the wheel and axle about its axis. If $K$ be the intersection with the axis of the line connecting the points $A_1$ and $B_1$, $KA_1$ and $KB_1$ are the arms of $P_1$ and $Q_1$, and equilibrium subsists about $K$, and also about $CD$, if:

$$P_1 \,.\, KA_1 = Q_1 \,.\, KB_1, \text{ or, since } \frac{KA_1}{KB_1} = \frac{CA_1}{DB_1}, \text{ if}$$

$P_1 \,.\, CA_1 = Q_1 \,.\, DB_1$, or, as $\frac{P_1}{P} = \frac{CA}{CA_1}$, and

$\frac{Q_1}{Q} = \frac{DB}{DB_1}$,

$\frac{P \,.\, CA}{CA_1} \,.\, CA_1 = \frac{Q \,.\, DB}{DB_1} \,.\, DB_1$, *i. e.*

$P \,.\, CA = Q \,.\, DB$, or $Pa = Qb$,

if $a$ and $b$ represent the arms of the power and weight, or the radii of the wheel and axle. In the wheel and axle, therefore, as in every lever, the moment of the power is equivalent to the moment of the weight.

§ 153. The forces $P_1$ and $Q_1$ give at $K$ a vertical pressure $P_1 + Q_1$, with which must also be associated the weight $G$ of the whole wheel and axle applied at the centre of gravity $S$. The supports of the gudgeons at $E$ and $F$ have also to sustain the vertical pressure $P_1 + Q_1 + G = P \sin. \alpha + Q \sin. \beta + G$. If we put the whole length of the wheel and axle measured from $E$ to $F = L$, the part $EC = l_1$, $CD = l$, $DF = l_2$, therefore $L = l + l_1 + l_2$, and the distances $ES$ and $FS$ of the centre of gravity $S$ from the supports $d$ and $d$, therefore also $L = d_1 + d_2$, we shall obtain since

$$\frac{DK}{DC} = \frac{P_1}{P_1 + Q_1}, \text{ as } DK = \frac{P_1 l}{P_1 + Q_1}$$

for the vertical pressure $X_1$ at the gudgeon $E$:

$$X_1 \,.\, EF = G \,.\, FS + (P_1 + Q_1)\, FK,$$

$$X_1 = \frac{Gd_2 + (P_1 + Q_1)\left(l_2 + \frac{P_1}{P_1 + Q_1} \,.\, l\right)}{L}, \textit{ i. e.}$$

$$X_1 = \frac{Gd_2 + (P_1 + Q_1)\, l_2 + P_1 l}{L}.$$

On the other hand, for the vertical pressure $X_2$ at $F$:

$$X_2 \,.\, EF = G.\, ES + (P_1 + Q_1)\, EK, \textit{ i. e.}$$

$$X_2 = \frac{Gd_1 + (P_1 + Q_1)\left(l_1 + \frac{Q_1}{P_1 + Q_1} \,.\, l\right)}{L}, \textit{ i. e.}$$

$$X_2 = \frac{Gd_1 + (P_1 + Q_1)\, l_1 + Q_1 l}{L}.$$

The horizontal forces $R$ and $S$ have the moments about $F$, $R \,.\, FC = R\,(l + l_2)$, and $S \,.\, FD = S \,.\, l_2$, and about $E : S \,.\, ED = S\,(l + l_1)$, and $R \,.\, EC = Rl_1$; if, therefore, we

put the horizontal pressures upon $E$ and $F$ effected by them $= Y_1$ and $Y_2$, we shall obtain:

$$Y_1 . FE = R . FC - S . FD, \text{ as}$$
$$Y_1 = \frac{R\,(l + l_2) - S l_2}{L}, \text{ and}$$
$$Y_2 . FE = S . ED - R . EC, \text{ as}$$
$$Y_2 = \frac{S\,(l + l_1) - R l_1}{L}.$$

From $X_1$ and $Y_1$ the total pressure at $E$ is:

$Z_1 = \sqrt{X_1^2 + Y_1^2}$, and likewise from $X_2$ and $Y_2$, the same at $F$:

$$Z_2 = \sqrt{X_2^2 + Y_2^2}.$$

Lastly, if $\phi$ and $\psi$ be the angles which the directions of these pressures make with the horizon, we shall then have

$$tang.\ \phi = \frac{X_1}{Y_1} \text{ and } tang.\ \psi = \frac{X_2}{Y_2}.$$

*Example.* The weight $Q$ of a wheel and axle pulls perpendicularly downwards, and amounts to 365 lbs.; the radius of the wheel $a = 1\frac{3}{4}$ ft.; that of the axle $b = \frac{3}{4}$ ft.; the weight of the machine itself is 200 lbs.; its centre of gravity $S$ lies distant from $E$ and $F$, $d_1 = 1\frac{1}{2}$, and $d_2 = 2\frac{1}{2}$ ft.; the middle of the wheel is about $l_1 = \frac{3}{4}$ ft. from the gudgeon $E$, and the vertical plane in which the weight acts is about $l_2 = 2$ ft. from the gudgeon $F$. Now if the force $P$ necessary for restoring the equilibrium at the wheel inclined to the horizon at an angle $50^0 = a$, pulls downwards, what will this be, and what will be the pressures on the gudgeons? $Q = 365$, $\beta = 90°$, consequently $Q_1 = Q\ sin.\ \beta = Q$ and $S = Q\ cos.\ \beta = 0$; further, $P$ being unknown, and $a = 50°$, consequently $P_1 = P\ sin.\ a = 0{,}7660 . P$ and $R = P\ cos.\ a = 0{,}6428 . P$; but now $a = 1\frac{3}{4} = \frac{7}{4}$ and $b = \frac{3}{4}$, it follows, therefore, $P = \frac{b}{a}\,Q = \frac{3}{7} . 365 = 156{,}4$ lbs., $P_1 = 119{,}8$ and $R = 100{,}5$. Further, because $G = 200$, $d_1 = \frac{3}{2}$, $d_2 = \frac{5}{2}$, $l_1 = \frac{3}{4}$, $l_2 = 2$, $L = \frac{3}{2} + \frac{5}{2} = 4$, and $l = L - (l_1 + l_2) - 4\frac{11}{4} = \frac{5}{4}$, so that the vertical pressure at $E$ is:

$$X_1 = \frac{200 . \frac{5}{2} + (365 + 119{,}8) . 2 + 119{,}8 . \frac{5}{4}}{4} = \frac{1619{,}35}{4} = 404{,}8 \text{ lbs.},$$

and that at $F$:

$$X_2 = \frac{200 . \frac{3}{2} + (365 + 119{,}8) . \frac{3}{4} + 365 . \frac{5}{4}}{4} = \frac{1119{,}85}{4} = 280{,}0 \text{ lbs.}$$

Both of these forces together give:

$$X_1 + X_2 = Q + G + P_1 = 684{,}8 \text{ lbs.}$$

The horizontal force at $E$ is:

$$Y_1 = \frac{100{,}5 . (\frac{5}{4} + 2) - 0 . 2}{4} = 81{,}7 \text{ lbs., and that at } F:$$

$$Y_2 = \frac{0 . (\frac{5}{4} + \frac{3}{4}) - 100{,}5 . \frac{3}{4}}{4} = -18{,}8 \text{ lbs.}$$

the sum of these is exactly $= R + S = 100{,}5$ lbs.

The pressure at $E$ is inclined at any angle $\phi$ to the horizon, for which we have:

$$tang.\ \phi = \frac{X_1}{Y_1} = \frac{404,8}{81,7},\ Log.\ tang.\ \phi = 0,69502,\ \phi = 78^\circ\ 35'.$$

The pressure itself: $Z_1 = \frac{X_1}{sin.\ \phi} = 413,0$ lbs.

On the other hand, for the inclination $\psi$ of the pressure at $F$:

$$tang.\ \psi = \frac{X_2}{Y_2} = \frac{280,0}{18,8},\ Log.\ tang.\ \psi = 1,17300,\ \psi = 86^\circ,\ 9',5;$$

and the pressure: $Z_2 = \frac{Y_2}{cos.\ \psi} = 280,6$ lbs.

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# CHAPTER V.

## ON THE RESISTANCES OF FRICTION AND RIGIDITY.

§ 154. We have hitherto assumed that two bodies can only act upon each other by forces at right angles to the plane of contact. If the surfaces at the point of contact were perfectly mathematical, *i. e.* not interrupted by the smallest irregular elevations or depressions, this law would also be fully confirmed by experience; but because every body possesses a certain degree of elasticity or softness, and because the surface of every body, even if it is smoothed or polished in a high degree, has still some small elevations or indentures, and in consequence of the porosity of matter, no continuity; therefore, by the reciprocal action of two bodies in contact, reciprocal impressions and partial penetration of the parts takes place at the point of contact, by which an adhesion of the two bodies is caused, which can only be overcome by a distinct force, whose direction coincides with the plane of contact.

This adhesion, produced by the impression and partial penetration of the bodies in contact and the resistance on the plane of contact arising from it, has obtained the name of friction. Friction presents itself in the motion of bodies as a passive power or resistance, because it only impedes and checks motion, but never produces nor promotes it. It is introduced into investigations in mechanics as a force which is opposed to every motion, whose direction lies in the plane of contact of two bodies. In whatever direction we move forward a body resting on a horizontal or inclined plane, friction will always act opposite to the direction of motion; for example, it will impede the ascent as much as the

descent of a body on an inclined plane. The smallest addition of force produces motion in a system of forces in equilibrium, so long as friction is not called into action; but when the same exerts its effect, a greater addition of force, dependant on the friction, is required to disturb the equilibrium.

§ 155. On overcoming friction, the parts in contact are compressed, and those which protrude, bent down, torn, or broken off, &c. Friction is not only dependant on the roughness or smoothness of the surfaces in contact, but also on the physical properties of the bodies themselves. Hard metals, for instance, cause less friction than soft. We can, however, lay down no general rules *a priori* of the dependance of friction on the physical properties of bodies; it is, on the contrary, necessary to make experiments on friction with bodies of different substances in order to find out the friction which takes place under various circumstances between bodies of the same substance.

The unguents which are applied to the rubbing surfaces exert a particular influence upon the friction and on the abrasions arising from the contact of bodies. The pores are filled up and other asperities diminished, and in general, the further penetration of the bodies prevented by the fluid or semifluid unguents, such as oil, tallow, fat, soap, &c., for which reason these occasion a considerable diminution of friction.

Friction must not, however, be confounded with adhesion, *i. e.* with that holding together of two bodies, which takes place when they come into contact at many points without reciprocal pressure. Adhesion increases with the size of the surface in contact, and is independent of the pressure, whilst the contrary is the case with friction. If the pressure be slight, the adhesion will be considerable in proportion to the friction; but if the pressure be considerable, then it will constitute but a small part of the friction, and therefore generally may be neglected. Unguents, like all fluid bodies, increase the adhesion, because they increase the number of the points of contact.

§ 156. *Kinds of friction.*—Two kinds of friction are distinguishable, viz. the rolling and the sliding. Sliding friction is that kind of resistance which is given out when a body so moves that all its points describe parallel lines. Rolling friction, on the other hand, is that resistance which arises from rolling, *i. e.* that motion of a body which moves progressively and rolls at the same time, and whose point of contact describes as great a space upon the body in

motion as upon the body at rest. A body $M$ supporting itself upon the plane $HR$, Fig. 158, for instance, moves sliding over the plane, and consequently has to overcome sliding friction when its points $A$, $B$, $C$ describe parallel spaces $AA_1$, $BB_1$, $CC_1$, &c., and

FIG. 158.

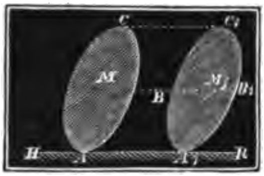


FIG 159.

therefore all these points of the moving body come into contact with others of the support. The body $M$, Fig. 159, on the other hand, rolls upon the plane $HR$, and has to overcome rolling friction, when the points $A$, $B$, &c., of the surface so move that the space $AB_1 = AB = A_1B_1$, likewise $AD = AE$, and $B_1E = B_1D_1$, &c.

The friction of axles is a particular kind of sliding friction, which arises when a cylindrical axle revolves in its bearing. We distinguish two kinds of axles, the gudgeon and the pivot. The gudgeon rubs against its support or envelop, whilst its other points always successively come into contact with the same points of the support. The pivot, on the other hand, presses with its circular base against its support, where its points revolve in concentric circles.

Further, particular frictions arise when a body oscillates upon a sharp edge, as in the balance, or when a vibrating body reposes upon a point, as in the magnetic needle.

Lastly, we distinguish the friction of quiescence which is to be overcome, when a body at rest is put into motion, from the friction of motion which opposes itself to the transmission of motion.

§ 157. *Laws of friction.*—The general laws to which friction is subject, are the following:

1. Friction is proportional to the normal pressure between the rubbing bodies. If a body be pressed against another by a double force, the friction is as great again; three times the pressure gives three times the friction. If in small pressures this law varies from observation, it must be attributed to the proportionately greater effect of adhesion.

2. Friction is independent of the extent of the surfaces of contact. The greater the surfaces are, the greater is the number

of parts which rub against each other; the smaller the pressure, the less the friction of each part; the sum of the frictions of all the parts is the same for a greater as for a less surface, in so far as the pressure and the other circumstances remain the same. If the side surfaces of a parallelepipedical brick are of the same quality, the force necessary to push it along a horizontal plane is the same, whether it rest upon the least, the mean, or the greatest surface. With very large side surfaces and with small pressures, this law has exceptions, in consequence of the effect of adhesion.

3. The friction of quiescence is indeed generally greater than that of motion; the last, however, is independent of the velocity; it is the same in small as in great velocities.

4. The friction of greased surfaces is generally less than that of ungreased, and depends less on the rubbing bodies than on the unguents.

5. The friction of gudgeons revolving on their bearings is less than the common sliding friction; the friction of rolling is in most cases so small, that it need hardly be taken into account in comparison with the sliding friction.

§ 158. *Co-efficient of friction.*—From the first law laid down in the former paragraph, the following may be deduced. A body $AC$, Fig. 160, presses against its support, first with the force $N$, and requires to draw it along, *i. e.* to overcome its friction, the exertion of a certain force $F$, and secondly with the force $N_1$, and requires the force $F_1$ to cause it to pass from a state of rest into one of motion. From the foregoing we have:

FIG. 160.


$$\frac{F}{F_1} = \frac{N}{N_1}, \text{ and therefore } F = \frac{F_1}{N_1} \cdot N.$$

If by experiment we have found the friction $F_1$ corresponding to a certain pressure $N_1$, we hence find, if the rubbing bodies, and the other circumstances are the same, the friction $F$ corresponding to another pressure $N$ when we multiply this pressure by the ratio $\left(\frac{F_1}{N_1}\right)$ of the values $F_1$ and $N_1$ corresponding to the first observation.

The ratio of the friction to the pressure, or the friction for a pressure = unity, a pound, for instance, is called the *co-efficient*

*of friction*, and will in the sequel be expressed by $f$, wherefore we may generally put $F = f \,.\, N$.

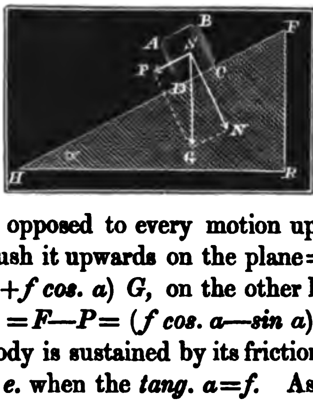
The co-efficient of friction is different for different substances and different conditions of friction, and must therefore be found out by experiment for each particular case.

When a body $AC$ is drawn a distance $s$ over a surface, there is a mechanical effect $Fs$ to perform; the mechanical effect or work required to overcome friction is, therefore, $fNs$, equal to the product of the co-efficient of friction, the normal pressure, and the distance along the plane of contact. When the plane is also moving, we must then understand by $s$ the relative distance.

*Example*—1. If by a pressure of 260 lbs. the friction amounts to 91 lbs., the corresponding co-efficient of friction is $f = \frac{91}{260} = \frac{7}{20} = 0{,}35$.—2. To draw a 500 lbs. heavy sledge along a horizontal and very smooth surface of snow, the co-efficient of friction is $f = 0{,}04$, the required force $F = 0{,}04 \,.\, 500 = 20$ lbs.—3. If the co-efficient of friction of a cart drawn over a paved road is 0,45 and the load amounts to 500 lbs. the mechanical effect required to draw it 480 feet is $= fNs = 0{,}45 \,.\, 500 \,.\, 480 = 108000$ ft. lbs.

§ 159. *The angle of friction and the cone of friction.*—A body $AC$, Fig. 161, lies on an inclined plane $FH$, whose angle of inclination $FHR = a$, its weight $G$ resolves itself into the normal pressure $N = G\ cos.\ a$ and into the parallel force $P = G\ sin.\ a$. From the first force there arises the friction $F = f\,G\ cos.\ a$, which is opposed to every motion upon the plane, wherefore the force to push it upwards on the plane $= F + P = f\,G\ cos.\ a + G\ sin.\ a = (sin.\ a + f\ cos.\ a)\ G$, on the other hand, the force to push it downwards is $= F - P = (f\ cos.\ a - sin\ a)\ G$; the last force is null, *i. e.* the body is sustained by its friction on the plane, when $sin.\ a = f\ cos.\ a$, *i. e.* when the $tang.\ a = f$. As long as the inclined plane has an angle of inclination, whose tangent is less than $f$, the body remains at rest on the plane, but when the tangent of this angle is a little greater than $f$, the body immediately begins to slide down. The angle, whose tangent is equal to the co-efficient of friction, is called the angle of friction or the angle of repose. The co-efficient of friction is given by observing the angle of friction $\rho$, (for the friction of repose) when $f$ is put $= tang.\ \rho$.

FIG. 161.




In consequence of friction, the surface $FH$, Fig. 162, reacts not

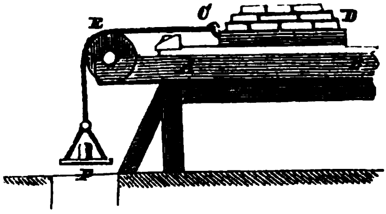
FIG. 162.

only against the normal pressure $N$ of another body $AB$; but also against its oblique pressure $P$, when the deviation $NBP = \phi$ of the direction of this pressure from the normal $BN$ does not exceed the angle of friction, for since the force $P$ gives the normal pressure $BN = P \cdot cos. \phi$, and the lateral or tangential pressure $BS = S = P\ sin.\ \phi$, and there arises from the normal pressure $P\ cos.\ \phi$ the friction $f\,P\ cos.\ \phi$ opposed to every motion in the plane $FH$, $S$ will therefore be unable to give rise to motion, and will remain in equilibrium so long as $f\,P\ cos.\ \phi > P\ sin.\ \phi$, or $f\ cos.\ \phi > sin.\ \phi$ *i. e.* $tang.\ \phi$ is $< f$, or $\phi < \rho$. If the angle of repose $CBD = \rho$ be made to revolve about the normal $CB$, it will describe a cone, which we may call the cone of friction or resistance. The cone of resistance includes all those directions of force by which a perfect counteraction of the oblique pressure takes place.

*Example.* To draw a filled and 200 lbs. heavy cask up an inclined wooden plane of 50°, the force required with a co-efficient of friction $f = 0{,}48$ is $= P = (f\ cos.\ a + sin.\ a)\ G = (0{,}48\ cos.\ 50^\circ + sin.\ 50^\circ) \cdot 200 = (0{,}308 + 0{,}766) \cdot 200 = 215$ lbs.; to let it down, or to prevent its sliding down, the force required, on the other hand, is: $P = (f\,cos.\ a - sin.\ a)\ G = -(sin.\ 50^\circ - 0{,}48 \cdot cos.\ 50^\circ)\ 200 = -(0{,}766 - 0{,}308) \cdot 200 = -91{,}5$ lbs.

§ 160. *Experiments on friction.*—Experiments on friction have been made by many philosophers, the most extensive of which and on the greatest scale, are those of *Coulomb* and *Morin*. To find out the co-efficients of friction for sliding motion, these two made use of a sledge sliding on a horizontal surface, which was pulled forward by a cord, passing over a fixed pulley, from which weights were suspended as in Fig. 163, where $AB$ represents the way, $CD$ the sledge, $E$ the pulley, and $G$ the weight. To obtain the co-efficient of friction for different substances, the surfaces in contact, not only of the sledge, but also of the way forming the support, were covered with the smoothest possible pieces of the substances under experiment, such as wood, iron, &c., &c. The co-efficients of the friction of repose were given by

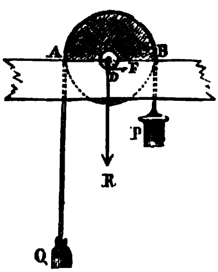
FIG. 163.



the weight which was necessary to cause the sledge to pass from a state of rest into one of motion, and the co-efficient of the friction of motion by the time $t$, which the sledge required to pass over a certain space $s$. If $G$ be the weight of the sledge and $P$ the weight required to draw it, we have the friction $= f\,G$, the motive force $= P - f\,G$, and the mass $M = \frac{P+G}{g}$, it therefore follows from § 65, that the acceleration of the uniformly accelerated motion arising, is: $p = \frac{P - f\,G}{P+G}\,g$, and inversely, the co-efficient of friction $f = \frac{P}{G} - \frac{P+G}{G} \cdot \frac{p}{g}$. But $s = \frac{1}{2}\,pt^2$ (§ 11), therefore, $p = \frac{2s}{t^2}$, and $f = \frac{P}{G} - \frac{P+G}{G} \cdot \frac{2s}{gt^2}$.

To measure the co-efficient of friction, for axle friction, a fixed pulley $ACB$, Fig. 164, is made use of, over which a cord passes, which is stretched by the weights $P$ and $Q$. From the sum of the weights, the pressure $P+Q$ is given, and from their difference $P-Q$ the force at the circumference of the pulley, which is in equilibrium with the friction of the axle, $F = f\,(P+Q)$, if now $CA = a$ the radius of the pulley, and $CD = r$ that of the axle, we have from the equality of moments $(P - Q)\,a = Fr = f\,(P+Q)\,r$, and, therefore, the friction of repose;

FIG. 164.




$f = \frac{P-Q}{P+Q} \cdot \frac{a}{r}$, on the other hand, for that of motion, if the weight $P$ falls a space $s$ in the time ($t$), and $Q$ rises as much, $f = \left(\frac{P-Q}{P+Q} - \frac{2s}{gt^2}\right)\frac{a}{r}$.

*Remark.* Before Coulomb, Amontons, Camus, Bülffinger, Muschenbroek, Ferguson, Vince, and others turned their attention to and made experiments on friction. The results of all these investigations are of little value in practice, because they were conducted upon too small a scale. The experiments of Ximenes, which were made about the same time as those of Coulomb, also fail in this respect. The results are to be found in a work, "Teoria e Pratica delle resistenze de' solidi ne' loro attriti," Pisa, 1782. The experiments of Coulomb are fully described in his work, "Théorie des Machines simples," 1821. The latest experiments upon friction are those of Rennie and Morin. Rennie used for his experiments partly, a sledge upon a horizontal surface, and partly upon an inclined plane, from which the bodies were allowed to slide down, and by which the amount of the friction was deduced from the angle of friction. Rennie's experiments extend to substances of various kinds met with in practice, as cloth, leather, wood, stones, and metals; they give important results

upon the abrasion of bodies, but from the apparatus and the mode of conducting these experiments, we cannot rely upon them for that accuracy which those of Morin appear to have attained. The experiments of Rennie are to be found in the "Philosophical Transactions" of 1818. The most extensive experiments, and promising a high degree of accuracy, have been completed by Morin, although it cannot be denied that they leave some doubts and uncertainties, and somewhat to be desired. This is not the place to describe the methods and apparatus of these experiments, we can only refer to the author's writings, "Nouvelles Expériences sur le Frottement," par Morin. An excellent article on Friction, and a full description of all the experiments upon it, especially those of Morin, is given by Brix in the "Transactions of the Society for the promotion of Manufacturing Industry in Prussia, 16 and 17 Jahrgang, Berlin, 1837—8.

§ 161. The following tables contain a condensed summary of the *co-efficients of friction* the most useful in practice.

TABLE I.

CO-EFFICIENTS OF THE FRICTION OF REPOSE.

| NAMES OF BODIES. | | Nature of the surfaces and unguents. | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | Dry. | Damped with water. | With olive oil. | Lard. | Tallow. | Dry soap. | Polished and greasy. | Greasy and wetted. |
| Wood upon wood | least, | 0,30 | 0,65 | — | — | 0,14 | 0,22 | 0,30 | — |
| | mean, | 0,50 | 0,68 | — | 0,21 | 0,19 | 0,36 | 0,35 | — |
| | greatest values. | 0,70 | 0,71 | — | — | 0,25 | 0,44 | 0,40 | — |
| Metal upon metal | least, | 0,15 | — | 0,11 | | | | | |
| | mean, | 0,18 | — | 0,12 | 0,10 | 0,11 | — | 0,15 | — |
| | greatest values. | 0,24 | — | 0,16 | | | | | |
| Wood upon metal | | 0,60 | 0,65 | 0,10 | 0,12 | 0,12 | — | 0,10 | — |
| Hempen ropes, twisted or matted, upon wood | least, | 0,50 | | | | | | | |
| | mean, | 0,63 | 0,87 | | | | | | |
| | greatest values. | 0,80 | | | | | | | |
| Thick sole leather, upon wood or iron | high at the edges, | 0,43 | 0,62 | 0,12 | — | — | — | — | 0,27 |
| | flat or smooth | 0,62 | 0,80 | 0,13 | | | | | |
| Black strap leather, upon pulleys | of wood, | 0,47 | — | — | — | — | — | 0,28 | 0,38 |
| | of iron. | 0,54 | | | | | | | |
| Stones or bricks upon stones or bricks, smooth worked | least, | 0,67 | | | | | | | |
| | greatest value. | 0,75 | | | | | | | |
| Stones and wrought iron | least, | 0,42 | | | | | | | |
| | greatest value. | 0,49 | | | | | | | |
| Oak upon muschekalk | | 0,64 | | | | | | | |

## TABLE II.

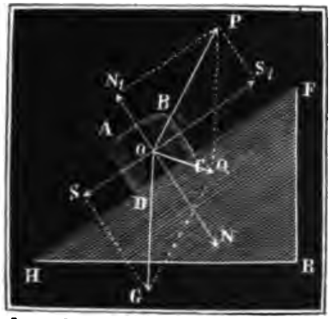
### CO-EFFICIENTS OF THE FRICTION OF MOTION.

| NAMES OF BODIES. | | Nature of the surfaces and unguents. | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | Dry. | Water. | Olive oil. | Lard. | Tallow. | Lard and black-lead. | Polished and greasy. | Dry soap. | Greasy and wetted. |
| Wood upon wood . | least, | 0,20 | — | — | 0,06 | 0,06 | — | — | 0,14 | 0,08 |
| | mean, | 0,36 | 0,25 | — | 0,07 | 0,07 | — | — | 0,15 | 0,12 |
| | greatest value. | 0,48 | — | — | 0,07 | 0,08 | — | — | 0,16 | 0,15 |
| Metal upon metal . | least, | 0,15 | — | 0,06 | 0,07 | 0,07 | 0,06 | 0,12 | — | 0,11 |
| | mean, | 0,18 | 0,31 | 0,07 | 0,09 | 0,09 | 0,08 | 0,15 | 0,20 | 0,13 |
| | greatest value. | 0,24 | — | 0,08 | 0,11 | 0,11 | 0,09 | 0,17 | — | 0,17 |
| Wood upon metal . | least, | 0,20 | — | 0,05 | 0,07 | 0,06 | — | — | — | 0,10 |
| | mean, | 0,42 | 0,24 | 0,06 | 0,07 | 0,08 | 0,08 | 0,10 | 0,20 | 0,14 |
| | greatest value. | 0,62 | — | 0,08 | 0,08 | 0,10 | — | — | — | 0,16 |
| Hemp, cords, twists, &c. | on wood, | 0,45 | 0,33 | | | | | | | |
| | on iron. | — | — | 0,15 | — | 0,19 | | | | |
| Sole leather, smooth, upon wood or metal. | raw, | 0,54 | 0,36 | 0,16 | — | 0,20 | | | | |
| | compressed, | 0,30 | — | | | | | | | |
| | greasy. | — | 0,25 | | | | | | | |
| The same, high at the edges, &c. | dry, | 0,34 | 0,31 | 0,14 | — | 0,14 | | | | |
| | greasy. | — | 0,24 | | | | | | | |

*Remark.* The co-efficients of friction for porous masses will be given in the Second Part, in the theory of the pressure of earth.

§ 162. *Inclined Plane.*—The theory of sliding friction has its chief application in the investigation of the equilibrium of a body $AC$, on an inclined plane $FH$, Fig. 165. If in accordance with § 135, $FHR = \alpha$, the angle of inclination of the inclined plane, and $POS_1 = \beta$, the angle which the force $P$ makes with the inclined plane, we have the normal force arising from the weight $G$ of the body, $N = G \cos. \alpha$, on the other hand, the force for sliding down $= S = G \sin. \alpha$, further the force $N_1$ with which

FIG. 165.



$P$ strives to draw the body ~~down~~ the plane is $= P\ sin.\ \beta$, and the force $S_1$ with which it pushes the body up the plane $= P\ cos.\ \beta$. The remaining normal pressure is: $N - N_1 = G\ cos.\ \alpha - P\ sin.\ \beta$, consequently the friction $F = f\ (G\ cos.\ \alpha - P\ sin.\ \beta)$. If it be required to find the force $P$ drawing the body up the plane, then there will be friction to overcome, and it must therefore be $S_1 = S + F$, *i. e.*

$$P\ cos.\ \beta = G\ sin.\ \alpha + f\ (G\ cos\ \alpha - P\ sin.\ \beta).$$

But if the force which is to prevent the body from sliding down is to be determined, then friction comes to its assistance, and the force is:

$$S_1 + F = S,\ i.\ e.\ P\ cos.\ \beta + f\ (G\ cos.\ \alpha - P\ sin.\ \beta) = G\ sin.\ \alpha.$$

From this the force may be determined:

$$\text{For the first case: } P = \frac{sin.\ \alpha + f\ cos.\ \alpha}{cos.\ \beta + f\ sin.\ \beta} \cdot G,$$

$$\text{For the second: } P = \frac{sin.\ \alpha - f\ cos.\ \alpha}{cos.\ \beta - f\ sin.\ \beta} \cdot G.$$

If the angle of friction $\rho$ be introduced, whilst we put $f = tang.\ \rho = \frac{sin.\ \rho}{cos.\ \rho}$, we shall obtain $P = \frac{sin.\ \alpha\ .\ cos.\ \rho \pm cos.\ \alpha\ .\ sin.\ \rho}{sin.\ \beta\ .\ cos.\ \rho \pm cos.\ \beta\ .\ sin.\ \rho} \cdot G$, or from the known rules of trigonometry: $P = \frac{sin.\ (\alpha \pm \rho)}{cos.\ (\beta \mp \rho)} \cdot G$, and the upper signs are to be taken, when motion is to be brought about; the lower, on the other hand, when motion is to be impeded.

The last formula is found by a simple application of the parallelogram of forces. Since a body counteracts that force of another body, which deviates by the angle of friction $\rho$ from the normal to its surface (§ 159), equilibrium in the foregoing case can subsist if the resultant $OQ = Q$ of the components $P$ and $G$ makes with the normal $ON$ the angle $NOQ = \rho$. If now we put in the general formula $\frac{P}{G} = \frac{sin.\ GOQ}{sin.\ POQ}$, $GOK = GON + NOQ = \alpha + \rho$, and $POQ = POS_1 + S_1OQ = \beta + 90^0 - \rho$, we then have

$\frac{P}{G} = \frac{\sin. (a+\rho)}{\sin. \left(\beta - \rho + \frac{\pi}{2}\right)} = \frac{\sin. (a+\rho)}{\cos. (\beta - \rho)}$, and for a negative value of $\rho$:

$\frac{P}{G} = \frac{\sin. (a-\rho)}{\cos. (\beta+\rho)}$, quite in accordance with the above.

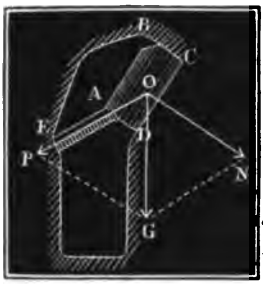
If the body reposes on a horizontal plane $a = o$, therefore, the force to push $P$ forward is: $P = \frac{f G}{\cos. \beta + f \sin. \beta} = \frac{G \sin. \rho}{\cos. (\beta - \rho)}$.

If the force acts parallel to the inclined plane then $\beta = o$, and therefore, $P = (\sin. a \pm f \cos. a)\ G, = \frac{(\sin.(a\pm\rho))}{\cos. \rho} . G.$ (compare § 159). If the force acts horizontally $\beta = -a$; $\cos. \beta = \cos. a$ and $\sin. \beta = -\sin. a$, therefore, $P = \frac{(\sin. a \pm f \cos. a}{\cos. a \mp f \sin. a} . G$ $= \frac{tang. a \pm f}{1 \mp f tang. a} . G$, also $= tang. (a \pm \rho)\ G$.

Again, the force to push a body upwards is least when the denominator $\cos. (\beta - \rho)$ is greatest, viz. $= 1$, therefore, $\beta - \rho = o$, *i. e.* $\beta = \rho$. When, therefore, the direction of force deviates by the angle of friction from the inclined plane, the force itself is the least and $= \sin. (a+\rho) . G$.

*Example.* What pressure on the axis has the prop $AE$, Fig. 166, to sustain, in order to prevent a block of stone (a wall) $ABCD$, of 5000 lbs. weight from slipping down the inclined plane $CD$, supposing the angle of the prop to the horizon to be 35°, that of the inclined plane $CD$, 50°, and the co-efficient of friction $f = 0,75$? Here $G = 5000$, $a = 50°$, $\beta = 35° - 50° = -15°$, and $f = 0,75$; therefore the formula gives:

FIG. 166.

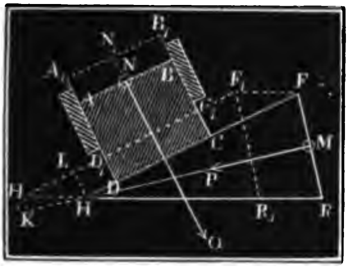



$$P = \frac{\sin. a - f \cos. a}{\cos. \beta - f \sin. \beta} . G = \frac{\sin. 50° - 0,75 \cos. 50°}{\cos. 15° + 0,75 \sin. 15°} . 5000$$
$$= \frac{0,766 - 0,482}{0,966 + 0,194} . 5000 = \frac{1420}{1,160} = 1224 \text{ lbs.}$$

If the prop were horizontal, we should have $\beta = -50°$, and $tang. \rho = 0,75$; hence $\rho = 36° 52'$; lastly, $P = G\ tang. (a - \rho) = 5000\ tang. (50° - 36° 52') = 5000\ tang. 13° 8' = 5000 . 0,2333 = 1166$ lbs. To push up the same wall upon the supporting one by a horizontal force, under otherwise similar circumstances a force $P$ would be necessary $= G\ tang. (a + \rho) = 5000\ tang. 86° 52' = 5000 . 18,2676 = 91338$ lbs.

§ 163. *Wedge.*—In the wedge, friction exerts a considerable influence upon the statical relations. The section of a wedge forms an isosceles triangle $FHR$, Fig. 167, with the edge $FHR = a$, the force $P$ acts at right angles to the back and the weight $Q$ at

FIG. 167.




right angles to the side $FH$. If we drive the wedge upon the base $HR$ a space $s = FF_1 = HH_1 = RR_1$, the weight $Q$ is raised through a space $CC_1 = DD_1 = HL = HH_1 \,.\, sin.\, HH_1 L = s\, sin.\, a$, and the force passes over $HK = HH_1 . cos.\, H_1 HK = s\, cos.\, \frac{a}{2}$; according to the principle of virtual velocities, and without regard to friction, $P \,.\, HK = Q \,.\, DD_1$, *i. e.* $P s\ cos.\ \frac{a}{2} = Qs\, sin.\, a$, therefore $P = \frac{Q\ sin.\ a}{cos.\ \frac{a}{2}}$

$$= \frac{2\, Q\, sin.\, \frac{a}{2}\, cos.\, \frac{a}{2}}{cos.\, \frac{a}{2}} = 2\ Q\ sin.\, \frac{a}{2},$$ which also follows from the formula in § 187, if we put in it $sin.\ \beta = 1$, and $cos.\ (a - \delta) = cos.\, \frac{a}{2}$.

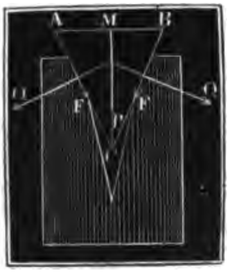
There are now, however, three frictions which come into play, viz. the friction against the sides $HF$ and $HR$, and the friction of the body $ABCD$ in its constrained motion. As the directions of the force on both sides of the wedge deviate equally, the pressure against both is equal, namely $= Q$, and the friction arising $= f\ Q$. The spaces of these frictions, however, are different. For the friction upon $HR$, $s = HH_1$, for that upon $HF = H_1 L = s\ cos.\ a$; accordingly the mechanical effects of both frictions are :$= f\ Q\ s + f\ Q\ s\ cos.\ a = f\ Q\ s\ (1 + cos.\ a) = 2\ f\ Q\ s \left(cos.\ \frac{a}{2}\right)^2$. Lastly, the friction between $CD$ and $FH$ presses upon the body $ABCD$ at right angles to its direction, and there produces the friction $f_1 . fQ$, if $f_1$ represent the co-efficient of friction for its constrained motion. This friction, however, has the same space as the weight $Q$, viz. $DD_1 = s\ sin.\ a$; and to it corresponds the mechanical effect $f_1 f Q s\, sin.\ a$. In order now to find the extreme limits of the condition of equilibrium, we must put the mechanical effect of the force $P$ equal to that of the weight $Q$, plus the mechanical effects of the friction, therefore,

$$Ps\ cos.\, \frac{a}{2} = Qs\ sin.\ a + 2\ Qfs \left(cos.\ \frac{a}{2}\right)^2 + ff_1\ Q s\,.\, sin.\ a,$$

and we obtain the force :

$$P = 2\,Q\left(sin.\,\frac{a}{2} + f\,cos.\,\frac{a}{2} + ff_1\,sin.\,\frac{a}{2}\right).$$

FIG. 168.



In a wedge $ABC$, Fig. 168, as it is used for the splitting asunder and compression of bodies, the force at the back corresponding to the normal pressure $Q$ against the sides $AC$ and $BC$, is $P = 2\,Q\left(sin.\,\frac{a}{2} + f\,cos.\,\frac{a}{2}\right)$, which is given if we put the sum of the vertical components of $Q$ and $F = fQ$, *i. e.* $2\,V_1 = 2\,Q\,sin.\,\frac{a}{2}$ and $2\,V_2 = 2\,f\,Q\,cos.\,\frac{a}{2}$ equivalent to the force $P$.

*Example.* The load of the wedge $Q$ in Fig. 167 = 650 lbs., the edge $a$ = 25°, the co-efficient of friction $f_1 = f = 0{,}36$. Required, the mechanical effect necessary to move the load $Q$ forward about ½ a foot.

The force is $P = 2\,.\,650\,[sin.\,12\tfrac{1}{2}° + 0{,}36\,cos.\,12\tfrac{1}{2}° + (0{,}36)^2\,sin.\,12\tfrac{1}{2}°]$
$= 1300\,.\,(0{,}2164 + 0{,}36\,.\,0{,}9763 + 0{,}1296\,.\,0{,}2164)$
$= 1300\,.\,(0{,}2164 + 0{,}3515 + 0.0281) = 1300\,.\,0{,}5960 = 774{,}8$ lbs. For, to the space of the load $CC_1$ = ½ foot, corresponds the space of the force $HK = s = \frac{CC_1}{sin.\,a}\,.\,cos.\,\frac{a}{2}$ $= \frac{CC_1}{2\,sin.\,\frac{a}{2}} = \frac{1}{4\,.\,0{,}2164} = 1{,}155$ feet; therefore the mechanical effect and weight is $Ps = 774{,}8\,.\,1{,}155 = 895$ ft. lbs. Without regard to friction, it would only be $650\,.\,\tfrac{1}{2} = 325$ ft. lbs. In consequence of friction, the mechanical effect expended would be nearly tripled.

§ 164. *Axle friction.*—In axles, the friction of motion only is of importance, on which account experiments on this only exist.

From the following table very important results for practice may be drawn, with axles of wrought or cast iron, moving in bearings of cast iron or brass, coated with oil, tallow or hog's lard, the co-efficient of friction is

By continuous greasing = 0,054,
In the usual manner = 0,070 to 0,080.

The values found by *Coulomb* vary partially from the annexed.

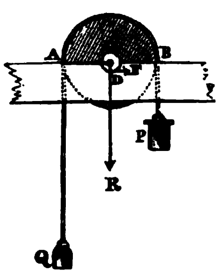
## TABLE III.

### CO-EFFICIENTS OF AXLE FRICTION, FROM MORIN.

| NAMES OF THE BODIES. | Nature of the surfaces and unguents. | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | Dry or a little greasy. | Greasy and wetted with water. | Greased and wetted with water. | Oil, tallow, or lard. In the usual way. | Oil, tallow, or lard. Continuously. | Very soft and purified carriage grease. | Hog's lard with plumbago. | Fatty. |
| Bell metal on the same | — | — | — | 0,097 | — | — | — | — |
| Cast iron upon bell metal . . . . . | — | — | — | — | 0,049 | | | |
| Wrought iron upon bell metal . . . | 0,251 | 0,189 | — | 0,075 | 0,054 | 0,090 | 0,111 | — |
| Wrought iron upon cast iron . . . . | — | — | — | 0,075 | 0,054 | — | — | — |
| Cast iron upon cast iron . . . . . | — | 0,137 | 0,079 | 0,075 | 0,054 | — | — | 0,137 |
| Cast iron upon bell metal . . . . . | 0,194 | 0,161 | — | 0,075 | 0,054 | 0,065 | — | 0,166 |
| Wrought iron upon lignum vitæ . . . | 0,188 | — | — | 0,125 | | | | |
| Cast iron upon lignum vitæ . . . . | 0,185 | — | — | 0,100 | 0,092 | — | 0,109 | 0,140 |
| Lignum vitæ upon cast iron . . . . | — | — | — | 0,116 | — | — | — | 0,153 |
| Lignum vitæ upon lignum vitæ . . . . | — | — | — | — | 0,070 | | | |

§ 165. If we know the pressure $R$ between an axle and its bearing, and if further the radius $r$ of the axle, Fig. 169, be given, the mechanical effect which the friction of the axle counteracts in every revolution may be calculated. The friction $F = fR$, the space corresponding to it, the circumference $2\pi r$ of the axle; it therefore follows that the mechanical effect lost by friction in each revolution is $= fR \cdot 2\pi r = 2\pi f R r$. If the axle makes ~~one~~ $u$ revolutions per minute $u$, the mechanical effect expended in each second

FIG. 169.



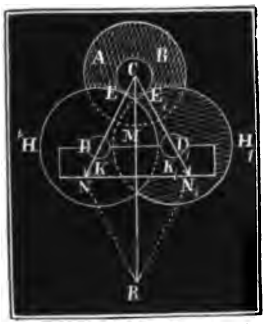
$$= 2\pi f R r \cdot \frac{u}{60} = \frac{\pi u f R r}{30} = 0{,}105 \, . u f R r.$$

The mechanical effect consumed by friction increases, therefore,

with the pressure on the axle, in proportion to the radius of the axle and the number of revolutions. It is, therefore, a rule in practice, not to augment unnecessarily the pressure on the axis in rotating machines by heavy weights, to make the axles no stronger than the solidity required for durability, and likewise not to make a great many revolutions in a minute, at least, not unless other circumstances require it.

By the application of friction wheels, which are substituted for the bearings, the mechanical effect of friction is much diminished. In Fig. 170, $AB$ is a wheel which reposes by its axle $CEE_1$ on the circumferences $EH$, $E_1H_1$ lying close to each other of the wheels (friction wheels) revolving about $D$ and $D_1$. From the given pressure $R$ of the wheel, there follow the pressures $N = N_1 = \frac{R}{2\cos.\frac{a}{2}}$, if $a$ be the angle $DCD_1$ which the central or lines of pressure $CD$ and $CD_1$ make between them. From the rolling friction between the axle $C$ and the circumferences of the wheels, these latter revolve with the axle, and there arise at the bearings $D$ and $D_1$ the frictions $fN$ and $fN_1$, which together amount to $\frac{fR}{\cos.\frac{a}{2}}$. If the radius of the wheel $DE = D_1E_1$ be represented by $a_1$, and that of the axle $DK = D_1K_1$ by $r_1$, we shall have the force at the circumference of the wheels, or at the circumference of the axle $C$ resting upon these, which is requisite to overcome $\frac{fR}{\cos.\frac{a}{2}}$; $F_1 = \frac{r_1}{a_1} \cdot \frac{fR}{\cos.\frac{a}{2}}$, whilst it will be $= fR$, if the axle $C$ rest immediately in a socket.

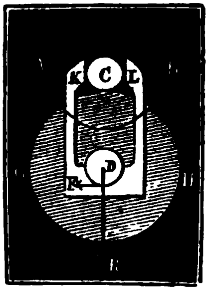
FIG. 170.




If we disregard the weights of the friction wheels, the mechanical effect of the friction by the application of these wheels is $\frac{r_1}{a_1 \cos.\frac{a}{2}}$ times as great as without them.

If we oppose to the pressure of the axle $R$ a single friction wheel $GH$, Fig. 171, and prevent any accidental lateral forces, by the fixed

FIG. 171.

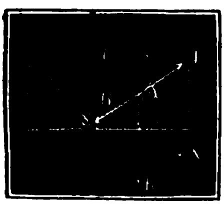


cheeks $K$ and $L$, $a = o$, $cos.\ \frac{a}{2} = 1$, and the above relations $= \frac{r_1}{a_1}$.

*Example.* A wheel weighs 30000 lbs., its radius $a = 16$ ft. and that of its axle $r = 5$ inches, what is the amount of force at the circumference of the wheel necessary to overcome the friction of the axle, and to maintain it in uniform motion, and what is the corresponding expenditure of mechanical effect if it makes 5 revolutions a minute? We may assume the co-efficient of friction $f$ here $= 0{,}075$, wherefore the friction $fR = 0{,}075 \cdot 30000 = 2250$ lbs. Since the diameter of the wheel is $\frac{16 \cdot 12}{5} = \frac{192}{5} = 38{,}4$ times as great as the diameter of the axle or the arm of the friction, the axle friction reduced to the circumference of the wheel $= \frac{fR}{38{,}4} = \frac{2250}{38{,}4} = 58{,}59$ lbs. The circumference of the axle is $\frac{2 \cdot 5 \cdot \pi}{12}$ $= 2{,}618$ feet; consequently the path of the friction in one second $= \frac{2{,}618 \cdot 5}{60}$ $= 0{,}2182$ feet, and its mechanical effect during one second $= 0{,}2182 \cdot fR = 0{,}2182 \cdot 2250$ $= 491$ ft. lbs. If the axles of this wheel rest upon friction wheels, whose radii are 5 times as great as those of the axle, and therefore $\frac{r_1}{a_1} = \frac{1}{5}$, the power at the circumference of the wheel will only be $\frac{1}{5} \cdot 38{,}4 = 7{,}68$ ft. lbs., and the mechanical effect of friction expended only $\frac{491}{5} = 98{,}2$ ft. lbs.

§ 166. The friction of an axle $ACB$, Fig. 172, which presses on its bearing in one point $A$ only, is less than that of a new axle resting on all points of the bearing. If no revolution takes place, the axle then presses on the point $B$, through which passes the direction of the mean pressure $R$; but if revolution begins in the direction $AB$, the axle by its friction will rise just so high in its bearing until the sliding force comes into equilibrium with the friction. The mean pressure $R$ is decomposed into a normal pressure $N$ and a tangential $S$; $N$ passes into the bearing and gives rise to $F = fN$, acting tangentially; $S$ puts itself in equilibrium with $F$; $S$ is therefore $= fN$. According to the Pythagorean doctrine, $R^2$ is $= N^2 + S^2$, therefore $R^2$ is here $= (1 + f^2)\, N^2$; inversely, the normal pressure $N = \frac{R}{\sqrt{1 + f^2}}$ and the friction $F = \frac{fR}{\sqrt{1 + f^2}}$; or, if the angle of friction $\rho$ be introduced, $f = tang.\ \rho$;

FIG. 172.



$$F = \frac{tang.\ \rho}{\sqrt{1 + tang.\ \rho^2}} \cdot R = tang.\ \rho\ cos.\ \rho\ R = R\ sin.\ \rho.$$

If the axle begins to move, the point of pressure $B$ moves forward in the bearing by the angle $ACB = \rho$ in the opposite direction.

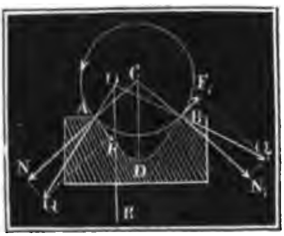
If no forward motion took place, $F$ would be $= fR = R\ tang.\ \rho = \frac{R\ sin.\ \rho}{cos.\ \rho}$; consequently the friction is the $cos.\ \rho$ times as great after moving forward as before the motion. Generally, $f = tang.\ \rho$ not quite $\frac{1}{10}$ and $cos.\ \rho > 0{,}995$, therefore the difference is not quite $\frac{5}{1000} = \frac{1}{200}$; therefore, in ordinary cases of application, we need have no regard to the effect of this motion.

If the wheel $AB$ revolves in a nave or eye, Fig. 173, about a fixed axis $AC$, the friction is the same as if the axis moves in a roomy nave, only the arm of the friction is the arm of the nave, not that of the fixed axle.

FIG. 173.


§ 167. If the axle lies in a prismatic bearing, there is greater pressure, and consequently more friction than in a round bearing. If the bearing $ADB$, Fig. 174, is triangular, the axle lies on two points $A$ and $B$, and at each there is the same friction to overcome, the mean pressure $R$ is decomposed into two lateral forces $Q$ and $Q_1$, and by the each of these gives a normal pressure $N$ and $N_1$, and each a tangential force $F = fN$ and $F_1 = f N_1$ equal to the friction. According to the former § these frictions may be put $= Q\ sin.\ \rho$ and $= Q_1\ sin.\ \rho$, we have then the whole friction $= (Q + Q_1)\ sin.\ \rho$. The forces $Q$ and $Q_1$ are given by the resolution of a parallelogram constructed from $Q$ and $Q_1$ with the aid of the mean pressure $R$, the angle of friction $\rho$, and the angle $ACB = 2\ a$, which corresponds to the arc $AB$, lying in the bearing. If $QOR = ACD - CAO = a - \rho$, $Q_1OR = BCD + CBO = a + \rho$; lastly, $QOQ_1 = a - \rho + a + \rho = 2\ a$. The application of the formula § 75, gives:

FIG. 174.



$$Q = \frac{sin.\ (a \pm \rho)}{sin.\ 2\ a}\ .\ R \text{ and } Q_1 = \frac{sin.\ (a \mp \rho)}{sin.\ 2\ a}\ .\ R;$$

hence the friction sought is:

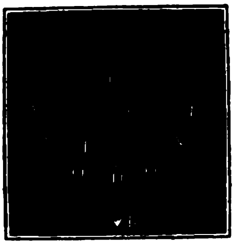
$$F + F_1 = (Q + Q_1)\ sin.\ \rho = (sin.\ [a - \rho] + sin.\ [a + \rho])\ \frac{R\ sin.\ \rho}{sin.\ 2\ a}.$$

But the $sin.\ (a - \rho) + sin.\ (a + \rho) = 2\ sin.\ a\ cos.\ \rho$ and $sin.\ 2\ a = 2\ sin.\ a\ cos.\ a$, therefore $F + F_1 = \frac{2\ sin.\ a\ R\ sin.\ \rho\ cos.\ \rho}{2\ sin.\ a\ cos.\ a}$ $= \frac{R\ sin.\ 2\ \rho}{2\ cos.\ a}$, which from the smallness of $\rho$ may be put $= \frac{R\ sin.\ \rho}{cos.\ a}$.

The friction of a triangular bearing is from this $\frac{1}{cos.\ a}$ times as great as that of a cylindrical one. If, for example, $ADB = 60^0$, $ACB = 180^0 - 60^0 = 120^0$, and $ACD = a = 60^0$, we then have: $\frac{1}{cos.\ 60^0} =$ twice the friction of that of a cylindrical bearing.

§ 168. With the aid of the last formula, the friction may be found for a new round bearing which the axle touches at all points. Let $ADB$, Fig. 175, be such a bearing. Let us divide the arc $ADB$, in which lie the axle and the bearing into many parts, such as $AN$, $NO$, &c., which have equal projections on the chord $AB$, and let us suppose that each of these parts supports an equal amount of the whole pressure $R$, viz: $= \frac{R}{n}$, ($n$ being the number of parts) of the axle on the bearing. According to the former §, the friction of two opposite parts $NO$ and $N_1 O_1 = \frac{R}{n} \cdot \frac{sin.\ 2\ \rho}{cos.\ NCD}$. But $cos.\ NCD = cos.\ ONP = \frac{NP}{NO}$, where $NP$ represents the projection of the part $NO$ and $NP = \frac{chord.\ AB}{n}$; it therefore follows that the friction corresponding to the parts $NO$ and $N_1 O_1 = \frac{R\ sin.\ 2\ \rho}{n}$ $\cdot \frac{n\ .\ NO}{chord} = \frac{R\ sin.\ 2\ \rho}{chord}\ .\ NO$. In order now to find the friction for the whole arc $ADB$ instead of $NO$, we must put in the arc $AD=$

FIG. 175.



$\frac{1}{2} ADB$, because the sum of all the frictions is $\frac{R \sin. 2\rho}{chord}$ times the sum of all the parts of the arc, it follows that the friction in a new bearing is : $F = R \sin. 2\rho \cdot \frac{arc\ AD}{chord\ AB}$, or if we put the angle $ACB$ subtended at the centre by $AB$, which corresponds to the arc of the bearing, $= 2\,a^0$, therefore the chord $AB = 2\,AC \cdot \sin.\ a$.

$$F = \frac{R \sin. 2\rho}{2} \cdot \frac{a}{\sin. a}, \text{ or } \sin. 2\rho = 2 \sin. \rho$$

taken approximately

$$F = R \sin. \rho \cdot \frac{a}{\sin.\ a}.$$

From this the friction is the greater the deeper the axle lies in its bearing. If, for instance, the bearing is half the circumference of the axle, $a$ is then $= \frac{1}{2}\pi$ and $\sin.\ a = 1$, we then have $F = \frac{\pi}{2} \cdot R \sin.\ \rho$, and because $\frac{\pi}{2} = 1{,}57$, therefore 1,57 times as great as that of the free bearing. In an axle which does not rest deep in its bearing, $a$ is small, therefore the $\sin.\ a$ may be put $= a - \frac{a^3}{6} = a\left(1 - \frac{a^2}{6}\right)$, whence it follows that:

$F = \left(1 + \frac{a^2}{6}\right) R \sin.\ \rho$, or $= R \sin.\ \rho$, if $a$ be very small.

§ 169. The axle pressure $R$ is given generally as the resultant of two forces $P$ and $Q$, directed at right angles to each other, and is therefore $= \sqrt{P^2 + Q^2}$. Provided we require it only for the determination of the friction $fR = f\sqrt{P^2 + Q^2}$, we may be satisfied with an approximate value of it, partly because the co-efficient $f$ can never be so accurately determined and depends upon so many accidental circumstances, and partly because the whole product of the friction $fR$ is mostly only a small part of the remaining forces of the machine resting upon the axle bearing, as the lever, pulley, wheel and axle, &c. The doctrine which teaches us to find an approximate expression of $\sqrt{P^2 + Q^2}$ is known under the name of Poncelet's theorem, and may be developed in the following manner:

$\sqrt{P^2 + Q^2} = P\sqrt{1 + \left(\frac{Q}{P}\right)^2} = P\sqrt{1 + x^2}$, whereby $x = \left(\frac{P}{Q}\right)$, $\frac{Q}{P}$ which supposes that $R$ is the smaller force, therefore, $x$ is a mere fraction. We may now put:

$\sqrt{1+x^2} = \mu + \nu x$, and determine the co-efficients $\mu$ and $\nu$, answering certain conditions. The relative error is:

$$y = \frac{\sqrt{1+x^2} - \mu - \nu x}{\sqrt{1+x^2}} = 1 - \frac{\mu + \nu x}{\sqrt{1+x^2}}.$$

For the smallest value of $x$, viz. $x = o$, $y = 1 - \mu$, and for the greatest, viz. $x = 1$, we have $y = 1 - \frac{\mu + \nu}{\sqrt{2}}$. If we make these errors, corresponding to the limits of $x$, equal, we then obtain an equation of condition $\mu = \frac{\mu + \nu}{\sqrt{2}}$, or $\nu = \mu\sqrt{2} - \mu = 0{,}414 \,.\, \mu$. If we take $x = \frac{\nu}{\mu}$, the result is, that $y = 1 - \sqrt{\mu^2 + \nu^2} = -(\sqrt{\mu^2 + \nu^2} - 1)$, as a negative error, is greater than any other which arises by assuming $x = \frac{\nu}{\mu} \pm \Delta$, that is, a little greater, or a little less than $\frac{\nu}{\mu}$; for in the latter case we have

$$y = -\left(\frac{\mu + \nu\left(\frac{\nu}{\mu} \pm \Delta\right)}{\sqrt{1 + \left(\frac{\nu}{\mu} \pm \Delta\right)^2}} - 1\right)$$

$$= -\left(\frac{\mu^2 + \nu^2 \pm \mu\nu\Delta}{\sqrt{\mu^2 + \nu^2 \pm 2\mu\nu\Delta + \mu^2\Delta^2}} - 1\right)$$

$$= -\left(\sqrt{\frac{(\mu^2 + \nu^2 \pm \mu\nu\Delta)^2}{\mu^2 + \nu^2 \pm 2\mu\nu\Delta + \mu^2\Delta^2}} - 1\right)$$

$$= -\left(\sqrt{\mu^2 + \nu^2 - \frac{\mu^4\Delta^2}{\mu^2 + \nu^2} \cdots} - 1\right).$$

If now we make this greatest negative error equal to the greatest positive error, we shall then obtain the following second equation of condition:

$$\sqrt{\mu^2 + \nu^2} - 1 = 1 - \mu\,;\ \text{or}\ \mu + \sqrt{\mu^2 + \nu^2} = 2.$$

But the first equation gives $\nu = 0{,}414\,\mu$; it, therefore, follows that

$$\mu\,(1 + \sqrt{1 + 0{,}414^2}) = 2,\ \textit{i. e.}$$

$$\mu = \frac{2}{1 + \sqrt{1{,}1714}} = 0{,}96 \text{ and } \nu = 0{,}414 \,.\, 0{,}96 = 0{,}40.$$

We may, therefore, put approximately $\sqrt{1 + x^2} = 0{,}96 + 0{,}40 \,.\, x$, and in like manner the resultant $R = 0{,}96\,P + 0{,}40\,Q$, knowing that

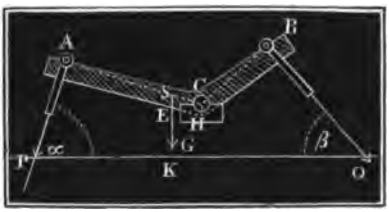
we thereby commit at most the error $\pm y = 1 - \mu = 1 - 0{,}96 = 0{,}04$ = four per cent of the true value.

This determination supposes that we know which is the greater of the forces; but if this be unknown, we may assume $\sqrt{1+x^2} = \mu\,(1+x)$, and so obtain $y = 1 - \frac{\mu\,(1+x)}{\sqrt{1+x^2}}$. Here not only the limit $x = o$ gives the error $= 1-\mu$, but also the limit $x = \infty$, the same error $= 1 - \frac{\mu x}{x} = 1 - \mu$; but if we put $x = \frac{\nu}{\mu} = 1$, we then obtain the greatest negative error $= -\left(\frac{2\,\mu}{\sqrt{2}} - 1\right) = -(\mu\sqrt{2} - 1)$ and by making these errors equal: $1 - \mu = \mu\sqrt{2} - 1$, therefore, $\mu = \frac{2}{1+\sqrt{2}} = \frac{2}{2{,}414} = \frac{1}{1{,}212} = 0{,}825$, for which 0,83 may be put. In the case where we do not know which is the greater of the forces, $R$ may be put $= 0{,}83\ (P+Q)$, and we know that the greatest error committed will be $\pm y = 1 - 0{,}83 = 0{,}17$ per cent $= \frac{1}{6}$ of the true value.

If, lastly, we know that $x$ does not exceed 0,2 we may disregard it altogether, and write $\sqrt{P^2+Q^2} = P$, but if $x$ exceeds 0,2, then $\sqrt{P^2+Q^2}$ is more accurately $= 0{,}888\,P + 0{,}490\,.\,Q$; in both cases the greatest error is about two per cent.*

§ 170. *Lever.*—The theory of friction above developed finds its application in the material lever, the wheel and axle, and in other machines. Let us in the first place treat of the lever, and take the general case, viz. that of the bent lever $ACB$, Fig. 176. Let us represent as before (§ 127) the arm $CA$ of the power $P$ by $a$, the arm $CB$ of the weight $Q$ by $b$, and the radius of the axle $CH$ by $r$, let us put the weight of the lever $= G$, its arm $CE = s$, and the angles $APK$ and $BQK$, by which the directions of the forces deviate from the horizon $= \alpha$ and $\beta$. The power $P$ gives the vertical pressure $P\ sin.\ \alpha$, and the weight $= Q\ sin.\ \beta$;

FIG. 176.




* Polytechnische Mittheilungen, Heft. 1, 1844.

the whole vertical pressure is, therefore, $V = G + P\ sin.\ a + Q\ sin.\ \beta$. The power $P$ gives further the horizontal pressure $P\ cos.\ a$, and the weight $Q$ an opposite pressure $Q\ cos.\ \beta$; since there remains for the horizontal pressure, $H = P\ cos.\ a - Q\ cos.\ \beta$, we may put the whole pressure on the axle:

$$R = \mu\ V + \nu\ H = \mu\ (G + P\ sin.\ a + Q\ sin.\ \beta) + \nu\ (P\ cos.\ a - Q\ cos.\ \beta),$$

of which the second part $\nu\ (P\ cos.\ a - Q\ cos.\ \beta)$ must never be taken negative, and, therefore, in the case where $Q\ cos.\ \beta$ is $>$ $P\ cos.\ a$, the sign must be changed, or rather $P\ cos.\ a$ must be subtracted from $Q\ cos.\ \beta$. In order to find that value of the power which corresponds to unstable equilibrium, so that the smallest addition of force produces motion, we must put the moment of power equal to the moment of weight, plus or minus the moment of weight of the machine (§ 127) plus the moment of friction, therefore,

$$Pa = Qb \pm Gs + fRr$$
$$= Qb \pm Gs + f\ (\mu\ V + \nu\ H)\ r, \text{ from which follows}$$
$$P = \frac{Qb \pm Gs + f\ [\mu\ (G + Q\ sin.\ \beta) \mp Q\ cos.\ \beta]\ r}{a - \mu\ fr\ sin.\ a \mp \nu fr\ cos.\ a}.$$

If $P$ and $Q$ act vertically, $R$ is simply $= P + Q + G$, therefore, $Pa = Qb \pm Gs + f(P + Q + G)\ r$. If the lever is one-armed, $P$ and $Q$ act opposite to each other, then $R = P - Q + G$, and consequently the friction is less. Besides $R$ must be put constantly positive in the calculation, because the friction $fR$ only impedes, but does not produce motion. From this we see that a one-armed is mechanically more perfect than a two-armed lever.

*Example.* If the arms of a bent lever, Fig. 176, are: $a = 6$ ft., $b = 4$ ft., $s = \frac{1}{2}$ ft. and $r = 1\frac{1}{2}$ inch, the angle of inclination $a = 70°, \beta = 50°$, and further the weight $Q = 5600$ lbs., and that of the lever $G = 900$ lb., the power required to restore the unstable equilibrium is the following. Without regard to friction $Pa + Gs = Qb$, therefore, $P = \frac{Qb - Gs}{a} = \frac{5600\ .\ 4 - 900\ .\ \frac{1}{2}}{6} = 3658$ lbs. If we put $\mu = 0{,}96$ and $\nu = 0{,}40$, we obtain $\mu\ (G + Q\ sin.\ \beta) = 0{,}96\ (900 + 5600\ sin.\ 50°) = 4982$ lbs., $\nu\ Q\ cos.\ \beta = 0{,}40\ .\ 5600.\ cos.\ 50° = 1440$ lbs.; $\mu\ sin.\ a = 0{,}96.\ sin.\ 70° = 0{,}902$, $\nu\ cos.\ a = 0{,}40.\ cos.\ 70° = 0{,}137$. It is easy to see, that here $P\ cos.\ a$ is less than $Q\ cos.\ \beta$, for since approximately $P = 3658$, we have $P\ cos.\ a = 1251$ lbs., and $Q\ cos.\ \beta = 3600$ lbs., let us therefore take for $\nu\ Q\ cos.\ \beta$, and $\nu\ cos.\ a$ the lower sign and put $P = \frac{5600\ .\ 4 - 900\ .\ \frac{1}{2} + fr\ (4982 + 1440)}{6 - fr\ (0{,}902 - 0{,}173)}$. Let us further take the co-efficient of friction $f = 0{,}075$, and we shall have $fr = 0{,}075\ .\ \frac{3}{24} = 0{,}009375$, and the power sought,

$P = \frac{22400-450+60}{6-0{,}00683} = \frac{22010}{5{,}9932} = 3673$ lb. Here the vertical pressure, if we substitute the value $P=3658$ lbs., and neglect friction, is $V=3658 \,.\, sin.\ 70^{\circ} + 5600\ sin.\ 50^{\circ} + 900 = 3437 + 4290 + 900 = 8627$ lbs., on the other hand, the horizontal pressure:

$$H = 5600\ cos.\ 50 - 3658\ cos.\ 70 = 3600 - 1251 = 2349 \text{ lbs.}$$

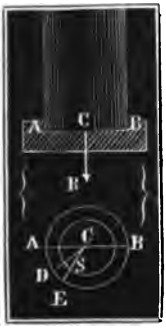
Here $H$ is $> 0{,}2\ V$, therefore, more correctly:

$$R = 0{,}888 \,.\, H + 0{,}490\ V = 0{,}888 \,.\, 8627 + 0{,}490 \,.\, 2349 = 8811,$$

and it follows that the moment of friction $= f r R = 0{,}009375 \,.\, 8811 = 82{,}6$ ft. lbs. and lastly, the power $P = \frac{22400-450+82{,}6}{6} = 3672$ lbs., which value varies little from the above.

§ 171. *Pivot friction.*—When in the wheel and axle a pressure takes place in the direction of the axis, as in the case, for example, of upright axles, in consequence of their weight, there is a friction on the base of the one axle. Because pressure is there exerted on points between the pivot and its step, this friction approximates to the simple sliding friction, and to the axle friction which we have hitherto considered, and we must put for it the co-efficients of friction given in Table II. To find the mechanical effect absorbed by this friction, we must know the mean space which the base $AB$, Fig. 177, of such an upright axle describes in a revolution. Let us assume that the pressure $R$ is equally distributed over the whole surface, let us also suppose that on equal parts of the bases the frictions are equal. Let us further divide the base by radii $CD$, $CE$, &c., into equal sectors or triangles $DCE$; to these will correspond not only equal amounts of friction, but also equal moments, therefore, it will be necessary only to find the moment of friction of one of these triangles. The frictions of such a triangle may be regarded as parallel forces, for they all act tangentially, *i. e.* at right angles to the radius $CD$, and since the centre of gravity of a body or a surface is nothing more than the point of application of the resultant of the parallel forces equally distributed over this body or surface, accordingly the centre of gravity $S$ of the sector or triangle $DCE$ is here the point of application of the resultant arising from its different frictions. If now the pressure on this sector $= \frac{R}{n}$, and the radius $CD = CE$, of the base $= r$, it follows that (from § 104) the moment of friction of this sector $= CS \,.\, \frac{fR}{n}$

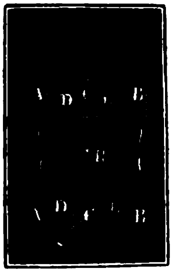
FIG. 177.



$= \frac{2}{3} r . \frac{fR}{n}$, and lastly, the moment of the entire friction of the axle $= n . \frac{2}{3} r \frac{fR}{n} = \frac{2}{3} f R r$.

Sometimes the rubbing surface is a ring $ABED$, Fig. 178.

FIG. 178.



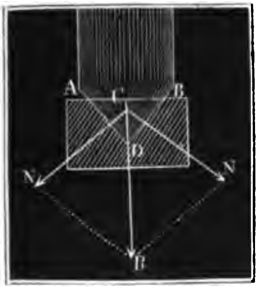
If its ~~diameters~~ are $CA = r_1$ and $CD = r_2$, we radii have then to determine the centre of gravity $S$ of a portion of a ring, and from § 109, obtain the arm $CS = \frac{2}{3} \left[ \frac{r_1^3 - r_2^3}{r_1^2 - r_2^2} \right]$ therefore, the moment of friction $= \frac{2}{3} f R \left( \frac{r_1^3 - r_2^3}{r_1^2 - r_2^2} \right)$. If we introduce the mean radius $\frac{r_1 + r_2}{2} = r$, and the breadth of the ring $r_1 - r_2 = b$, we obtain this moment of friction also $= f R \left( r + \frac{b^2}{12 r} \right)$.

The mechanical effect of friction for a revolution of the axle is in the first case $= 2 \pi . \frac{2}{3} f R r = \frac{4}{3} \pi f R r$, and in the second $\frac{4}{3} \pi f R \left( \frac{r_1^3 - r_2^3}{r_1^2 - r_2^2} \right)$. Here we easily see that to diminish this loss of mechanical effect, the upright axle or shaft must be made as light as possible, and that a greater loss of mechanical effect would arise if under otherwise similar circumstances, the friction were to take place in a ring instead of a complete circle.

*Example.* In a turbine making 100 revolutions a minute and 1800 lbs. weight, the size of the pivot at the base, is $\frac{1}{2}$ inch; how much mechanical effect does the friction of this pivot consume in one second? The co-efficient of friction being taken $= 0,1$ we have the friction $f R = 0,1 . 1800 = 180$ lbs.; the space per revolution $= \frac{4}{3} \pi r = \frac{4}{3} . 3,14 . \frac{1}{24} = 0,1745$ ft. ~~lbs.~~, hence the mechanical effect per revolution $= 180 . 0,1745 = 31,41$ ft. lbs. But now this machine makes in a second $\frac{100}{60} = \frac{5}{3}$ of a revolution, hence it follows that the loss of mechanical effect sought $= \frac{314,1}{6} = 52,3$ ft. lbs.

§ 172. *Pointed axles.*—If the axle $ABD$, Fig. 179, has conical ends, the friction comes out greater than if it has plane ends,

FIG. 179.




because the pressure of the axle $R$ is resolved into the normal forces $N, N_1$, which produce the friction, and which together are greater than $R$ alone. If the half of the convergent angle $ADC = BDC = a$, we have $2\,N = \frac{R}{sin.\,a}$, and consequently the friction of the pointed axle $= f\frac{R}{sin.\,a}$. Let the radius of the axle $CA = CB$ at the entrance into the step be represented by $r_1$, we shall then have as before the moment of friction $= \frac{2}{3} f \frac{R r_1}{sin.\,a}$. Let this axle dip a little only into the step, the mechanical effect of the friction of this axle will be less than that of an axle with a plane base, and on this account the application of the pointed axle is of service. When, for example, $\frac{r_1}{sin.\,a} = \frac{r}{2}$, therefore, $r_1 = \frac{1}{2}\, r\, sin.\,a$, the pointed axle of the radius $r_1$ causes only half the loss of mechanical effect through friction which the truncated axis of the radius $r$ does.

If the pivot forms a truncated cone, Fig. 180, friction takes place as well at the envelop as the truncated surface, and the moment of friction comes out $= \left(r_1^{\,3} + \frac{r^3 - r_1^{\,3}}{sin.\,a}\right) \frac{2}{3} \frac{f R}{r^2}$, if $r$ be the radius of the place of entrance into the step, and $r_1 = DE$ that of the base, and $a^0$ the half of the convergent angle.

FIG. 180.

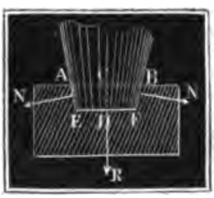



FIG. 181.

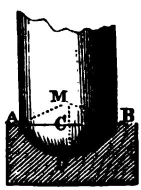



FIG. 182.


Lastly, the pivot or upright axles (Figs. 181, 182) are very often rounded. Although by this rounding, the friction itself is by no means diminished, there arises nevertheless a diminution of the moment of friction, from the extremity not dipping far into the step.

If we suppose a spherical rounding, we obtain by the aid of the higher calculus for a semi-spherical step, the moment of friction $= \frac{f\pi}{2} . Rr$; but for that of a step having a less segment $= f\frac{2}{3}\left[1 + 0{,}3\left(\frac{r_1}{r}\right)^2\right] Rr_1$, $r$ being the radius of the sphere $MA = MB$, $r_1$ the radius of the step $CA = CB$.

*Example.* If the weight of the armed axle of a horse capstan $R = 6000$ lbs., the radius of the conical pivot $= r = 1$ inch, and the angle of convergence of the cone $2\alpha = 90°$, then the moment of friction of this pivot $= \frac{2}{3} . f . \frac{Rr}{sin. \alpha} = \frac{2}{3} . 0{,}1 . \frac{6000}{sin. 45°} . \frac{1}{12} = \frac{100}{3\sqrt{\frac{1}{2}}} = 47{,}1$ ft. lbs. This axle makes during the lifting up of a ton from a shaft or mine $= u = 24$ revolutions, then the mechanical effect which is expended at the pivot in this time by friction $= 2\pi u . \frac{2}{3} f \frac{Rr}{sin. \alpha} = 2\pi . 24 . 47{,}1 = 7103$ ft. lbs.

§ 178. *Points and knife edges.*—To avoid as much as possible the friction of the axle, rotatory bodies are supported on pointed pivots, knife edges, &c. If we had only to do with perfectly rigid and inelastic bodies, no loss of labour would arise through friction by this method of support or suspension, because no measurable space here is described by the friction; but since every body possesses a certain degree of elasticity, by resting of such a body on a point or knife edge, a slight penetration takes place, and a rubbing surface is thereby caused, upon which space is described by the friction which gives rise to a certain loss of labour, although very small. In rotations and vibrations long sustained, bodies supported in this manner, present similar surfaces of friction arising from the abrasion of their points or knife edges, and the friction must then be estimated by what has been already mentioned. For these reasons this mode of support is applicable only to such instruments as the compass, the balance, &c., where it is of importance to diminish the amount of friction, and where motion is only allowed from time to time.

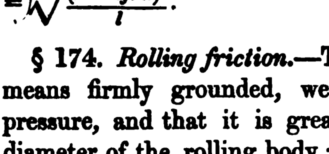
Experiments on the friction of a body resting upon a hard steel point, and revolving about it, have been made by *Coulomb*. From these, it results that the friction increases somewhat more than the pressure, and varies with the thickness of the supporting pivot. It is least for a granite surface, greater for one of agate and of rock

crystal, greater still for a glass surface, and greatest of all for a steel one. For a very small pressure, as in the magnetic needle, the pivot may be pointed to 10° or 12° of convergence. But if the pressure is great, we must then apply a far greater angle of convergence, viz. 30° to 45°. The friction is less when the body having a plane surface reposes upon a point than when it lies in a conical or spherical step. Similar relations take place in the knife edge as applied to the balance, and the beams of balances, that are intended for heavy loads, require sharp axes of 90° convergence, while an edge of 30° is sufficient for the lighter ones.

FIG. 183.


*Remark.* If we assume that the needle *AB*, Fig. 183, rests on the point *DCE* of the pivot *FCG*, of the height $CM = h$, and radius $DM = r$, and suppose that the volume $\frac{1}{3} \pi r^2 h$ is proportionate to the pressure *R*, the amount of friction may be found in the following manner. If we put $\frac{1}{3} \pi r^2 h = \mu R$, where $\mu$ is a number resulting from experience, and introduce the angle of convergence $DCE = 2\alpha$, and, therefore, put $h = r\ cotg.\ \alpha$, we obtain the radius of the base $r = \sqrt[3]{\frac{3 \mu R\ tang.\ \alpha}{\pi}}$, and $f R r = \sqrt[3]{\frac{3 \mu R^4\ tang.\ \alpha}{\pi}}$ $= f \sqrt[3]{\frac{3 \mu}{\pi}} \cdot R^{\frac{4}{3}} (tang.\ \alpha)^{\frac{1}{3}}$. We must, therefore, assume that the friction on the pivot increases equally with the cubic root of the fourth power of the pressure, and the cubic root of the tangent of half the angle of convergence. The amount of friction of a beam *AB*, Fig. 184, which oscillates on a sharp edge $CC_1$, may be found in like manner. If $\alpha$ be half the angle of convergence *DCM*, *l* the length $CC_1$ of the edge, and *R* the pressure, this amount is given $= f \sqrt{\frac{(P\ tang.\ \alpha)^3}{l}}$.

FIG. 184.




§ 174. *Rolling friction.*—The theory of rolling friction is by no means firmly grounded, we know that it increases with the pressure, and that it is greater for a smaller than for a larger diameter of the rolling body; but in what algebraical dependence this friction stands to the pressure and diameter of the rolling body, cannot as yet be considered as determined. *Coulomb* made only a few experiments with rollers, from two to twelve inches thick, of lignumvitæ and elm, which were rolled along a surface of oak, by means of a thin thread passing over the roller *AB* whose extre-

and $AB = R$ ∴ the pressure upon

of depth

$R$ $R$

a normal

$\int_{-\alpha}^{\alpha} r\cos$ $\frac{R}{r\alpha \sin\alpha}$

but

$i\cos\varphi$

$F$ $\frac{R}{\sin\alpha}\int_{-\alpha}^{\alpha}\cos$ $R\sin 2\varphi$ $\frac{\alpha}{\sin\alpha}$

---

## Pivot friction.

$\frac{R}{\pi r^2}$ pressure on unit

$\rho\, d\rho\, d\varphi$

$\frac{R\rho\, d\rho\, d\varphi}{\pi r^2}$

friction $f\frac{R}{\pi r^2}\, d\rho\, \rho\, d\varphi$

moment of friction $f\frac{R}{\pi r^2}\rho^2\, d\rho\, d\varphi$

$f\frac{R}{\pi r^2}\int_0^{2\pi}\int_0^{r}\rho^2\, d\rho\, d\varphi = f\frac{R}{3\pi}\cdot\frac{2\pi r^3}{r^2} = \frac{2}{3} f R r$

[illegible]

[illegible]

$\pi(r^2 - r_1^2)$ ∴ $F = \frac{fR}{\pi(r^2-r_1^2)}\int_0^{2\pi}\int_{r_1}^{r}\rho^2\, d\rho\, d\varphi =$

$\frac{2\pi f R}{\pi(r^2-r_1^2)}\cdot\frac{r^3-r_1^3}{3} = \frac{2}{3} f R\left(\frac{r^3-r_1^3}{r^2-r_1^2}\right)$

Pointed Axle.

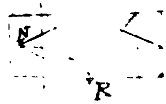
[illegible] point at its

[illegible] into the [illegible]

[illegible] R repr[illegible]

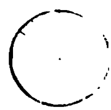
[illegible]

[illegible], $\rho\, d\rho\, d\varphi$ [illegible].



[illegible] $R = 2N\cos\alpha$ [illegible]

[illegible] ∴ $2N = \frac{R}{\sin\alpha}$ ∴ [illegible]



[illegible]

$F = \frac{fR}{r^2\pi\sin\alpha}\int_0^{2\pi}\int_0^{r}\rho^2\, d\rho\, d\varphi$ ∴

$F = \frac{2}{3} f R r_1$ [illegible] $R$ [illegible]

[illegible]

[illegible]

$F = \frac{fR}{\pi r^2}\left[\int_0^{2\pi}\int_0^{r_1}\rho^2\, d\rho\, d\varphi + \frac{1}{\sin\alpha}\int_0^{2\pi}\int_{r_1}^{r}\rho^2\, d\rho\, d\varphi\right]$

$= \frac{fR\,2\pi}{\pi r^2}\cdot\frac{r_1^3}{3} + \frac{1}{\sin\alpha}\cdot\frac{fR\,2\pi}{\pi r^2}\cdot\frac{r^3 - r_1^3}{3}$

$$= \frac{2fR}{3r^2}\left[r^3 + \frac{r^3 - r_1^3}{2\sin\alpha}\right] = \frac{2}{3}\left[r^3 + \frac{r^3 - r_1^3}{2\sin\alpha}\right]\frac{fR}{r^2}$$

[illegible] and

$\pi r^2$ [illegible] $\frac{K}{\pi r^2}$

[illegible]

A [illegible] $\frac{K}{\pi r}$

[illegible]

[illegible] we have $R - 2N$ [illegible] $2N$ [illegible] $\frac{K}{2}$ [illegible]

[illegible] $\frac{fK}{\pi r^2}$ [illegible]

[illegible]

[illegible] $\frac{fK}{\pi r^2}\,\rho^2\, d\rho\, d\phi$ [illegible]

[illegible]

[illegible]

[illegible]

[illegible] $\frac{fK}{\pi r^2}$ [illegible] $d\rho\, d\phi$ [illegible]

[illegible]

[illegible] $d\rho$ [illegible]

[illegible] $\int_0^{\frac{1}{2}\pi}$ [illegible] $r^3$ [illegible] $d\phi$ —

[illegible] $= fKr\int_0^{\frac{1}{2}\pi}$ [illegible]

[illegible] is $fKr(\phi$ [illegible]

[illegible] $\int$ [illegible] $\frac{\pi}{2}$

… hemisphere …

… $r_1 = r \sin \alpha \therefore \alpha = \sin^{-1}\frac{r_1}{r}$

… the preceding case we have the …

… element

$\frac{f K}{r_1^2}$ … we also have … values …

… $2 f \frac{K}{r^2} r_1^3 \sin^2\varphi \cos\varphi \, d\varphi$ … we have

$f K \, 2 r^2 \sin^2\varphi \, d\varphi = f r^2 K (1 - \cos 2\varphi)$; now

… integration … then the …

… we have $F = \frac{f r_3 K}{r_1^2} \int_0^{\alpha} (1 - \cos\varphi)$ …

… is $\frac{f R r^3}{r_1^2}$ …

… but $\frac{\sin 2\alpha}{2} = \sin\alpha\cos\alpha$

$F = \frac{f R r^3}{r_1^2}\left(\alpha - \frac{\sin 2\alpha}{2}\right)$ … $F = \frac{f r^3}{r_1^2}\left(\sin^{-1}\frac{r_1}{r} - \frac{r_1}{r}\sqrt{1 - \frac{r_1^2}{r^2}}\right)$

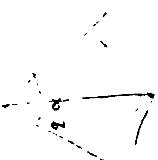
$= \sin\alpha\sqrt{1 - \sin^2\alpha}$

… $\sin^{-1}\frac{r_1}{r} = \frac{r_1}{r} - \frac{1}{6}\frac{r_1^3}{r^3} + \frac{1}{40}\frac{r_1^5}{r^5}$

$\frac{r_1}{r}\sqrt{1 - \frac{r_1^2}{r^2}} = \frac{r_1}{r^2}\left[r - \frac{1}{2}\frac{r_1^2}{r} - \frac{1}{8}\frac{r_1^4}{r^3} \text{ etc}\right] = \left[\frac{r_1}{r} - \frac{1}{2}\frac{r_1^3}{r^3} - \frac{1}{8}\frac{r_1^5}{r^5}\right]$

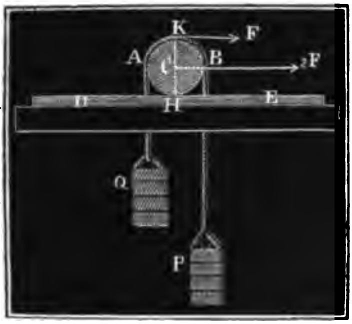
$F = \frac{f R r^3}{r^2}\left[\frac{r_1}{r} + \frac{1}{6}\frac{r_1^3}{r^3} + \frac{3}{40}\frac{r_1^5}{r^5} - \frac{r_1}{r} + \frac{1}{2}\frac{r_1^3}{r^3} + \frac{1}{8}\frac{r_1^5}{r^5}\right]$

$F = \frac{f R r^3}{r^2}\left(\frac{4}{6}\frac{r_1^3}{r^3} + \frac{8}{40}\frac{r_1^5}{r^5}\right) = \frac{2}{3}\left[1 + 3\frac{r_1}{r}\right] R$,



$N = \frac{P}{\cos\theta}$

FIG. 185.




mities were stretched by unequal weights $P$ and $Q$, Fig. 185. From the results of these experiments, rolling friction appears to increase directly with the pressure, and inversely with the ~~diameter~~ radius of the roller, so that the force necessary to overcome this friction may be expressed by $F = f \cdot \frac{R}{r}$, if $R$ be the pressure, $r$ the radius of the roller, and $f$ the co-efficient of friction derived from experiment. If $r$ be given in inches, then from these experiments

For rolling upon compressed wood $f = 0{,}0189$,
" " elm $f = 0{,}0310$.

These formulas suppose that the force $F$ acts at the circumference of the roller, but if the force be applied to the axis $C$ of the rolling bodies, by which, as in every description of carriage, axle friction ensues, the required force is $2\,F$, because here the arm $CH$ is only half that of $KH$ with respect to the point of application $H$.

A body $ABS$ is moved forwards, Fig. 186, lying on the rollers $C$ and $D$, the required force $P$ here comes out very small, because two rolling frictions only, viz. that between $AB$ and the rollers, and that between the rollers and the way $HR$ are to be overcome. The progressive space of the rollers is only half that of the load $Q$, and on this account for farther progression, the rollers must be replaced under it from before, because the points of contact $A$ and $B$, by virtue of the rolling, recede as much as the axis of the rollers advances.

FIG. 186.


If the roller $AH$ has revolved about an arc $AO$, the roller has then moved over a space $AA_1$ equal to this arc, and $O$ comes into contact with $O_1$, the new point of contact $O_1$ has, therefore, receded by $AO_1 = AO$ behind the former ($A$). If the co-efficients

of friction are $f$ and $f_1$, the power necessary to draw the load $R$ forward is $P = (f + f_1) \frac{R}{r}$.

*Remark.* The extended experiments of *Morin* on the resistance of carriages upon roads, accord with the law by which this resistance increases equally with the pressure, and inversely with the thickness of the roller. Another French engineer, *Dupuit*, on the contrary, deduces from his experiments, that rolling friction increases indeed directly with the pressure, but for the rest, only inversely proportional to the square root of the radius of the roller. Particular theoretical views upon rolling friction may be found in *Gerstner's* Mechanics, vol. 1, § 537, and developed in *Brix's* treatise upon friction, art. 6. We shall treat this more fully in the second part, when we come to roads and railways.

FIG. 187.

§ 175. *Friction of cords.* — We have now to investigate the friction of a flexible body. When a perfectly flexible cord is stretched by a weight $Q$ over the edge $C$ of a rigid body $ABE$, Fig. 187, and thereby deviates from its original direction by an angle $DCB = a^0$, there arises at the edge a pressure $R$ from which a friction $F$ takes place, and requires for the restoration of unstable equilibrium that the force $P$ should be greater or less than $Q$. The pressure is (§ 74):

$$R = \sqrt{P^2 + Q^2 - 2\,PQ\,cos.\,a},$$

consequently the friction:

$$F = f\,\sqrt{P^2 + Q^2 - 2\,PQ\,cos.\,a.}$$

If further we put $P = Q + F$ and $P^2$ approximately $= Q^2 + 2\,QF$, we then obtain:

$$F = f\,\sqrt{Q^2 + 2\,QF + Q^2 - 2\,Q^2\,cos.\,a - 2\,FQ\,cos.\,a}$$

$$= f\,\sqrt{2\,(1 - cos.\,a)\,(Q^2 + QF)} = 2 f\,sin.\,\frac{a}{2}\,\sqrt{Q^2 + Q\,F},$$ which

again may be put $= 2\,f\,sin.\,\frac{a}{2}\,(Q + \frac{1}{2}F)$, if we have regard to the two first members of the square root only. Now if $F - f\,F\,sin.\,\frac{a}{2}$ is given $= 2\,f\,Q\,sin.\,\frac{a}{2}$, consequently the friction sought is

$P = Q + F$ $\quad R = 2Q \ldots x \quad F = f \cdot Q$

$\therefore P = Q(1 + 2f \ldots x$

$$F = \frac{2 f Q \sin. \frac{a}{2}}{1 - f \sin. \frac{a}{2}}, \text{ which generally is } = 2 f Q \sin. \frac{a}{2} \left(1 + f \sin. \frac{a}{2}\right)$$

and indeed very often $= 2 f Q \sin. \frac{a}{2}$. In order to draw the cord over the edge, a force $P = Q + F = \left(1 + \frac{2 f \sin. \frac{a}{2}}{1 - f \sin. \frac{a}{2}}\right) Q$ is necessary; and inversely, to prevent the descent of the weight $Q$ by the cord, a force $P_1 = Q \div \left(1 + \frac{2 f \sin. \frac{a}{2}}{1 - f \sin. \frac{a}{2}}\right)$ is requisite; therefore approximately $P = \left[1 + 2 f \sin. \frac{a}{2} \left(1 + f \sin. \frac{a}{2}\right)\right] Q$, or more simply, we may put $P = \left(1 + 2 f \sin. \frac{a}{2}\right) Q$ and $P_1$

$$= \frac{Q}{1 + 2 f \sin. \frac{a}{2} \left(1 + f \sin. \frac{a}{2}\right)}, \text{ or more simply}$$

$$P_1 = \frac{Q}{1 + 2 f \sin. \frac{a}{2}} = \left(1 - 2 f \sin. \frac{a}{2}\right) Q.$$

If the cord passes over several edges, the forces $P$ and $P_1$ at the other extremity of the cord may be in like manner calculated by the repeated application of these formulæ. Let us take the simple case of a cord $ABC$, Fig. 188, passing over a body of $n$ edges, and at each edge making the same small angle $a$. The tension of the first portion of the cord will be $Q_1 = \left(1 + 2 f \sin. \frac{a}{2}\right) Q$, that of the extremity be $= Q$, that of the second

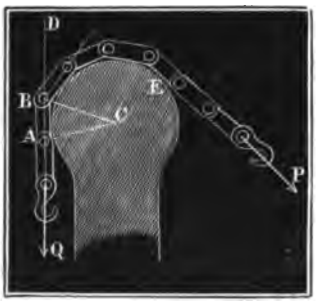
$$Q_2 = \left(1 + 2 f \sin. \frac{a}{2}\right) Q_1$$

FIG. 188.

$=\left(1+2f\,sin.\frac{a}{2}\right)^2 Q$; that of the third $Q_3 = \left(1+2f\,sin.\frac{a}{2}\right) Q_2$ $=\left(1+2f\,sin.\frac{a}{2}\right)^3 Q$; therefore, the force at the remaining extremity $P=\left(1+2f\,sin.\frac{a}{2}\right)^n Q$, in so far as motion takes place in the direction of the force $P$. If we change $P$ into $Q$, and $Q$ into $P$, we obtain $P_1=\frac{Q}{\left(1+2f\,sin.\frac{a}{2}\right)^n}$, provided only a motion in the direction of $Q$ is to be prevented.

The friction $F=P-Q$ is in the first case $=\left[\left(1+2f\,sin.\frac{a}{2}\right)^n-1\right] Q$ and in the second $= Q - P_1 = \left[\left(1+2f\,sin.\frac{a}{2}\right)^n-1\right] P_1$ $=\left[1-\left(1+2f\,sin.\right)^{-n}\right] Q.$

FIG. 189.




The same formulæ are applicable to a body winding round a cylinder, and consisting of members, as for instance, a chain $ABE$, Fig. 189, where $n$ is the number of links in contact, the length $AB$ of a link $= l$, and the distance $CA$ of the axis $A$ of a link from the centre of the arc covered $= r$, we then have $sin.\frac{a}{2}=\frac{l}{2r}$.

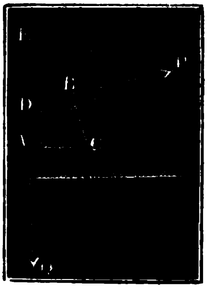
*Example.*—What is the amount of friction at the circumference of a wheel 4 feet in diameter, if twenty links of a chain, five inches long and one inch thick, pass over it, one end of which is fixed, and the other stretched by a force of 50 lbs? Here $P_1 =$ 50 lb. $n=20$, $sin.\frac{a}{2}=\frac{5}{48+1}=\frac{5}{49}$, let us now put for $f$ the mean value 0,35, we then obtain the friction with which the chain acts against the wheel in its revolution:

$$F=\left[\left(1+2\,.\,035\,.\,\frac{5}{49}\right)^{20}-1\right].\,50=\left[\left(1+\frac{35}{490}\right)^{20}-1\right].\,50$$

$$=\left[\left(\frac{15}{14}\right)^{20}-1\right].\,50=2,974.50=149 \text{ lbs.}$$

§ 176. A stretched cord $AB$, Fig. 190, lies about a fixed and cylindrically rounded body $ACB$, the friction may be likewise found

FIG. 190.



from the rule of the former paragraph. Here the angle of deviation $EDB = a^0 =$ the angle $ACB$ at the centre subtended by the arc of the cord $AB$; if we divide this into equal parts, and consider the arc $AB$ as consisting of $n$ straight lines, we have then $n$ corners, each with a deviation of $\frac{a^0}{n}$, and consequently the equation between the power and weight, as in the former §.: $P = \left(1 + 2f\,sin.\,\frac{a}{2n}\right)^n Q.$

From the smallness of $\frac{a}{2n}$ we may put the $sin.\,\frac{a}{2n} = \frac{a}{2n}$, whence $P = \left(1 + \frac{fa}{n}\right)^n Q.$ If further we make use of the binomial series, we obtain:

$$P = \left(1 + n\frac{fa}{n} + \frac{n(n-1)}{1.2}\frac{(fa)^2}{n^2} + \frac{n(n-1)(n-2)}{1.2.3}\frac{(fa)^3}{n^3} + \ldots\right) Q,$$

but as $n$ is very great, therefore $n-1 = n-2 = n-3 \ldots . = n$ it may be put.

$$P = \left(1 + fa + \frac{1}{1.2}(fa)^2 + \frac{1}{1.2.3}.(fa)^3 + \ldots\right) Q.$$

But now $1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \ldots . = e^x$, where $e$ denotes the base 2,71828... of the hyperbolic system of logarithms, therefore, it may also be put:

$$P = e^{fa}.\,Q, \text{ as also } Q = Pe^{-fa}, \text{ lastly } a = \frac{1}{f}\,hyp.\,Log.\,\frac{P}{Q}$$

$$= \frac{2,3026}{f}\,(Log.\,P - Log.\,Q).$$

If the arc of the cord is not given in parts of $\pi$, but in degrees, we have then to substitute $a = \frac{a^0}{180^0}.\,\pi$; if lastly, it be expressed by the number of coils $u$, we have then to put $a = 2\,\pi\,u$.

The formula $P = e^{fa}.\,Q$ expresses that the friction of the cord $F = P - Q$ upon a fixed cylinder is not dependant on the diameter of the same, but on the number of coils of the cord, and moreover shews that it may very easily be increased, almost to infinity. If we put $f = \frac{1}{3}$, we have:

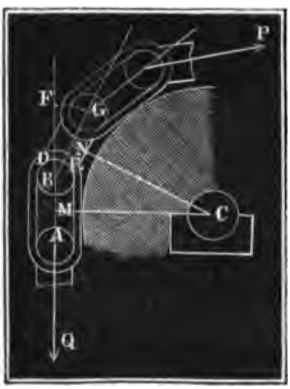
For $\frac{1}{4}$ of a winding $P = 1,69\ Q$
„ $\frac{1}{2}$ „ $P = 2,85\ Q$
„ 1 „ $P = 8,12\ Q$
„ 2 „ $P = 65,94\ Q$
„ 4 „ $P = 4348,56\ Q$, &c.

FIG. 191.


*Example.* To let down a shaft a load $P$ of 1200 lbs. from a certain height, the rope to which this weight is attached is wrapped $1\frac{3}{8}$ times about a round firmly clamped holder $AB$, Fig. 191, and the other extremity of the rope is held by the hand. With what force must this extremity be stretched, that the load may slowly and uniformly descend? If we put $f = 0,3$, we obtain this power $Q = Pe^{-f\alpha} = 1200 \,.\, e^{-0,3 \,.\, \frac{11}{8} \,.\, 2\pi} = 1200 \,.\, e^{-\frac{33}{40}\pi}$, therefore, $hyp.\ Log.\ Q = hyp.\ Log.\ 1200 - \frac{33}{40}\pi = 7,0901 - 2,5918 = 4,4983$. $Log.\ Q = 1,9536$, $Q = 89,9$ lbs.

§ 177. *Rigidity of chains.*—If ropes, or other similar bodies, &c., are placed over a pulley, or on the circumference of other cylinders revolving about an axis, the cord or chain friction considered in the foregoing paragraph ceases, because the circumference of the wheel has the same velocity as the rope; but now the force of bending by the winding of the rope about the pulley, and also that of unbending by the unwinding, becomes perceptible. If it is a chain which winds round a drum, there arises the resistance of the winding and unwinding manifested in a friction of the chain pins, while these last are revolving through a certain angle. If $AB$, Fig. 192, is one link, and $BG$ the one lying next, if further $C$ is the axis of revolution of the wheel on which the chain stretched by the weight $Q$ winds itself, if lastly $CM$ and $CN$ are let fall perpendicularly to the longer axes of the links $AB$ and $BG$, $MCN = \alpha^0$ is the angle through which the wheel revolves whilst a fresh link is laid on,

FIG. 192.



$FBG = 180^0 - ABE$ is the angle by which the link $BG$ with its bolt $BD$ revolves about the link $AB$. If now $BD = BE = r_1$ is the radius of the bolt, the point of friction or pressure $D$ describes an arc $DE = r_1\alpha$, and the mechanical effect of friction $f_1 Q$ hereby produced at the point $B$ is $f_1 Q \,.\, r_1 \alpha$. The force $P_1$ expended in overcoming this friction, acting in the direction of the longer axis $BG$, describes the simultaneous space $s = CN$ times the arc of the angle $MCN = CN \,.\, a$, and the mechanical effect $= P_1 \,.\, CN \,.\, a$; by equating both labours we have $P_1 \,.\, CN \,.\, a = f_1 \,.\, Qr_1 a$, and the required force, if $a$ represent the radius of the drum $CN$ increased by half the thickness of the chain: $P_1 = f_1 Q \,.\, \frac{r_1}{a}$.

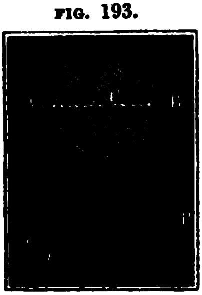
Without regard to friction, the force for a revolution of the wheel would be $P = Q$, having regard to the friction in the winding up of the chain $P = Q + P_1 = \left(1 + f_1 \frac{r_1}{a}\right) Q$. If the chain unwinds itself from the drum, an equal resistance takes place; if therefore, a winding on one side, and an unwinding on the other takes place, the force $P = \left(1 + f_1 \frac{r_1}{a}\right)^2 Q$, or approximately:

$$= \left(1 + 2 f_1 \frac{r_1}{a}\right) Q.$$

Lastly, if the pressure on the axle $= R$, and its radius $= r$, it follows that the force, taking into account all resistances, is:

$$P = \left(1 + 2 f_1 \frac{r_1}{a}\right) Q + f \frac{r}{a} R.$$

FIG. 193.



*Example.* What is the magnitude of a force $P$ at the extremity of a chain passing over a pulley $ACB$, Fig. 193, if the weight $Q$ drawing vertically downwards $= 110$ lbs., the weight of the pulley with the chain 50 lbs., the radius of the pulley measured to the middle of the chain $= 7$ in. that of the axle $C$ $\frac{5}{8}$ inch, and that the chain bolts $= \frac{3}{8}$ in.? The co-efficients of friction $f = 0{,}075$ and $f_1 = 0{,}15$, therefore from the last formula we obtain the force:

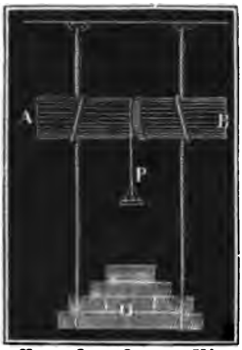
$$P = \left(1 + 2 \,.\, 0{,}15 \,.\, \frac{3}{8 \,.\, 7}\right) . \, 110 + 0{,}075 \,.\, \frac{5}{8 \,.\, 7}(110 + 50 + P),$$

or, if we assume $P$ on the right hand nearly $= 110$ lbs.
$P = 1{,}016 \,.\, 110 + 0{,}0067 \,.\, 270 = 111{,}76 + 1{,}81 = 113{,}6$ lbs.

§ 178. *Rigidity of cords.* — In bending a cord over a pulley or wheel, rigidity comes in as a resistance opposed to motion. The same takes place, but in a far less degree, in the unrolling from cylinders. Amontons and Coulomb set about mea-

suring the amount of this resistance by experiment. The results obtained by them are by no means satisfactory; partly because they are not in sufficient accordance with each other, and partly because they have not that extension so desirable for practical application. The experiments of Coulomb, which are those only of which we shall speak, were mostly made with hempen cords, of $\frac{1}{4}$ to $\frac{3}{4}$ inch thick, and with pulleys of from 1 to 4 inches diameter. Other experiments must be made before we can know what is the resistance of rigidity of a hempen rope of from 2 to 3 inches thick, when wrapped round a drum of from 1 to 6 feet in height; and also what is the amount of this resistance in the case of the wire-ropes, now come generally into use.*

FIG. 194.




Coulomb made his experiments in two ways; at one time with the apparatus of Amontons, Fig. 194, where $AB$ is a roller, with two cords winding round it, the tension is effected by a weight $Q$, and the rolling down of the cylinder by a second one $P$, which pulls, by means of a thin string at this roller; at another time, with a cylinder, which was allowed to roll upon a horizontal line, and round which a cord was wound, and from the difference of the weights suspended at both extremities, which effected a slow rolling forward, and after abstraction of the rolling friction, the resistance of the rigidity was deduced.

It results from the experiments of Coulomb, that the rigidity increases equally with the tension of the winding cord; that it consists, moreover, of a constant part $K$, which is no more than might be expected, because a certain force is necessary to bend an unstretched cord. It also appears that this resistance increases inversely as the diameter of the pulley; that it is, therefore, with twice the diameter of the pulley, only half as great; with three times the diameter, one-third, &c. The relation between the thickness and the rigidity of the cord is only approximately given from these experiments, since the rigidity depends upon the quality of the materials, the twisting of the strings, &c. For new ropes, the rigidity was found proportional to the power $d^{1.7}$, for old $d^{1.4}$, $d$ being the diameter of the rope. It is, therefore, only an approximation, when some assume that this resistance increases propor-

* See Appendix.

tionally with the thickness, others with the square of the thickness of the rope.

§ 179. The rigidity of cords may be therefore expressed by the formulæ:

$S = \frac{d^n}{a}(K + \nu Q)$, where $d$ is the thickness of the cord, $a$ the radius of the pulley measured to the axis of the cord, $n$, $K$ and $\nu$, numbers from experiment. *Prony* deduced from *Coulomb's* experiments that for new cords

$$S = \frac{d^{1,7}}{a}(2,45 + 0,053\,Q), \text{ and for old}$$

$$S_1 = \frac{d^{1,4}}{a}(2,45 + 0,053\,Q),$$ $a$ and $d$ being expressed in lines, $Q$ and $S$ in pounds. These expressions refer to the Paris measure; expressed in ~~English~~ Prussian inches and pounds, they become,

$$S = \frac{d^{1,7}}{a}(14,23 + 0,295\,Q) \text{ and } S_1 = \frac{d^{1,4}}{a}(6,83 + 0.141\,Q).$$ Not wrong

As these complicated formulæ do not always give the results in accordance with experiment, we may, until other experiments supersede them, put with *Eytelwein*

$S = \nu \,.\, \frac{d^2}{a} Q = \frac{d^2\,Q}{3500\,a}$, provided that $a$ be expressed in Prussian feet and $d$ in Prussian lines, $Q$ and $S$ in the same weight which, however, may be arbitrary. For the metrical standard $S = 18,6 \,.\, \frac{d^2\,Q}{a}$. This formula as might be expected will give satisfactory approximative results only for great tensions, as they generally occur in practice.

The rigidity of tarred ropes is found to be about $\frac{1}{6}$th greater than that of untarred; for wetted ropes, however, there is no determinate relation of this kind,

*Example.* With a tension of 350 lbs., and a radius of the pulley of $2\frac{1}{4}$ inches, the rigidity of a new rope of 9 lines $= 0,78$ (English) inches, according to *Prony* is: $S = \frac{4}{9}\left(\frac{4}{5}\right)^{1,7} 14,23 + 0,295 \,.\, 350) = 0,613 \,.\, 47,0 = 28,8$ lbs.; (according to *Eytelwein* $S =$) $\frac{9^2 \,.\, 350 \,.\, 24}{3500 \,.\, 5} = 38,9$ lbs. Were the tension $Q$ only 150 lbs., we should have from *Prony*, $S = 0,613 \,.\, 23,4 = 14,34$ lbs.; from *Eytelwein*:

$= \frac{81 \,.\, 24 \,.\, 3}{350} = 16{,}7$ lbs., therefore, here a better accordance. We see from these examples, how little reliance is to be placed on the formula.

*Remark.* A farther extension of this subject, viz. in respect to the rigidity of wire ropes, will be given under the article, windlass and capstan.

§ 180. Let us now apply the formula given for the rigidity of cords, to the theory of pullies. The radius $CA$ of a fixed pulley $= a$,

FIG. 195.


Fig 195, the radius of the axle $= r$, the thickness of rope $= d$, the weight $Q$ at one extremity of the cord $= G$, and the power which must be applied to the other extremity to draw it slowly up $= P$. Without friction on the axle and without rigidity, $P$ would be $= Q$, but because the axle exerts a pressure $P + Q + G$ against its bearing, there arises a friction $f (P + Q + G)$ which, since it acts at the radius $r$, makes an increase of power $\frac{f\,r}{a}\,(P + Q + G)$ necessary; since the rigidity of the rope must be added to this, which manifests itself thus, that the cord does not at once take the curvature of the circumference of the pulley, but lays itself upon the pulley with an increasing curvature, and in this manner causes an extension of the arm of $Q$; the arm, therefore, of the weight $Q$ is not $CA$ but $CD$, and the force at the arm $CB$

$$= CA = a,\ P = \frac{CD}{CA} \,.\, Q = \left(1 + \frac{AD}{CA}\right) Q = Q + S = Q + \frac{d^n}{a}(K + \nu Q).$$

The complete equation between the power and the weight is now

$$P = Q + \frac{d^n}{a}(K + \nu Q) + \frac{f\,r}{a}\,(P + Q + G).$$

In the wheel and axle the power $P$ acts at a different arm $a$ to that of the weight, whose arm $= b$, therefore,

$$Pa = Qb + d^n\,(K + \nu Q) + fr\,(P + Q + G), \text{ and}$$

$$P = \frac{b}{a}\,Q + \frac{d^n}{n}(K + \nu Q) + \frac{f\,r}{a}\,(P + Q + G).$$

Hence the force

$$P = \frac{(b + \nu d^n + fr)\ Q + d^n \,.\, K + fr\,G}{a - fr}.$$

*Example.*—A weight $Q = 200$ lbs. is to be raised with the wheel and axle by a power $P = 50$ lbs.; suppose the wheel to be $1\frac{1}{2}$ feet, and the pivot $\frac{3}{8}$ inch radius, and

the rope applied $\frac{1}{2}$ an inch thick, and the weight of the whole machine 70 lbs., what radius must we give to the axle? It must be:

$$b = [Pa - d^n (K + \nu Q) - fr (P + Q + G)] : Q,$$

therefore, in numbers if we put $f = 0{,}075$,

$$b = [50 \,.\, 18 - (\tfrac{1}{2})^{1,7} \,.\, (14{,}23 + 0{,}295 \,.\, 200) - 0{,}075 \,.\, \tfrac{1}{2} \,.\, 320] : 200$$
$$= [900 - 0{,}308 \,.\, 73{,}23 - 12] : 200 = 865{,}4 : 200 = 4{,}327 \text{ inches.}$$

Without additional resistances $b$ would be

$$= Pa : Q = 75 : 200 = 0{,}375 \text{ feet} = 4\tfrac{1}{2} \text{ inches.}$$

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# CHAPTER VI.

## ELASTICITY AND RIGIDITY.

§ 181. *Elasticity.*—The parts of a rigid body adhere to each other with a certain force, which is called *cohesion,* and which must be overcome when bodies are changed in their figure and extension, or broken. The first effect which forces produce in a body, is a change in the position of their parts relatively to each other, and a resulting change of form or volume of the body. If the forces acting upon a body exceed certain limits, a separation of the parts, and a breaking of the whole body ultimately takes place. The capability of bodies, which suffer a change of form by the action of forces, to resume perfectly their former state after the withdrawal of the forces, is called *elasticity.* The elasticity of every body has a certain limit. If the change of form or volume exceeds a certain amount, the body retains an alteration of its volume, even when the forces which have effected it cease to act. The limit of elasticity is different for different bodies. Bodies which suffer a considerable change of form before this limit is attained, are called *perfectly elastic.* Those, on the other hand, in which there is scarcely any appreciable change of form preceding the limit, are called *inelastic,* although in reality there exist no bodies of this kind.

It is an important rule in building and in machinery never to load the materials to that extent, that any alteration of their form should attain, much less exceed, the limits of elasticity.

§ 182. *Elasticity and strength.*—Different bodies present different phenomena when their form is changed beyond the limits of elasticity. If a body be brittle it flies into pieces. If it be ductile, as many of the metals, it will admit of alterations of form beyond the limits of elasticity, without suffering a separation of its parts.

Many bodies are hard, others soft; the one opposes a great resistance to a separation of their parts, whilst the others easily allow of this to be brought about.*

In the restricted sense of the word, we understand by *elasticity*, the resistances which a body opposes to a change of form; on the other hand by *strength*, the resistance which a body opposes to a separation of its parts. We will accordingly consider each of these in the following.

According to the way in which external forces act upon a body, and change their form and dimensions, we distinguish the elasticity and strength of bodies, into:

1. The *absolute resistance*,
2. The *relative resistance*,
3. The *resistance to compression*, and
4. The *resistance to torsion*.

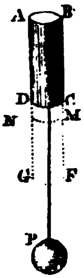
If two external forces act by tension in the direction of the axis of a body, it resists by its *absolute elasticity* and *strength* any extension or rupture. If, on the other hand, these forces act at right angles to the axis of a body, the body will resist by its relative elasticity and strength any bending or fracture. If, further two forces act in the direction of the axis of a body by compression, so that the body becomes either compressed or crushed, then there is the *elasticity* and *strength of compression* to be overcome. If, lastly, forces strive to turn a body in opposite directions about an axis, or which do not act in the same plane normal to the axis, than there is the *elasticity* and *strength of torsion* to be overcome.

§ 183. *Modulus of elasticity.*—The change of volume within the limits of elasticity, *i. e.* the extension or compression of a body, is pretty nearly proportional to the force exerted, but if this change exceeds that limit, this proportionality ceases, and the change goes on rapidly to that of rupture or crushing. As a measure of the elasticity, the *modulus of elasticity* $E$, is that which expresses the force which is necessary to elongate a prismatic body of a transverse section, unity = *i. e.* a *square foot*, to double, or to compress it to ~~one half of its original length~~. A different modulus corresponds to different materials: for each substance it must be determined by experiment. For the rest we must bear in mind that the modulus of elasticity only holds good for extensions and com-

* See Appendix.

pressions within the limit of elasticity, and its measure is one, not of observation, but of hypothesis and calculation, because it is not easy to find a body which, without exceeding the limit of elasticity, allows so great a change of form as the modulus of elasticity supposes.*

FIG. 196.

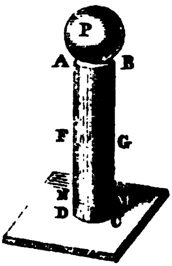



A body $AC$, Fig. 196, which has the initial length $AD=BC=l$, and the transverse section 1, requires for its extension $DG=l$, the force $E$, if, however, its transverse section is $F$, that is, if it consists of $F$ contiguous prisms, this force is then $F\,.\,E$. If, on the other hand, this body is to be extended a length $DN=CM=\lambda$, then for the force $P$

$P : F\,.\,E=\lambda : l$, it therefore follows

1. That $P=\frac{\lambda}{l}F\,.\,E$, and inversely, 2. $\lambda=\frac{P}{F.E}\,.\,l$.

FIG. 197.




The same formulæ are also applicable to a body $AC$, Fig. 197, of the length $AD = l$, and the transverse section $AB = F$, if it become shortened a length $\lambda$ by the compression of a force $P$.

By the aid of these formulæ we may calculate from the change of volume ($\lambda$) the corresponding force $P$, or from the force $P$ the quantity of the extension or compression.

*Example.* If the modulus of elasticity of brass wire amounts to 14625000 lbs. what force is necessary to stretch $\frac{1}{12}$ inch a wire 5 feet in length and $\frac{1}{6}$ inch in thickness? $l = 5\,.\,12 = 60$ inches, $\lambda = \frac{1}{12}$ inch, consequently $\frac{\lambda}{l} = \frac{1}{720}$; further, $F = \frac{\pi d^2}{4} = 0{,}7854\left(\frac{1}{6}\right)^2 = 0{,}0218$ square inches, the required force accordingly is $P = \frac{1}{720}\,.\,0{,}0218\,.\,14625000 = 442$ lbs. — 2. The modulus of elasticity of iron wire is 263250000 lbs.; if an iron chain, 60 feet long and 0,2 inches thick, be stretched by a force of 150 lbs., the same will be increased by a length $\lambda = \frac{150\,.}{0{,}7854\,.\,(0{,}2)^2}\,.\,\frac{60\,.\,12}{263250000} = \frac{108000}{31416\,.\,263{,}25} = 0{,}013$ inches $= 0{,}156$ lines.

§ 184. *Modulus of working load and strength.*—The force $T$, which a body of the transverse section unity accumulates when its extension attains the limit of elasticity, is easily determined from the modulus of elasticity $E$ and the elongation

* See Appendix.

$\lambda$ corresponding to this limit; for $T:E=\lambda:1$, therefore, $T=\lambda E$. l
This is the strain beyond which materials used in construction and machinery must not be loaded if they are to maintain sufficient safety together with durability. If the transverse section of a body, which has to sustain a tensile strain $P$ be $= F$, we have then

$$1.\ P=FT,\ \text{and}\ 2.\ F=\frac{P}{T}.$$

The force $T$ by which we judge of the working load of bodies, may be introduced into calculations under the name of ***modulus of working load.***

The ***modulus of strength*** $K$, which expresses the force by which a body of the transverse section unity becomes ruptured, is entirely different from this modulus. If the transverse section of a prismatic body, or its least section $= F$, it follows that the force, for the rupture of this body, is:

$$1.\ P_1=FK,\ \text{and inversely,}\ 2.\ F=\frac{P_1}{K}.$$

Generally the strength of materials of construction and parts of machines are calculated by the co-efficient $K$, which is divided for security's sake, by one of the numbers 3, 4 to 10. This makes little difference in the result, as we may see from a comparison of the values found in the succeeding table, but the supposition is incorrect, or to be justified only in so far as the modulus of strength is from 3, 4 to 10 times that of the modulus of tenacity, or generally bears a constant relation to it.

If the section of the body be a circle of the diameter $d$, we have therefore,

$$\frac{\pi d^2}{4}=F,\ \text{so that}\ d=\sqrt{\frac{4F}{\pi}}=1{,}128\ \sqrt{F}=1{,}128\sqrt{\frac{P}{T}}$$

and hence, from the load or strain $P$ on a body, and the modulus of tenacity $T$ of its material, the ~~strength~~ may be found, for which the body will not be strained beyond the limit of elasticity.

*Example.* What load will a column of fir sustain, if it be 5 inches in breadth and 4 inches in thickness? The modulus of tenacity being taken at 3000 lbs. and the section $F$ being $= 5 \,.\, 4 = 20$ square inches, we obtain $P = 20 \,.\, 3000 = 60000$ lbs. for the power of tenacity of this column. But if we take the modulus of strength $K = 12000$ lbs., and assume a triple security, we obtain $P = 20 \,.\, \frac{12000}{3} = 80000$ lbs.; but to maintain security for a long period, we must only take one-tenth of $K$, and we shall then have $P = 20 \,.\, 1200 = 24000$ lbs.—2. A round and wrought-iron pump-

rod is to sustain a weight of 4500 lbs.; what diameter ought it to have? Here $T = 20000$ lbs., therefore, $d = 1{,}128 \sqrt{\frac{4500}{20000}} = 1{,}128 \cdot \sqrt{\frac{9}{40}} = 0{,}535$ feet. The modulus of strength for wrought iron of the medium kind = 58000 lbs., and if we take one-sixth for the security, we then obtain $K = 10000$ lbs. and $d = 1{,}128 \sqrt{\frac{4500}{10000}} = 0.756$ inch, the requisite thickness of the rod.

§ 185. *Strongest form of body.*—If a vertically suspended prismatic body, for example, a pole or cord, is very long, its weight $G$ must be added to the force of rupture, and, therefore $P + G$ must be put $= FT$. If now $l$ be the length of the body, and $\gamma$ the weight of a cubic inch of its mass, we have then $G = Fl\gamma$, and, therefore, $P = F(T - l\gamma)$, as inversely $F = \frac{P}{T - l\gamma}$. If a body $ABC..G$, Fig. 198, consists of equal portions, each of the length $l$, its successive transverse sections are as follows. The section of the first portion is as before $F_1 = \frac{P}{T - l\gamma}$. For the second portion, whose section is $F_2$ and weight $F_2 l\gamma$, $P + F_1 l\gamma + F_2 l\gamma = F_2 T$, hence $F_2 = \frac{P + F_1 l\gamma}{T - l\gamma} = F_1 + \frac{F_1 l\gamma}{T - l\gamma} = F_1\left(1 + \frac{l\gamma}{T - l\gamma}\right)$. For the third portion it follows that $F_3 = F_2 \left(1 + \frac{l\gamma}{T - l\gamma}\right) = F_1 \left(1 + \frac{l\gamma}{T - l\gamma}\right)^2$, for the fourth $F_4 = F_3 \left(1 + \frac{l\gamma}{T - l\gamma}\right) = F_1 \left(1 + \frac{l\gamma}{T - l\gamma}\right)^3$, and generally for the $n$th portion: $F_n = F_1 . \left(1 + \frac{l\gamma}{T - l\gamma}\right)^{n-1}$ or $F_n = \frac{P}{T - l\gamma}\left(1 + \frac{l\gamma}{T - l\gamma}\right)^{n-1}$, the corresponding section.

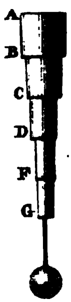
If $l$ is very small, the portions therefore very short, we may then put:

$$F_n = \frac{P}{T}\left(1 + \frac{l\gamma}{T}\right)^{n-1}.$$

If the number of portions is very great, or if the thickness of the body $AG$, Fig. 199, increases uniformly from below upwards, we may then (from the reasons in § 175, put the cross section

$$F_n = \frac{P}{T} \cdot e^{(n-1)\frac{l\gamma}{T}} = \frac{P}{T} \cdot e^{\frac{nl\gamma}{T}} = \frac{P}{T} \cdot e^{\frac{L\gamma}{T}}$$

FIG. 198. FIG. 199.





where $e$ represents the base 2,71828... of the Naperien logarithms, and $L$ the entire length of the body.

A body of uniform thickness to have the same tenacity throughout, must have a transverse section $F = \frac{P}{T - L\gamma}$. If $L\gamma$ is small as compared with $T$, $\frac{L\gamma}{T}$ is a small fraction, so that we may put:

$$F_n = \frac{P}{T}\left[1 + \frac{L\gamma}{T} + \frac{1}{2}\left(\frac{L\gamma}{T}\right)^2\right] \text{ and}$$

$$F = \frac{P}{T}\left[1 + \frac{L\gamma}{T} + \left(\frac{L\gamma}{T}\right)^2\right],$$

further, the weight of the first body is

$$= \frac{F_1 + F_n}{2} . L\gamma = \left[1 + \frac{1}{2}\frac{L\gamma}{T} + \frac{1}{4}\left(\frac{L\gamma}{T}\right)^2\right] \frac{P}{T} L\gamma;$$

and that of the second $= F . L\gamma$

$$= \left[1 + \frac{L\gamma}{T} + \left(\frac{L\gamma}{T}\right)^2\right] \frac{P}{T} L\gamma;$$

hence the prismatic body is heavier, and on that account more costly than one having at each point in its length a cross section corresponding to the load it has to bear, and which may therefore be called a body of uniform resistance, or ***a body of the strongest form.***

*Examples.*—1. What cross section ought a wrought iron shaft to have, when besides its own weight it has to sustain a load $P = 75000$ lbs.? The modulus of tenacity or strain is taken at $T = \frac{1}{8} K = 10311$ lbs., and the weight of a cubic inch of wrought iron $\gamma = \frac{7,60 . 52,4}{12 . 12 . 12} = \frac{75000}{10311 - 12000 . 23046} = 9,939$ square inches, the section sought is $F = \frac{P}{T - L\gamma}$, and the weight of the shaft $G = F . L\gamma$ $9,39 . 1200 . 0,23046 = 2747$ lbs.—2. If we were to give to this shaft the form of a body of uniform resistance, we should obtain for the least section $F = \frac{P}{T} = \frac{75000}{10311} = 7,28$ square inches, for the greatest section $F_n = 7,28$ square inches $= 7,28 . e^{0,23066 . 1,2} = 7,28 . e^{0,27654} = 9,73$ square inches, and the weight $= \left(\frac{7,28 + 9,73}{2}\right) . 2765 = 22502$ lbs. (approximately).

§ 186. *Numerical values.*—In the following table are given the

mean values of the different moduli, of elasticity, tenacity, and strength of the materials most commonly occurring in construction.

## TABLE. I.

### THE MODULI OF ELASTICITY AND STRENGTH.

| NAMES OF THE SUBSTANCES. | Extension at the limits of the elasticity. $\frac{\lambda}{l}$. | Modulus of elasticity. $E$. | Modulus of working load. $T$ | Modulus of strength. $K$. | Modulus of safety. $K_1$. |
|---|---|---|---|---|---|
| Box, oak, fir, firm Scotch fir . | $\frac{1}{600}$ | 1856005 | 3094 | 12373 | 1237 |
| Iron in wires . . . . . . . | $\frac{1}{1250}$ | 26808964 | 21650 | 87645 | 14436 |
| Iron in bars . . . . . . . . | $\frac{1}{1520}$ | 29902306 | 20622 | 59805 | 10311 |
| Iron in plates . . . . . . . | | 26808969 | | 56712 | 9280 |
| Cast iron . . . . . . . . . | $\frac{1}{1200}$ | 17528938 | 14436 | 19592 | 3094 |
| Steel . . . . . . . . . . | $\frac{1}{835}$ | 30933420 | 37120 | 123700 | 20622 |
| Hard cast steel . . . . . . | $\frac{1}{4500}$ | 45369016 | 98987 | 150543 | 24740 |
| Copper . . . . . . . . . | | | | 38151 | 6187 |
| Copper wire . . . . . . . . | | | | 75271 | 12370 |
| Brass . . . . . . . . . . | $\frac{1}{1320}$ | 97955830 | 7218 | 18560 | 3093 |
| Brass wire . . . . . . . . | $\frac{1}{742}$ | 149511530 | 20622 | 75271 | 12370 |
| Bell metal . . . . . . . . | $\frac{1}{1590}$ | 48462358 | 3093 | 35058 | 5774 wrong |
| Lead . . . . . . . . . . | $\frac{1}{477}$ | 721779 | 1547 | 928 | 329 do. |
| Leaden wire . . . . . . . . | $\frac{1}{1500}$ | 1031114 | 722 | 2062 | 351 |
| Marble . . . . . . . . . | | 2680896 | | 2062 | 206 |
| Ropes under 1 inch . . . . | | | | 9280 | 3093 |
| „ 1 — 3 „ . . . . | | | | 7218 | 2371 |
| „ above 3 „ . . . . | | | | 5156 | 1753 |
| Straps . . . . . . . . . | | | | | 299 |

The values contained in the second vertical column of this table, of the relative extension $\left(\frac{\lambda}{l}\right)$ at the limits of elasticity, give likewise the relation $\frac{T}{E}$ of the values of the fourth and third columns. The sixth column is derived from the fifth, if we divide the woods by 10, the metals by 6, and the cords by 3. The strength of wires is always greater than that of rods, because the enveloping crust of wires is stronger than their nucleus.

§ 187. *Flexure of bodies.*—A prismatic body *ABCD*, Fig. 200, is fixed at one extremity, for instance, imbedded in a wall, and at the other extremity acted upon by a force *P*; strains then take place in this body, in consequence of which, one part is extended, and the other compressed, and the whole becomes deflected. If we imagine the whole body to be decomposed into thin laminæ by planes parallel to the axis, and at right angles to the direction of force, we may then assume that there is a certain mean lamina *KLM*, which is called the ***neutral surface*** or the ***neutral axis*** of the laminæ, which is not strained by this flexure, and remains unaltered in length, while the laminæ on the convex side undergo an extension, and those on the concave side a compression. Let $A\,B\,C_1\,D_1$, Fig. 201, be the longitudinal section of the body, *KL* its neutral axis, $NO_1$ an extended and $UV_1$ a shortened or compressed lamina. If the flexure had taken place without any change of volume, *KL* would be = *AD* = *NO*, &c.; *i. e.* the length of all the laminæ would be one and the same; the body also would have the form *ABCD*, but because the body has sustained extensions and compressions, certain laminæ, such as *AD*, *NO*, &c., have undergone the elonga-

FIG. 200.

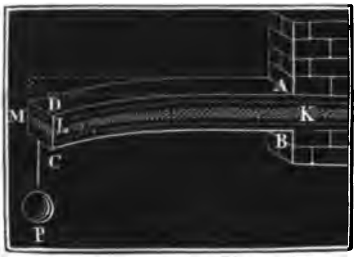
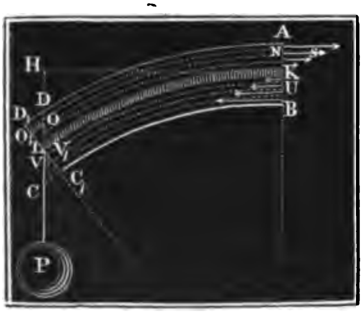



FIG. 201.



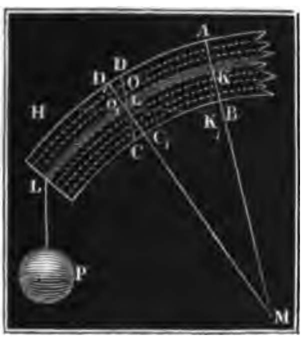

* See Appendix.

tions $DD_1$, $OO_1$, &c., and others, as $BC$ and $UV$, the compressions $CC_1$, $VV_1$, &c., and the form of the body has changed to that of $ABC_1D_1$. In every case the elongations $DD_1$, $OO_1$, and the compressions $CC_1$, $VV_1$, &c., are proportional to the distances $LD$, $LO$, $LC$, $LV$, &c., from the neutral axis. But the strains in the direction of the laminæ are in the ratio of the elongations and compressions effected by them; we must, therefore, assume that these strains are proportional to the distances from the neutral axis. If, then, we put the strain on a fibre, or layer of fibres, of a transverse section equal to unity (a square inch), and at a unit of distance (one inch) from the neutral axis $= S$; the strain for the distance $KN=z$ is $Sz$, and for the section $F$, it is $FSz$. If now the experimental number $S$ represents both the extension and compression, we know the sum of all the strains $= (F_1z_1+F_2z_2+\ldots)\ S$, where $F_1$, $F_2$, &c. are the sections and $z_1$, $z_2$, &c. the distances from the neutral axis. In order that the tensions may produce no pressure, and therefore no alteration in the length, at the extremity $K$ of the neutral axis, which we may regard as the fulcrum of a lever, the sum of the tensions $(F_1z_1+F_2z_2+\ldots)\ S$, and therefore also $F_1z_1+F_2z_2+\ldots$ must be $= 0$; *i. e. the neutral axis or the neutral lamina must pass through the centre of gravity of the cross section of the body.*

We may now compare the condition of the body with the equilibrium of a bent lever. The force $P$ acts at the arm $KH=l$, the moment is, therefore, $M = Pl$, and balances the collective forces of extension and compression, whose moments are $z_1 \,.\, F_1Sz_1$, $z_2.F_2Sz_2$, &c., or $F_1z_1^2.S$, $F_2z_2^2.S$, &c.; consequently we must put

$$M = Pl = (F_1z_1^2 + F_2z_2^2 + \ldots)\,.\,S.$$

FIG. 202.




This formula holds good for each cross section of the body, only for $l$ we must substitute its distance each time from the point of application $L$ of the force $P$. The factor $F_1z_1^2 + F_2z_2^2 + \ldots$ is dependant only on the cross section of the deflected body, and may be represented by the letter $W$. Hence we may put $M=Pl=WS$, and assert that the tension or strain of a trans-

verse section is proportional to its distance $l$ from the point of application of the force.

§ 188. From the modulus of elasticity $E$, the length of a fibre $l$ at a unit of distance (an inch) from the neutral axis, and the elongation $\lambda$ which it undergoes, the corresponding tension $S = \frac{\lambda}{l} E$ is known. If now $ABC_1D_1$, Fig. 202, is a short portion of the deflected body, $KL = l$ its length, and $MK = ML = \rho$ its radius of curvature, we have then $DD_1 : KL = LD : ML$, and also $OO_1 : KL = LO : ML$; *i. e.* $OO_1 : l = LO : \rho$. If we now assume $LO = 1$ and $OO_1 = \lambda$, we obtain $\lambda : l = 1 : \rho$, and hence $S = \frac{\lambda}{l} E = \frac{E}{\rho}$. If, finally, we substitute this value of $S$ in the formula $M = WS$, we have the moment $M = \frac{WE}{\rho}$, and inversely, $WE = M\rho$.

The product $WE$ is called the moment of flexure, ***and hence the product of the moment M and the radius of curvature ρ is equivalent to the moment of flexure for all cross sections.***

If we divide the neutral axis $KL$, Fig. 204, into $n$ equal parts,

FIG. 203.

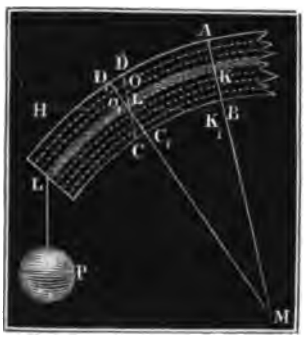
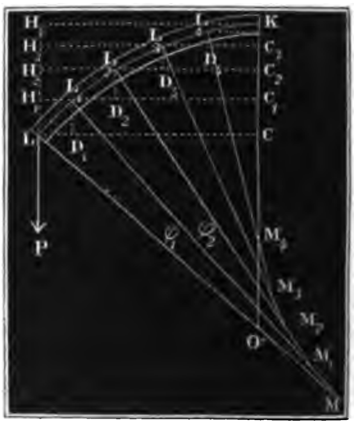



FIG. 204.




as $LL_1$, $L_1L_2$, $L_2L_3$, &c. $= \frac{l}{n}$, and determine the radii of curvature $\overline{ML_1} = \rho_1$, $\overline{M_1L_2} = \rho_2$, &c., corresponding to these parts, the angle of curvature $\overline{LML_1} = \phi_1{}^0$, $\overline{L_1M_1L_2} = \phi_2{}^0$, &c., which

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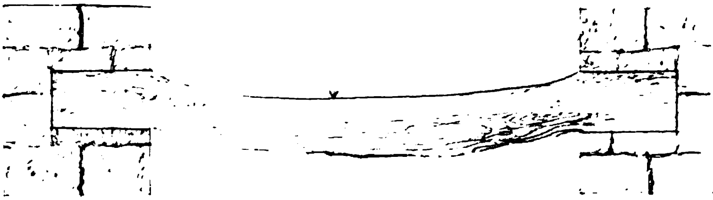


every two radii of curvature include, are known, viz. $LL_1 = \frac{l}{n}$ $= \rho_1 \phi_1$, $L_1L_2 = \frac{l}{n} = \rho_2 \phi_2$, &c., and therefore $\phi_1 = \frac{l}{n\rho_1}$, $\phi_2 = \frac{l}{n\rho_2}$, &c. If, further, we substitute $\rho_1 = \frac{WE}{M_1}, \phi_2 = \frac{WE}{M_2}$ &c., we then obtain $\phi_1 = \frac{M_1 l}{nWE}, \phi_2 = \frac{M_2 l}{nWE}$, &c.; and by the summation of all these angles we find the angle $LOK = a^0$, by which a greater portion, or the whole neutral axis is deflected.

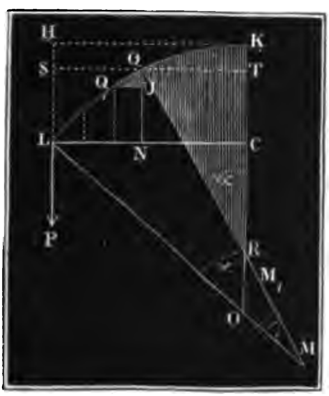
§ 189. *Elastic curve.*—If we suppose a small flexure, we may take the projection $CL = KH_4$, parallel to the initial direction of the undeflected beam, and equal to the length of the beam itself, and likewise the projections $LD_1$, $L_1D_2$, &c., equal to the parts $LL_1$, $L_1L_2$, &c., of the neutral axis, *i. e.* $= \frac{l}{n}$, and we obtain the moments $M_1 = \frac{Pl}{n}$, $M_2 = \frac{2\,Pl}{n}$, $M_3 = \frac{3\,Pl}{n}$, &c. If we substitute these values in the formulæ for $\phi_1$, $\phi_2$, &c., then the measures of the angles of curvature are given:

$$\phi_1 = \frac{Pl^2}{n^2 WE},\ \phi_2 = \frac{2\,Pl^2}{n^2 WE},\ \phi_3 = \frac{3\,Pl^2}{n^2 WE},\ \text{\&c.};$$

and by addition, the measure of the whole angle of curvature $KOL = a$ of the neutral axis:

$$a = \frac{Pl^2}{n^2 WE}(1+2+3+\ldots+n) = \frac{Pl^2}{n^2 WE} \cdot \frac{n^2}{2} = \frac{Pl^2}{2\,WE}.$$

FIG. 205.




With the assistance of the last formula, we may now find the equation to the curve formed by the neutral axis, $KL$ Fig. 205. Let us divide the absciss $LN = x$, commencing at the point $L$, into $m$ equal parts and find the parts of the ordinate $NQ = y$ corresponding to them. Since the radius of curvature $QR$ is perpendicular to the part of the arc $QQ_1$, the angle $QQ_1U = QRK = a_2$, and therefore the part $QU$ of the ordinate $y_1 = Q_1U \,.\, tang.\, a_2$,

or $Q_1U$ being put $= \frac{x}{m}$ and $tang.\ a_2 = a_2$, $QU\ \frac{xa_2}{m}$. Now

$$a_2 = LOK - LMQ = a - a_1 = \frac{Pl^2}{2WE} - \frac{Px^2}{2WE} = \frac{P}{2WE}(l^2 - x^2);$$

it follows, therefore, that $QU = \frac{x}{m} \cdot \frac{P}{2WE}(l^2 - x^2)$. If for $x^2$ we substitute successively $\left(\frac{x}{m}\right)^2, \left(\frac{2x}{m}\right)^2, \left(\frac{3x}{m}\right)^2$, &c., we then obtain by the last formula all the parts of $y$, and by the addition of these, the whole ordinate:

$$NQ = y = \frac{x}{m} \cdot \frac{P}{2WE}\left[l^2 - \left(\frac{x}{m}\right)^2 + l^2 - \left(\frac{2x}{m}\right)^2 + l^2 - \left(\frac{3x}{m}\right)^2 + \dots\right] = \frac{x}{m} \cdot \frac{P}{2WE}\left[ml^2 - \left(\frac{x}{m}\right)^2 (1^2 + 2^2 + 3^2 + \dots + m^2)\right], \text{ i. e. } y = \frac{Px}{2WE}\left(l^2 - \frac{x^2}{3}\right).$$

By this formula we may calculate for every absciss $x$ the corresponding ordinate $y$, and likewise for the whole length $CL = l$, the height of the arc $CK = a$. This last is:

$$a = \frac{Pl}{2\ WE}\left(l^2 - \frac{l^2}{3}\right) = \frac{Pl^3}{3\ WE}.$$

Therefore, *the height of the arc increases as the force and the cube of the length.*

If we have $a$ by measurement, we may find from this formula the modulus of elasticity, $E = \frac{Pl^3}{3\ Wa}$.

§ 190. If the whole load is uniformly distributed over the beam, and if each unit of length sustains a portion $= q$, therefore, for the whole length $l$, $Q = lq$, we must substitute for the moments $\frac{1}{n}Pl, \frac{2}{n}Pl, \frac{3}{n}Pl$, &c., the moments $\frac{1}{2}q\left(\frac{l}{n}\right)^2$, $\frac{1}{2}q\left(\frac{2\,l}{n}\right)^2$, $\frac{1}{2}q\left(\frac{3\,l}{n}\right)^2$, &c., because the centres of gravity of the loads $q\frac{l}{n}, q \cdot \frac{2\,l}{n}, q \cdot \frac{3\,l}{n}$, &c., lie in the middle of $\frac{l}{n}, \frac{2\,l}{n}, \frac{3\,l}{n}$, the arms are, therefore, $\frac{1}{2} \cdot \frac{l}{n}$, $\frac{1}{2} \cdot \frac{2\,l}{n}, \frac{1}{2} \cdot \frac{3\,l}{n}$. Hence we obtain

$$\phi_1 = \frac{1}{2} \cdot \frac{ql^3}{n^3\,WE}, \phi_2 = \frac{1}{2} \cdot \frac{2^2 \cdot ql^3}{n^3\,WE}, \phi_3 = \frac{1}{2} \cdot \frac{3^2 \cdot ql^3}{n^3\,WE}, \text{ \&c.}$$

And, therefore,

$$a = \tfrac{1}{2} \cdot \frac{ql^3}{n^3 WE}(1^2+2^2+3^2+\ldots+n^2) = \frac{ql^3}{2n^3 WE} \cdot \frac{n^3}{3} = \frac{ql^3}{6\,WE},$$

If again we take $x=l$, we obtain the height of the arc

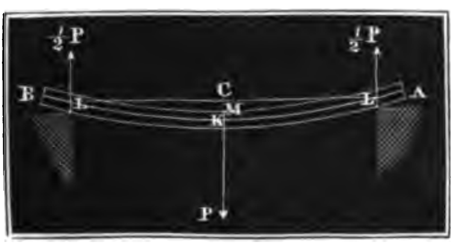
$$a = \frac{ql}{6\,WE} \cdot \tfrac{3}{4}\, l^3 = \frac{ql^3}{8\,WE} = \frac{Ql^3}{8\,WE} = \tfrac{3}{8} \cdot \frac{Ql^3}{3\,WE},\ i.\,e.$$

$\frac{3}{8}$ths as great as if the load $Q$ were suspended at the extremity of the beam.

If the beam is loaded by a weight $Q$, uniformly distributed and by a force $P$ at the extremity, the height of the arc is then

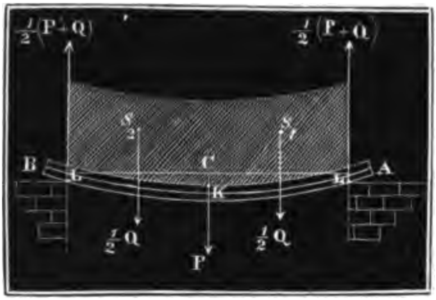
$$a = \frac{Pl^3}{3\,WE} + \frac{Ql^3}{8\,WE} = \left(\frac{P}{3} + \frac{Q}{8}\right)\frac{l^3}{WE}.$$

FIG. 206.




If a beam $AMB$, Fig. 206, is supported at both extremities, and loaded in its middle by a weight $P$, both the extremities are deflected upwards by the reactions $\frac{1}{2}P$ and $\frac{1}{2}P$, as was in the former case (§ 189) the one extremity downwards, the formula then found here holds

FIG. 207.




good, if instead of $P$, we put $\frac{P}{2}$, and instead of the whole length $LL = l$, half the length $KL = \frac{l}{2}$. Hence the height of the arc is:

$$a = \frac{\frac{1}{2}P.(\frac{1}{2}l)^3}{3\,WE} = \tfrac{1}{16}\cdot\frac{Pl^3}{3\,WE}$$

= a sixteenth of the height of the arc of the beam, which is loaded at its extremity.

If lastly the load $Q = ql$ is uniformly distributed over the body $AB$, Fig. 207, supported at both extremities, we must put in the formula $a = \left(\frac{P}{3} + \frac{Q}{8}\right)\frac{l^3}{WE}$ in place of $l$, $\frac{l}{2}$, in place of $P$, $\frac{P+Q}{2}$ and $Q, -\frac{Q}{2}$, because with respect to $K$, the weight $\frac{Q}{2}$ at the arm $\frac{l}{4}$ is opposed to the reaction $\frac{P+Q}{2}$ at the arm $\frac{l}{2}$. Consequently

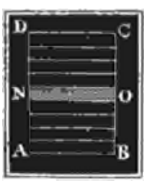
$$a = \left(\frac{P+Q}{6} - \frac{Q}{16}\right)\frac{l^3}{8\,WE} = \left(P + \frac{5}{8}\,Q\right)\frac{l^3}{48\,WE}.$$

For $P = o$, $a = \frac{5}{8}\cdot\frac{Ql^3}{48\,WE}$; the load is, therefore, uniformly distributed over the whole arc, and the height of the arc is $\frac{5}{8}$ times as great as if the weight acted at the middle of the beam.

§ 191. *Rectangular beams.*—In order to give the relations of flexure of a beam or other prismatic body, and the elastic curve formed by its neutral axis, the transverse section of the body must be known, and the moment of flexure $WE$, calculated from it.

If the section of the beam be a rectangle $ABCD$, Fig. 208, of the width $AB = CD = b$, the height $AD = BC = h$, the moment of flexure $WE = (F_1 z_1^2 + F_2 z_2^2 + \ldots)\,E$ will be known if we decompose this cross section by lines parallel to the neutral axis $NO$ into $2\,n$ equal laminæ, each having the area $b\,.\,\frac{h}{2n} = \frac{b\,h}{2n}$; and determine the moments

FIG. 208.



these laminæ, and add them together. If we put successively $\frac{1}{n} \cdot \frac{h}{2}, \frac{2}{n} \cdot \frac{h}{2}, \frac{3}{n} \cdot \frac{h}{2}$ for $z$ in $\frac{bh}{2n} \cdot z^2 E$, we shall then obtain the moments of the laminæ on one side of the neutral axis; but if we double their sum, we have the complete moment of flexure

$$WE = 2 \cdot \frac{bh}{2n} \left[\left(\frac{h}{2n}\right)^2 + \left(\frac{2h}{2n}\right)^2 + \left(\frac{3h}{2n}\right)^2 + \ldots\right] E$$

$$= \frac{bh}{n} \cdot \left(\frac{h}{2n}\right)^2 (1^2 + 2^2 + 3^2 + \ldots + n^2)\, E = \frac{bh^3}{4 \cdot 3} E = \frac{bh^3}{12} \cdot E.$$

The moment of flexure, therefore, of a rectangular beam *increases as the width and the cube of the depth of the beam.*

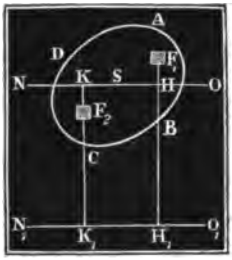
If we put this value of $WE$ into the formula $a = \frac{Pl^3}{3WE}$ of § 189, we shall obtain $a = 4 \cdot \frac{Pl^3}{bh^3 E}$, but if into the formula $a = \frac{1}{48} \frac{Pl^3}{WE}$ of § 190, then $a = \frac{Pl^3}{4\, bh^3 E}$. Inversely, the modulus of elasticity follows from the height of the arc $a$, $E = \frac{4Pl^3}{abh^3}$ for the one, and $E = \frac{Pl^3}{4\, abh^3}$ for the other case.

*Example.*—1. A wooden beam, 10 feet = 120 inches in length, 8 inches in width, and 10 inches in height, is to be supported at both its ends, and bear a uniform load $Q$ = 10000 lbs; what flexure will it undergo? The height of the arc is $a = \frac{5}{8} \frac{Ql^3}{4bh^3E} = \frac{5}{32} \cdot \frac{10000 \cdot 120^3}{8 \cdot 10^3 \cdot E} = \frac{50000 \cdot 12^3}{32 \cdot 8\, E} = \frac{1350000}{4 \cdot E}$. Now $E$ being put = 1800000 lbs. it follows that $a = \frac{135}{4 \cdot 180}$ = 0,1875 inches.—2. If a rectangular cast iron bar, 2 inches wide and ½ inch thick, has been deflected ¼ inch by a weight $P$ = 18 lbs. lying in the middle of it, whilst the distance of the supports amounts to 5 feet, the modulus of elasticity of cast iron will be $E = \frac{Pl^3}{4\, abh^3}$ $= \frac{18 \cdot 60^3}{4 \cdot \frac{1}{4} \cdot 2 \cdot (\frac{1}{2})^3} = \frac{18 \cdot 60^3}{\frac{1}{4}}$ = 72,216000 = 15552000 lbs.

§ 192. *Reduction of the moment of flexure.*—If we know the moment of flexure of a body, $ABCD$, Fig. 209, about an axis $N_1O_1$, lying without the centre of gravity, the moment about another axis $NO$, passing through the centre of gravity $S$, and running parallel with the former, may be found. If the distance $HH_1 = KK_1$ of both axes $= d$, and the distances of the elementary surfaces $F_1$, $F_2$, &c. from the neutral axis $NO = z_1$, $z_2$, &c., we shall have the distances from the axis

$N_1O_1, = d+z_1,\ d+z_2$, &c., and the moment of flexure will be

FIG. 209.



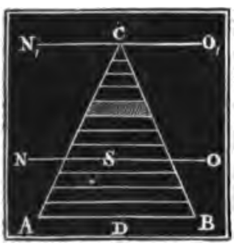

$W_1E = [F_1(d+z_1)^2 + F_2(d+z_2^2) + \ldots]E = [F_1(d^2 + 2dz_1 + z_1^2) + F_2(d^2 + 2dz_2\ z_2^2) + \ldots]E = [d^2(F_1+F_2+\ldots) + 2d(F_1z_1 + F_2z_2+\ldots) + (F_1z_1^2 + F_2z_2^2 + \ldots)]$. But $F_1+F_2+\ldots$ as the sum of all the elements $=$the transverse section $F$ of the whole body; further, $F_1z_1+F_2z_2+\ldots$ as the sum of the moments about an axis passing through the centre of gravity of the body $= 0$, and $F_1z_1^2+F_2z_2^2+\ldots$ is the moment of flexure $WE$ about the neutral axis $NO$; it follows, therefore, that $W_1E = (Fd^2 + W)E$, or $W_1 = Fd^2 + W$; and inversely, $W = W_1 - Fd^2$.

*The measure $W$ of the moment of flexure about the neutral axis is equal to the measure $W_1$ of the moment of flexure about a second parallel axis, less the product of the transverse section $F$ and the square ($d^2$) of the distance of both axes.* Hence it follows, that of all the moments of flexure, that about the neutral axis is the least.

The moments of flexure of many bodies about any axis may be easily found; we may therefore avail ourselves of these to determine by means of the formulæ found, the moments about the neutral axis.

§ 193. To find the moment of flexure of a prism having a triangular transverse section $ABC$, we must decompose this section by lines parallel to the base $AB$ into $n$ thin laminæ, and determine the moments of these about the axis $N_1O_1$ passing through the point $C$ parallel to $AB$. If $h$ is the height $CD$, and $b$ the breadth $AB$ of the triangular section $ABC$, we have the height of these laminæ $= \frac{h}{n}$, their lengths $= \frac{b}{n}$,

FIG. 210.




$\frac{2b}{n}, \frac{3b}{n}$, &c., to $\frac{nb}{n}$, and their distances from $N_1O_1 = \frac{h}{n}, \frac{2h}{n}$, $\frac{3h}{h}$, &c., to $\frac{nh}{n}$. From these the areas of the laminæ are $F_1 = \frac{bh}{n^2}, F_2 = \frac{2\,bh}{n^2}, F_3 = \frac{3\,bh}{n^2}$, and their moments $F_1z_1^2 = \frac{bh^3}{n^4}$,

$F_2 z_2^2 = 2^3 \cdot \frac{bh^3}{n^4}$, $F_3 z_3^2 = 3^3 \cdot \frac{bh^3}{n^4}$, &c., and the moment of flexure about the axis $N_1 O_1$:

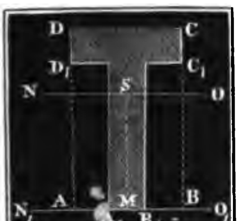
$$W_1 = \frac{bh^3}{n^4}(1^3+2^3+3^3+\ldots+n^3) = \frac{bh^3}{n^4} \cdot \frac{n^4}{4} = \frac{bh^3}{4}.$$

The distance of the centre of gravity $S$ from the point $C$ is $d = \frac{2}{3} h$, and the area of the whole triangle $F = \frac{bh}{2}$; therefore $Fd^2 = \frac{bh}{2} \cdot \frac{4}{9} h^2 = \frac{2\,b\,h^3}{9}$, and the moment of flexure about the neutral axis $NO$ sought is:

$$WE = (W_1 - Fd^2)\,E = \left(\frac{bh^3}{4} - \frac{2\,b\,h^3}{9}\right) E = \frac{1}{3} \cdot \frac{bh^3}{12} E =$$

a third of the moment of flexure of the rectangular beam, which has the same depth and width as the triangular one. But since this beam has double the volume, it then follows, that under otherwise similar circumstances, the triangular beam has $\frac{2}{3}$ of the moment of flexure of the rectangular.

FIG. 211.

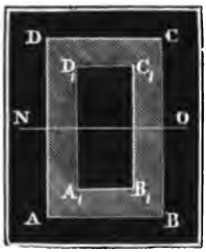



We may find in the same manner the moments of flexure of many other bodies used in construction. For the transverse section of a T-shaped body $A_1B_1CD$, Fig. 211, whose dimensions are $AB = b$, $AB - A_1B_1 = AA_1 + BB_1 = b_1$, $AD = BC = h$ and $AD_1 = BC_1 = BC - CC_1 = h_1$, the moment of flexure about the lower edge $A_1B_1$ = the moment of the rectangular figure $ABCD$, less the moments of the rectangles $A_1D_1$ and $B_1C_1$, *i. e.* $W_1 = \frac{1}{2} \cdot \frac{b\,(2h)^3}{12} - \frac{1}{2} \cdot \frac{b_1\,(2\,h_1)^3}{12} = \frac{bh^3 - h_1^3 b_1}{3}$, as follows, if we consider each of these rectangles as the half of rectangles having double the height with the neutral axis $N_1O_1$. Now the area $A_1C_1D = bh - b_1h_1$, and its moment $Fd = bh \cdot \frac{h}{2} - b_1h_1 \cdot \frac{h_1}{2} = \frac{1}{2}(bh^2 - b_1h_1^2)$; hence it follows that the arm $MS = d = \frac{bh^2 - b_1h_1^2}{2\,(bh - b_1h_1)}$, the moment $Fd^2 = \frac{1}{4}(bh^2 - b_1h_1^2)^2 : (bh - b_1h_1)$, and the moment of flexure about the neutral axis passing through the centre of gravity $S$:

$$W = W_1 - Fd^2 = \frac{bh^3 - b_1h_1^3}{3} - \tfrac{1}{4}\,(bh^2 - b_1h_1^2)^2 : (bh - b_1h_1)$$
$$= \frac{4\,(bh^3 - b_1h_1^3)\,(bh - b_1h_1) - 3\,(bh^2 - b_1h_1^2)^2}{12\,(bh - b_1h_1)}$$
$$= \frac{(bh^2 - b_1h_1^2)^2 - 4\,bh\,b_1h_1\,(h - h_1)^2}{12\,(bh - b_1h_1)}.$$

§ 194. *Hollow beams.*—The moment of flexure of a hollow rectangular beam $ABCD$, Fig. 212, is determined, if we deduct from the moment of the complete beam that of the hollow part. $AB = b$ is the external breadth, and $BC = h$ the height, and $A_1B_1 = b_1$ the internal breadth, and $B_1C_1 = h_1$ the height, we then have the moments of flexure of both $= \frac{bh^3}{12}$ and $\frac{b_1h_1^3}{12}$, and by subtraction we get the moment of flexure of the hollow beam $W = \frac{bh^3 - b_1h_1^3}{12}$.

FIG. 212.

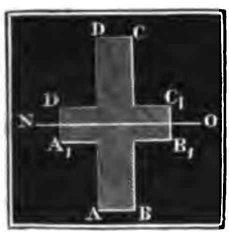


We may find in an exactly similar manner the moment of flexure of a body $ABCD$, Fig. 213, hollowed out at the sides. $AB = b$ is the outer breadth and $BC = h$ the height; and if $AB - A_1B_1 = b_1$, and $B_1C_1 = h_1$, the sum of the breadth and the heights of both hollows, by subtraction we have again:
$$W = \frac{bh^3 - b_1h_1^3}{12}.$$

FIG. 213.

The moment of flexure of a body $ABCD$, Fig. 214, of a cross-shaped section, may be obtained in the same manner. Here $AB = b$ the width, and $BC = h$ the height of the middle piece, and if $A_1B_1 - AB = b_1$ and $A_1D_1 = h_1$ are the sum of the breadths and the height of the side ribs; by addition we have the moment of flexure: $W = \frac{bh^3 + b_1h_1^3}{12}$.

FIG. 214.



It is besides easy to see, that deep, hollow, and ribbed or flanged sections of the same area have a greater moment of flexure than square sections. Because this moment increases with the transverse section $F$ and the square ($z^2$) of the distance from the neutral axis, one and the same fibre

affords, therefore, a greater resistance to flexure, the further it is distant from the neutral axis. If, for example, the height $h$ of a massive rectangular beam be equal to double its breadth $b$, its moment of flexure will be either $W = \frac{b \,.\, (2\,b)^3}{12} = \frac{2}{3}\, b^4$ or $= \frac{2\,b \,.\, b^3}{12}$ $= \frac{1}{6}\, b^4$, according as we put up the beam with the lesser breadth $b$, or the greater $2\,b$; in the first case, therefore, the moment of flexure is four times greater than in the second. If we replace the massive beam of the cross section $bh$ by a hollow one, whose hollow $bh$ is equal to the massive part of the section $b_1h_1 - bh$, if, therefore, $b_1h_1 - bh = bh$, *i. e.* $b_1h_1 = 2bh$, or $b_1 = b\sqrt{2}$ and $h_1 = h\sqrt{2}$, we shall obtain the moment of flexure of the last $\frac{b_1h_1^3 - bh^3}{12}$ $= \frac{b\sqrt{2}\;(h\sqrt{2})^3 - bh^3}{12} = \frac{3}{12}\, bh^3$, *i. e.* three times as great as for the first.

§ 195. *Cylinders.*—The moment of flexure of a cylinder is determined in the following manner. Let $AOBN$, Fig. 215, be the circular transverse section, and $NO$ the neutral axis of the cylinder. The diameter $AB$, divides this section into two equal parts, having equal moments of flexure, and the moment of flexure of the whole may be found by doubling the moment of the half $ANB$. The half may be divided by sections $DE$, $FG$, &c. parallel to $AB$, and at right angles to $NO$ into thin lamina, which may be considered as rectangular. The moment of flexure of such a portion $DEFG$, $= \frac{KL \,.\, \overline{DE}^3}{12}$. Now $CA = CN = r$ the radius of the circular section, a quadrant $AN$ has, therefore, the area $\frac{\pi r}{2}$, and if we divide this into $n$ equal parts, any such part $DG = \frac{1}{n} . \frac{\pi r}{2} = \frac{\pi r}{2n}$. The projection parallel to $CN$, $GH = KL$ corresponds to this part, and may be determined by putting, $GH : GD = GK : CG$, and, therefore, $GH = \frac{GD \,.\, GK}{CG} = \frac{\pi}{2n} . GK$. Hence we have for the moment of flexure of the part

FIG. 215.

$$DEFG=\frac{\pi}{2n}\cdot GK\cdot\frac{(2GK)^3}{12}=\frac{\pi}{3n}(GK)^4.$$

If we put the variable angle corresponding to the section $GF$, $ACG=\phi^0$, we shall obtain the ordinate $GK=r\,cos.\,\phi$, and for the last moment of flexure $=\frac{\pi}{3n}r^4\,cos.\,\phi^4=\frac{\pi}{3n}r^4\cdot\frac{3+4\,cos.\,2\,\phi+cos.\,4\,\phi}{8}$. The moment of flexure of the half cylinder will be now found, if for $\phi$ we successively put the values $\frac{1}{n}\frac{\pi}{2}$, $\frac{2}{n}\cdot\frac{\pi}{2}$, $\frac{3}{n}\cdot\frac{\pi}{2}$, &c., to $\frac{n}{n}\cdot\frac{\pi}{2}$, and add the results. But $\frac{\pi\,r^4}{3\,n}\cdot\frac{1}{8}=\frac{\pi\,r^4}{24n}$ is a common factor, we have, therefore, only to consider the sum of such values, as $3+4\,cos.\,2\,\phi+cos.\,4\,\phi$. The number 3 added $n$ times gives $3\,n$; the sum of all values of the $cos.\,2\,\phi$ which present themselves, when $\phi$ is made to increase from 0 successively to $\frac{\pi}{2}$, and, therefore, $2\,\phi$ from 0 to $\pi$, equal to 0, because the cosines in the second quadrant are equal and opposite to the cosines in the first; lastly, the sum of all the cosines of all angles from 0 to $2\,\pi=0$, hence the sum of all values of $3+4\,cos.\,2\,\phi+cos.\,4\,\phi$ taken between the limits $\phi=0$ to $\phi=\frac{\pi}{2}$ is $=3\,n$, and the measure of the moment of flexure of the half cylinder $=\frac{\pi\,r^4}{24n}\cdot 3\,n=\frac{\pi r^4}{8}$, and, lastly, that of the whole cylinder:

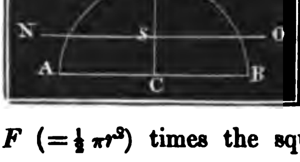
$$W=\frac{\pi}{4}r^4=0{,}7854\,r^4.$$

For a tube or hollow cylinder with the outer radius $r_1$ and the inner $r_2$

$$W=\frac{\pi}{4}\,(r_1^4-r_2^4).$$

To find the moment of flexure of a body having a semi-circular transverse section $ADB$, Fig. 216, we may make use of the rule found in § 192, from which the moment about the axis $NO$ passing through the centre of gravity $S$ is equivalent to the moment about the diameter $AB$, considered as a second axis, less the transverse section $F$ $(=\frac{1}{2}\,\pi r^2)$ times the square of the distance $CS$ of both axes. From this we obtain the moment sought $=\frac{1}{2}\cdot\frac{\pi}{4}r^4-\frac{1}{2}\,\pi\,r^2\,.\,\overline{CS}^2$

FIG. 216.



$$=\frac{\pi r^4}{8}-\frac{1}{2}\pi r^2 \cdot \left(\frac{4r}{3\pi}\right)^2 \text{ (§ 108)} = \pi r^4 \left(\frac{1}{8}-\frac{8}{9\pi^2}\right)=0{,}110 \,.\, r^4.$$

§ 196. *Relative strength.*—When we know the moment of flexure of a prismatic body, we may determine from it by simple multiplication the *working load* and the *absolute* strength of the body. If a single fibre, or layer of fibres, is extended or compressed to the limits of elasticity, the body has then attained the limits of its tenacity. If we again represent by $T$ the modulus of tenacity and the distance of the furthermost fibre from the neutral axis by $e$, we shall have $T=\frac{\lambda}{l}E$ and $\frac{\lambda}{l}$, or the relative elongation $=\frac{e}{\rho}$, hence $\frac{E}{\rho}=\frac{T}{e}$. *If we substitute $\frac{T}{e}$ for $\frac{E}{\rho}$ in the formula for the moment of flexure, it will then give the statical moment of the tenacity.* We have $Px=SW=\frac{EW}{\rho}$, therefore, also, $Px=\frac{TW}{e}$. It is evident that this moment is a maximum when $x=l$, or when the arm $=l$; from this we may conclude, that at the extremity where the beam is fixed, the greatest flexure ensues, and the limit of elasticity is first attained. Accordingly, the *working load of a beam* is determined by the formula

$$P=\frac{TW}{el}.$$

In like manner, the *strength* or the resistance to rupture of the beam may be determined. If a fibre is strained to the point of rupture, the breaking of the whole beam takes place, because the beam has now a section smaller by the section of these fibres and therefore a greater deflexion ensues, and thus a rupture of the succeeding fibres or layer of fibres follows. If we put the modulus of strength $=K$, we have $\frac{E}{\rho}=\frac{K}{e}$, and, therefore, the force for the rupture of the beam:

$$P=\frac{KW}{el}.$$

In a uniform rectangular beam, the distance of the outermost lamina of fibres from the neutral axis $=\frac{h}{2}$, hence the formula $Pl=\frac{E}{\rho}\cdot\frac{bh^3}{12}$ (§ 191) gives the resistance to rupture

$$P = \frac{2\,K}{h} \cdot \frac{bh^3}{12\,l} = \frac{bh^2}{6\,l} \cdot K.$$

If the beam is hollow, as in Fig. 212, we have $P = \frac{bh^3 - b_1h_1^3}{6\,hl} \cdot K$, so that the formula also holds good for a body, as in Fig. 213, hollowed out at the sides.

In a prismatic body of a triangular cross section, as in Fig. 210, $e = \frac{2}{3}\,h$, hence $P = \frac{K}{\frac{2}{3}h} \cdot \frac{bh^3}{36\,l} = \frac{bh^2}{24\,l} \cdot K$. According to this, rectangular beams for a similar section have twice the tenacity of triangular beams.

For a cylinder of radius $r$, $e = r$, therefore,

$$Pl = \frac{K}{r} \cdot \frac{\pi}{4} r^4 = \frac{\pi}{4} r^3\,K.$$

If the cylinder is hollow, we have $Pl = \frac{\pi}{4}\left(\frac{r_1^4 - r_2^4}{r_1}\right) K.$

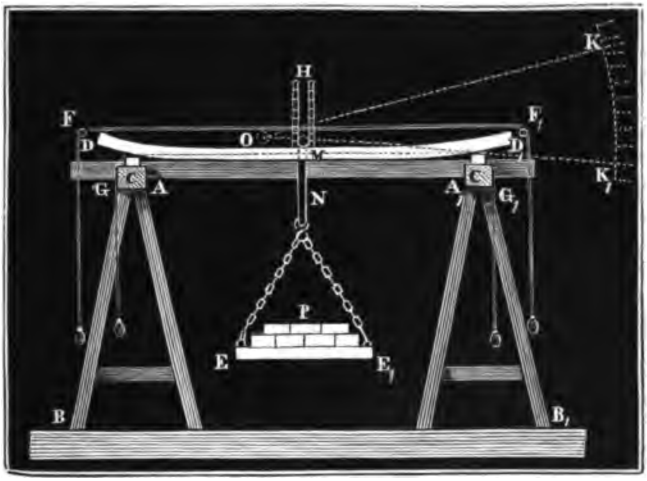
If we substitute the modulus of the working load $T$ for that of the strength, or for $K$, an aliquot part, *i. e.* $\frac{1}{10}$th, the working load is given by the formula already found.

§ 197. *Experiments**.—To find the deflexion and tenacity of beams, we may make use of the experimental values for $E$ and $T$ in § 186; but as concerns the strength of beams, it is safer to replace the modulus of strength there given and derived from experiments on tensile strain by those values of $K$ which have been found from experiments on compression. A perfect accordance cannot exist between the moduli found by these two methods, because in rupture, not only an extension, but also a compression takes place, and both of these not only in the direction of the axis, but also in the transverse section, though here not to the same amount. Besides, many other circumstances affect the elasticity, tenacity and strength of bodies, on which account, considerable variations in the results always present themselves. Timber, for example, is stronger at the core and at the root than at the sap and the top. Timber will also bear a greater strain when the force acts perpendicular to the annual rings, than when parallel to them. Lastly, the soil and the situation where it has grown, temperature, dryness, age, &c., affect the resistance of woods. Besides, the deflexion of a body after it has been loaded for a long time is always somewhat greater than on the immediate application of the load.

* See Appendix.

Experiments upon elasticity and strain were made by *Eytelwein* and *Gerstner*, with the apparatus represented in Fig. 217. $AB$ $A_1B_1$ are two tressels, $C$ and $C_1$ two iron supports. $DD_1$ the

FIG. 217.




rectangular beam for experiment resting upon them. The load $P$ for the flexure of the body lies upon a scale-pan $EE_1$ suspended to a stirrup $MN$, whose upper and rounded extremity lies in the middle $M$ of the beam. In order to find the deflexion corresponding to a load $P$; *Eytelwein* applied two fine horizontal threads $FF_1$ and $GG_1$ and likewise a scale $M$ resting upon the middle of the beam; *von Gerstner*, on the other hand, availed himself of a long one-armed delicate lever $OK$, whose fulcrum was at $M$, and whose extremity, like the hand of a watch, indicated upon a vertical scale $KK_1$ the deflexion of $M$ to fifteen times its amount.

*Remark.* Experiments on elasticity, &c. have been made by Banks, Barlow, Buffon, Burg, Ebbels, Eytelwein, Finchan, von Gerstner, Gauthey, Muschenbroek, Rennie, Rondelet, Tredgold, &c. An ample summary of these, and besides a theory somewhat different from the above, is given by Burg in the 19th and 20th vol. of the "Jaharbücher des polytechnischen Institutes in Wien." The experiments of Eytelwein and von Gerstner are described in Eytelwein's "Handbuch der Statik fester Körper," vol. 2, and in von Gerstner's "Handbuch der Mechanik," vol. 1. The Treatise printed from the transactions of the Association of Prussian Industry, "Elementare Berechnung des Widerstandes prismatischer Körper gegen Biegung," by Brix, has been used for the preparation of the foregoing article.

§. 198. *Modulus of relative strength.*—The following table contains the mean values of the modulus of rupture for several

bodies met with in the arts. To find with the assistance of these, the pressures which bodies can sustain with safety for a long duration, we must put for wood the tenth, for metals and stones, from the third to the fourth of $K$.*

TABLE II.

THE MODULUS OF FRACTURE OR MODULUS OF STRENGTH FOR THE FLEXURE OF BODIES.

| Names of Substances. | Modulus of Fracture K. | Names of Substances. | Modulus of Fracture K. |
|---|---|---|---|
| Box . . . . . | 10000 to 24000 | Elm. . . . . | 6000 to 12000 |
| Oak . . . . . | 8000 „ 24000 | Cast Iron . . . | 24000 „ 56000 |
| Pine . . . . | 8000 „ 13000 | Limestone . . . | 700 „ 1700 |
| Scotch Fir. . . | 7000 „ 17000 | Sandstone . . . | 600 „ 800 |
| Deal . . . . | 7000 „ 14000 | Brick . . . . | 180 „ 340 |

According to this we may assume for wood as a mean $K = 12000$ and for cast-iron $K = 40000$ pounds, and we shall then obtain for a rectangular beam imbedded in a wall at one extremity and loaded at the other:

1. $Pl = 200 \,.\, bh^2$, if it consist of wood and tenfold security be allowed,
2. $Pl = 1000 \,.\, bh^2$, if the beam be of cast-iron and fourfold security be given.

If the body be cylindrical, we then have for wood

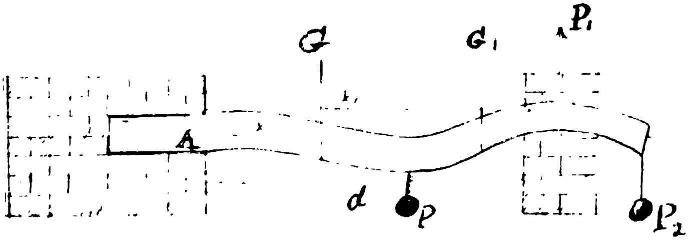
3. $Pl = 950\, r^3$, and for cast-iron
4. $Pl = 4700\, r^3$.

$P$, $l$, $b$, $h$, $r$, have the denominations hitherto used.

For wrought iron $K$ is taken 20 per cent less, because this bends more than cast iron, here therefore we must put $Pl = 800\, bh^2 = 3600\, r^3$.

If the load $Q$ be uniformly distributed over the beam, the beam will bear as much again, wherefore the above coefficients must be *doubled.* If the beam rest at its extremities on points of support, whose distance is $l$, and if the load $P$ act in the middle between these points, then for $P$ we must put $\frac{P}{2}$ and for $l$, $\frac{l}{2}$, wherefore $Pl$ becomes $\frac{Pl}{4}$, and the tenacity quadrupled. But if the load between the points be uniformly distributed over the beam, we

* See Appendix.


G
G1
P1
A
d
P
P2

bodi
the
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$2^d$

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[illegible]

[illegible] $P_1 \ell^2 + P$ [illegible]

[illegible] $P_1 = [P\ell^2 + P$ [illegible] $]$ [illegible]

[illegible] $16\ell_1 - 16$ [illegible] $\frac{P}{4}$ [illegible]

[illegible]

$$\begin{cases} P_1 = P\ell \ldots \\ P_2 = P \ldots \end{cases}$$

[illegible] $d$ [illegible]

[illegible] $x^2$ [illegible] $x^3$ [illegible] $P_1$ [illegible] $x^2$ [illegible] $x^3$ [illegible]

[illegible] $x^3$ [illegible] $P_2$ [illegible]

[illegible]

[illegible] $\frac{\ell}{4}$ [illegible] $P$ [illegible]

[illegible]

[illegible] $P$ [illegible]

[illegible]

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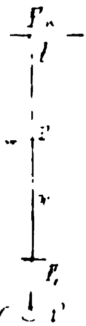

$T = \ldots$

$T_n + P - \ldots T \therefore T_1 = \frac{P}{T}$

T

$\therefore \frac{dx}{T} = \frac{dT}{T}$

$\frac{\gamma}{T}\int_0^l dx = \int_{\frac{P}{T}}^{T_n} \frac{dT}{T} = \log \frac{T_n}{\frac{P}{T}} = \frac{\gamma}{T} l \quad \therefore \quad \frac{T_n}{\frac{P}{T}} = \varepsilon^{\frac{l\gamma}{T}} \quad \text{or } T_n = \frac{P}{T}\cdot\varepsilon^{\frac{l\gamma}{T}}$

section.

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Weight sustained?

$\gamma T_x dx =$ wt of element $\quad \gamma\int T_x dx = \ldots$

but $T_n = \frac{P}{T}\varepsilon^{\frac{l\gamma}{T}} \qquad T_x = \frac{P}{T}\varepsilon^{\frac{x\gamma}{T}}$

$\therefore$ Wt $= \gamma\frac{P}{T}\varepsilon^{\frac{x\gamma}{T}}dx = \gamma\frac{P}{T}\int_0^l \varepsilon^{\frac{x\gamma}{T}}dx = P\left(\varepsilon^{\frac{l\gamma}{T}} - 1\right.$

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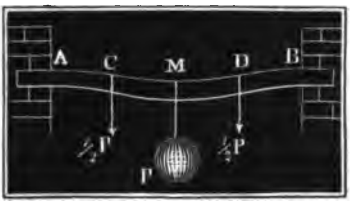
then shall have for the pressure $\frac{Q}{2}$, which acts from below upwards at a point of support, the moment $\frac{Q}{2} \cdot \frac{l}{2}$; and for the opposite pressure $-\frac{Q}{2}$ as the half of the load pulling downwards at the centre of gravity, the moment $-\frac{Q}{2} \cdot \frac{1}{2} \cdot \frac{l}{2} = -\frac{Ql}{8}$; hence there will remain as the pressure for rupture at the middle, the moment $\frac{Ql}{4} - \frac{Ql}{8} = \frac{Ql}{8}$, and therefore $Ql = 8 \cdot \frac{bh^2}{6} K$, also $= 8 \cdot \frac{\pi}{4} br^3 K$, therefore the strength or tenacity is twice as great as if the load acted at the middle, and eight times as great as if it pulled downwards at one extremity whilst the other remained fixed.

FIG. 218.




If a beam, Fig. 218*, is embedded in a wall at both extremities, or if its extremities are fixed, then the beam sustains as much again as if it rested freely at its extremities; for in this case the greatest flexure is not only in the middle, but likewise at the extremities; the beam, therefore, breaks at the same time in the middle and at the extremities; whilst at the intermediate points $C$ and $D$, where the convexity passes into concavity, no flexure at all ensues. Consequently, for a portion $AC$, the pressure $= \frac{P}{2}$, the arm $= \frac{l}{4}$, and the moment $= \frac{P}{2} \cdot \frac{l}{4} = \frac{Pl}{8}$. If, finally, in this last case the load $Q$ is uniformly distributed over the beam, the moment presents itself $= \frac{Ql}{16}$, because we may suppose, that the one half of $Q$ is immediately sustained by the points of support, and that the other half acts in the middle of $Q$. the beam.

The weight $G$ of a beam acts exactly as if the load $Q$ were distributed uniformly over the beam; for a beam fixed at one extremity, therefore, the moment $= Pl + \frac{1}{2} Gl$; but for a beam

* See Appendix.

resting on both extremities and loaded in the middle, it is $=\frac{P}{2}\cdot\frac{l}{2}+\frac{G}{2}\cdot\frac{l}{2}-\frac{G}{2}\cdot\frac{l}{4}=(P+\frac{1}{2}G)\frac{l}{4}$, &c.

*Example.*—1. A rectangular beam of fir, 7 inches thick and 9 inches in depth, is to rest on both its extremities, so that the distance of the points of support may amount to 20 feet; what load, suspended from the middle, will it sustain? $b = 7$, $h = 9$, $l = 20$ feet $= 240$ inches; hence $240 . P = 4 . 200 . 7 . 9^2$; consequently this load $P = 70 . 27 = 1890$ lbs—2. A round wooden water-wheel, and its axle, 10 feet long, is to sustain at the wheel, together with its own weight, a uniformly distributed load $Q = 10000$ lbs.; what diameter must the ~~wheel~~ axle have? $Ql = 10000 . 120 = 1200000, = 8 . 950 . r^3$, or $r^3 = \frac{1200000}{8 . 950} = 157{,}9$;

hence the radius sought $r = \sqrt[3]{157{,}9} = 5{,}4$ inches, and the diameter of the axle $2r = 10{,}8$ inches, for which we may assume one foot.—3. To what height may the corn in a granary be heaped up if the bottom rest upon beams of 25 feet in length, 10 inches in breadth, and 12 in depth, the distance between the axes of any two beams $= 3$ feet, and one cubic foot of corn weighs 48,5 lbs.? If we apply the formula $Ql = 16 . 200 . bh^2$, we must put $b = 10$, $h = 12$, $l = 25 . 12 = 300$; consequently $Q = \frac{16 . 200 . 10 . 144}{300} = 15360$ lbs. A parallelepiped, 25 feet long, 3 feet broad, $x$ feet deep, weighs $= 25 . 3 . x . 48{,}5$ lbs.; hence, if we put this value $= Q$, it follows that $x = \frac{15360}{75 . 48{,}5} = 4{,}22$ feet, the requisite height to which the grain may be heaped up.

$p \cdot \frac{bl^2}{3}$

$\frac{Q\,l}{12}$

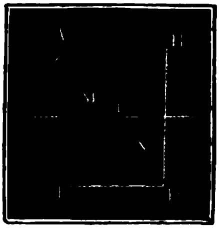
§ 199. *Strongest beams.*—Bodies of equal section very often possess different relative strengths, the formula $Pl=\frac{K}{6} . bh^2$ shews that the *strength increases, as the breadth, as the square of the depth, and inversely as the length of the beam.* The depth has consequently a greater influence upon the tenacity than the breadth; a beam of double the breadth bears twice as much, *i. e.* as much as two single beams; on the other hand, a beam of double the depth, four times that of a beam of the same depth. For this reason beams are made, namely, when they are of cast iron, much deeper than broad, they are hollowed out near the middle, and what is taken away replaced by parts at a greater distance from the neutral axis; but this rule must be particularly attended to, viz. always to lay the beam ~~less~~ on the least side, or rather so to lay it, that the pressure may act in the direction of the greater side.

The strength of the round trunk or of any other cylindrical body is $P=\frac{\pi}{4}\cdot\frac{r^3}{l}K$, that of a square with equal breadths and depths

$\therefore r, = P_1 = \frac{2r\,.\,(2r)^2}{l}\,.\,\frac{K}{6} = \frac{4}{3}\,.\,\frac{r^3}{l}\,K$; if we compare both pressures with each other, $\frac{P}{P_1} = \frac{\pi}{4}\,.\,\frac{3}{4} = 0{,}588$ ; the cylindrical body has, therefore, only about 59 per cent., the strength of a beam having a square transverse section. Wooden beams are hewn or cut from round trunks of trees, and thereby are much weakened. But the question now is, which is the strongest form of beam that can be cut from a cylindrical trunk?

Let $ABDE$, Fig. 219, be the section of the trunk, $AD = d$ its diameter, further $AB = DE = b$ the breadth, and $AE = BD = h$ the depth of the beam. Then $b^2 + h^2 = d^2$, or $h^2 = d^2 - b^2$, and the moment of rupture

FIG. 219.



$$Pl = \frac{K}{6}\,.\,bh^2 = \frac{K}{6}\,b\,(d^2 - b^2).$$

The problem amounts to making $b\,(d^2 - b^2) = bd^2 - b^3$ as great as possible. If instead of $b$, we put $b \pm x$, where $x$ is very small, we then obtain for the last expression

$$(b \pm x)\,d^2 - (b \pm x)^3 = bd^2 - b^3 \pm (d^2 - 3\,b^2)\,x - 3\,bx^2,$$

provided we neglect $x^3$, and the difference of the two $= (\pm d^2 - 3\,b^2)\,x + 3\,bx^2$. That the first value $bd^2 - b^3$ may in every case be greater than the last, the difference $\mp\,(d^2 - 3\,b^2)\,x + 3\,bx^2$ must be but positive, whether we take $b$ greater or less than $x$. But this is only possible if $d^2 - 3\,b^2 = 0$, for the difference then $= 3\,bx^2$, therefore positive, whereas, if $d^2 - 3\,b^2$, is a real positive or negative value, $3\,bx^2$ may be neglected, and the difference may be put $= \mp(d^2 - 3\,b^2)\,x$, which if $x$ has the same sign, is at one time positive, at another negative. But if we put $d^2 - 3\,b^2 = 0$, we obtain the breadth sought $b = d\,\sqrt{\tfrac{1}{3}}$, and the corresponding depth $h = \sqrt{d^2 - b^2} = d\,\sqrt{\tfrac{2}{3}}$; therefore, the ratio of the depth to the breadth : $\frac{h}{b} = \frac{\sqrt{2}}{\sqrt{1}} = 1{,}414$ or about $\frac{7}{5}$. The trunk must be so fashioned that it shall produce a beam whose depth to its breadth is as 7 to 5. To find the section corresponding to greatest strength, let us divide the diameter $AD$ into three equal parts, raise at the points of division $M$ and $N$ perpendiculars $MB$ and $NE$, and finally connect the points of intersection $B$ and $E$ by the circle with the extremities $A$ and $D$ by straight lines. $ABDE$ is the section of greatest

resistance; for since $AM : AB = AB : AD$ and $AN : AE = AE : AD$, $AB = b = \sqrt{AM \cdot AD} = \sqrt{\frac{1}{3} d \cdot d} = d\sqrt{\frac{1}{3}}$ and $AE = h$ $= \sqrt{AN \cdot AD} = \sqrt{\frac{2}{3} d \cdot d} = d\sqrt{\frac{2}{3}}$, therefore $\frac{h}{b} = \frac{\sqrt{2}}{1}$, which is actually requisite.

*Remark.* The trunk has the moment of rupture $Pl = \frac{\pi K}{4} \cdot r^3$, but the beam of greatest resistance formed from it $Pl = \frac{K}{6} \cdot d\sqrt{\frac{1}{3}} \cdot \frac{2}{3} d^2 = \frac{K}{\sqrt{243}} \cdot d^3$ $= \frac{8K}{\sqrt{243}} r^3$; the trunk, therefore, loses by squaring about $1 - \frac{8}{\sqrt{243}} \cdot \frac{4}{\pi}$ $= 1 - 0{,}65 = 0{,}35$, *i. e.* 35 per cent. of its strength. To spare this loss, the trunk is often hewed not quite square, but the corners rounded off. A beam with a square section formed from the same trunk, has the moment $Pl = \frac{K}{6} \cdot d\sqrt{\frac{1}{2}} \cdot \frac{d^2}{2}$, because here the breadth = the depth $= d\sqrt{\frac{1}{2}} = 0{,}707\, d$, hence the loss here $= 1 - \frac{8}{6 \cdot 2\sqrt{2}} \cdot \frac{4}{\pi} = 1 - \frac{8}{3\pi\sqrt{2}} = 1 - 0{,}60 = 0{,}40$, *i. e.* 40 per cent.

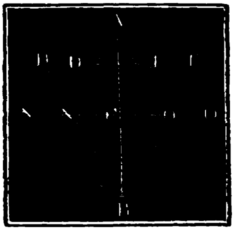
§ 200. *Hollow and elliptical beams.*—Very frequently bodies are hollowed at the inside or outside, and provided with ribs or flanges, either with a view to save material, or what comes to the same thing, to gain in strength. For a hollow rectangular beam of iron $P = 1000 \cdot \frac{bh^3 - b_1 h_1^3}{lh}$, the hollow may be of the depth $h_1$ and breadth $b_1$, made within or without at the sides. For a hollow cylindrical body $P = 4700 \cdot \frac{r_1^4 - r_2^4}{l r_1}$. In such cases the thickness of the solid part $r_1 - r_2$ is commonly made $= \frac{2}{5}$ of the outer radius $r_1$; whence it follows:

$$P = 4700 \cdot \frac{r_1^4 - (0{,}6\, r_1)^4}{l r_1} = 4700 \cdot \frac{0{,}8704\, r_1^3}{l} = 4090\, \frac{r_1^3}{l}.$$

An equally heavy solid cylinder has the radius $r = \sqrt{r_1^2 - r_2^2}$ $= \sqrt{r_1^2 - 0{,}36\, r_1^2} = 0{,}8\, r_1$; hence its moment of resistance $= 4700 \cdot (0{,}8\, r)^3 = 2406\, r^3$, namely, about 41 per cent. less than that of the hollow cylinder.

We gain also in strength, when instead of a cylinder we apply a prismatic body with an elliptical section, and place its greater axis upright or parallel to the direction of the pressure. If we suppose a circle $AO_1BN_1$ whose radius $CA = CB = a$ the semi-axis major, described about this elliptical section $AOBN$, Fig. 220, the

FIG. 220.



strength of resistance of the body having an elliptical section may be calculated simply from that having a circular section. The length of any element $DE$ of the elliptic elements parallel to its minor axis $NO = 2b$ is always $\frac{b}{a}$ of the length of the circular element $D_1E_1$; but now the elasticity and strength are proportional to these dimensions singly; therefore, also, the strength of the elliptic element to that of the circular element, is as $b$ to $a$, and, finally, the strength for the whole ellipse $= \frac{b}{a}$ times the strength of the whole circle, *i. e.*

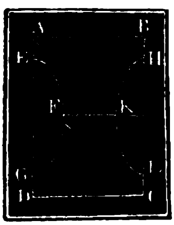
$$Pl = \frac{\pi}{4} \cdot \frac{b}{a} \cdot a^3 K = \frac{\pi}{4} b\, a^2 K, \text{ for cast iron} = 4700\, a^2\, b.$$

If now it be an elliptical hollowing whose axes are $a_1$ and $b_1$, there will remain

$$Pl = \frac{\pi}{4} \cdot \frac{ba^3 - b_1 a_1^3}{a} K, \text{ for cast iron.}$$

$$= 4700 \cdot \frac{a^3 b - a_1^3 b_1}{a}.$$

FIG. 221.



If, lastly, a body having a rectangular section $ABCD = bh$, Fig. 221, be hollowed at the flanks by the semi ellipses $EFG$, $HKL$, and if the semi axes of these are $= a_1$ and $b_1$, we shall have then

$$Pl = bh^2 \frac{K}{6} - \frac{\pi\, b_1 a_1^3}{4h} K = \frac{2\,b\,h^3 - 3\,\pi\, b_1 a_1^3}{12\,h} . K,$$

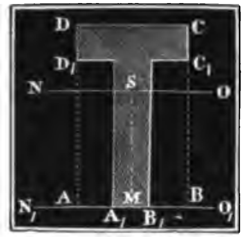
for cast iron

$$Pl = 200 \cdot \frac{bh^3 - \cancel{4{,}712}\; 4{,}7124\; a_1^3 b_1}{h}.$$

*Examples.*—1. A transverse beam of oak, 9 inches broad and 11 inches deep, of known sufficient tenacity, is to be replaced by a hollow cast iron beam, of 5 inches in outer breadth and 10 in depth; of what thickness of metal must it be cast? Let this thickness $= x$, we have then for the breadth of the hollowing $= 5 - x$, and its depth $= 10 - x$; consequently, for the hollow beam $b_1 h_1^3 - b_2 h_2^3 = 5 \cdot 10^3 - (5 - x)(10 - x)^3 = 2500\, x - 450\, x^2 + 35\, x^3 - x^4$. Since the moment of resistance of the wooden beam $= 200 \cdot 9 \cdot 11^2 = 217800$, we shall have to put: $\frac{1000}{10}(2500\, x - 450\, x^2 + 35\, x^3 - x^4) = 217800$, or $2500\, x - 450\, x^2 + 35\, x^3 - x^4 = 2178$. As a first approximation $x = \frac{2178}{2500}$

$= 0{,}9$ inches. But this value gives $450 \,.\, x^2 = 450 \,.\, 0{,}81 = 364{,}5$; $35\, x^3 = 25{,}5$, $x^4 = 0{,}7$; we may therefore put: $x = \frac{2178 + 364{,}5 - 25{,}5 + 0{,}7}{2500} = \frac{2517.7}{2500}$ $= 1{,}01$ inch for the requisite thickness of iron.—2. If in a T-shaped girder of cast iron, the breadth $AB = CD = b$ is equal to the depth $h$ and the thickness $A_1 B_1 = CC_1 = \frac{1}{5} b$, therefore $b_1 = \frac{4}{5} b$, and $h_1 = \frac{4}{5} b$; we shall then have for the moment of resistance (§ 193):

FIG. 222.



$$Pl = \frac{K}{12} \,.\, \frac{(bh^2 - b_1 h_1^2)^2 - 4\, bhb_1 h_1\, (h - h_1)^2}{(bh - b_1 h_1)\, e},$$

or by substituting $e = \frac{1}{2} \,.\, \frac{bh^2 - b_1 h_1^2}{bh - b_1 h_1}$,

$$Pl = \frac{K}{6} \,.\, \frac{(bh^2 - b_1 h_1^2)^2 - 4\, bb_1\, hh_1\, (h - h_1)^2}{bh^2 - b_1 h_1^2}$$

$$= 1000 \,.\, \frac{(b^3 - 0{,}512\, b^3)^2 - 4 \,.\, 0{,}64\, b^4 \,.\, (b - 0{,}8\, b)^2}{b^3 - 0{,}512\, b^3}$$

$= 1000 \,.\, \frac{0{,}2381 - 2{,}56 \,.\, 0{,}04}{0{,}488} \,.\, b^3 = \frac{135{,}6}{0{,}488} \,.\, b^3 = 278 \,.\, b^3$. If, now, such a girder, 4 feet in length, rest on both its extremities, and is to bear a load in its middle of 7400 lbs., $Pl$ would then $= 7400 \,.\, 4 \,.\, 12 = 355200$, and therefore, $4 \,.\, 278\, b^3 = 355200$; whence we should have the extreme depth and breadth $b = h = \sqrt[3]{\frac{355200}{1112}} = 6{,}84$ in. and the thickness of iron $\frac{1}{5}\, b = 1{,}35$ inches.

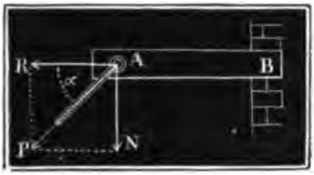
201. *Oblique pressure.*—If the pressure $P$ act obliquely to the axis of a beam, which for example, is inclined to the horizon whilst the pressure acts vertically, we have then only to take into account its components directed at right angles to the axis. If, for example, the inclined stretcher $AB$, Fig. 223, supports an accumulated load $Q$, this may be decomposed into the components $Q_1$ and $N$, and for an inclination $a$ to the horizon of the stretcher, the pressure $Q_1$, counteracted by the stretcher $= Q \,.\, cos.\, a$, and the pressure $N$ counteracted by the lateral wall $BC = Q\, sin.\, a$. Taking the friction into account $\#\, Q_1 = Q.\, (cos.\, a - f\, sin.\, a)$ and hence for a round stretcher: $Q\, (cos.\, a - f\, sin.\, a) = 8 \,.\, \frac{950\, r^3}{l}$, $r$ being the radius and $l$ the length of the stretcher.

FIG. 223.

If the pressure $P$ be applied directly to the beam. $AB$, Fig. 224, deviating from the axis by the angle $PAR = a$ two

components present themselves $N = P\ sin.\ a$ and $R = P\ cos.\ a$, of which the one brings into play the relative, and the other the absolute elasticity of the beam. If $F$ be the cross section of the beam, every unit of it is stretched by the force $\frac{P\ cos.\ a}{F}$, and, therefore, the modulus of ~~elasticity~~ $K$

FIG. 224.




must be taken at $\frac{P\ cos.\ a}{F}$ less, therefore, we must substitute for $K$, $K - \frac{P\ cos.\ a}{F}$, whence it follows that:

$$P\ sin.\ a = \left(K - \frac{P\ cos.\ a}{F}\right)\frac{W}{el};$$

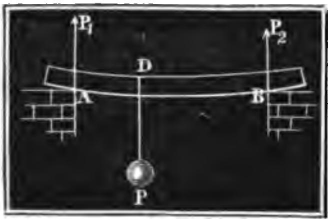
therefore, for a rectangular beam $P\ sin.\ a = \left(K - \frac{P\ cos.\ a}{F}\right)\frac{bh^2}{6\,l}$, and the pressure for rupture: $P = \frac{K}{\frac{6\,l\ sin.\ a}{bh^2} + \frac{cos.\ a}{F}}$. For $a^0 = 90^0$, $sin.\ a = 1$, $cos.\ a = 0$, hence $P = \frac{K}{6}.\frac{bh^2}{l}$, for $a^0 = 0$, $sin.\ a = 0$, $cos.\ a = 1$, hence $P = KF$, as it should be, for we have here only to consider the absolute strength.

*Example.* What distance from each other must the 10-inch stretchers of $ASB$, Fig. 222, be laid, if it be $4\frac{1}{2}$ feet wide, and run for 60 feet up a vein having a slope or inclination of 70°; the weight of a cubic foot of the ground to be supported being 65 lbs., the co-efficient of friction upon the supports is taken at $\frac{1}{3}$? Let $x$ feet be the distance of two rafters, the weight sustained by one rafter $= 4{,}5 \,.\, 60 \,.\, 65\, x = 17550.x$ lbs., and from the theory of the inclined plane, this rafter will have only to sustain the pressure $Q_1 = (sin.\ 70^0 - \frac{1}{3}\ cos.\ 70) \,.\, 17550\, x = (0{,}9397 - 0{,}1140) \,.\, 17550\, x = 0{,}8257 \,.\, 17550\, x = 14492\, x$ lbs. But the rafter sustains $8 \,.\, 950 \,.\, \frac{r^3}{l} = 8 \,.\, \frac{950 \,.\, 5^3}{54} = 17592$; we must therefore put: $14492\, x = 17592$, and $x = \frac{17592}{14492} = 1{,}214$ feet $= 14{,}6$ inches. We must, therefore only leave an interval between any two rafters of 14,6 inches.

§. 202. *Loading beyond the middle.* — If a pressure $P$ acts upon a beam, supported at both ends, not at the middle, but at a point $D$ at distances $DA = l_1$ and $DB = l_2$ from the points of support, the beam, then can bear a greater load. According to the equality of certain statical moments, the point of support $A$ sustains the pressure $P_1 = \frac{l_2}{l_1 + l_2}P$, and the point $B$ the pressure $P_2 = \frac{l_1}{l_1 + l_2}P$,

hence the moment of rupture at the point of application $D = DA \,.\, P_1$ $= DB \,.\, P_2 = \frac{l_1 l_2 P}{l_1 + l_2}$. For any other point $E$ this moment $EB \,.\, P_1$ is less, because the arm $EB$ is less than the arm $DB = l_2$; the greatest deflexion also takes place at $D$, and fracture first occurs at this point. Accordingly we must put $\frac{P l_1 l_2}{l_1 + l_2}$

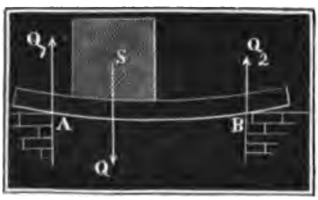
FIG. 225.




$= \frac{KW}{e}$, or the whole length $l_1 + l_2$ being represented by $l$, $\frac{P l_1 l_2}{l}$ $= \frac{K}{6} \,.\, bh^2$, if the beam is rectangular. The pressure $P = \frac{K}{6} \,.\, \frac{l}{l_1 l_2} \,.\, bh^2$ is moreover $= \infty$ when $l_1$ or $l_2$ ~~very nearly~~ $= 0$, and is infinitely less, the more $l_1$ and $l_2$ approach to equality. If, lastly, $l_1 = l_2$, *i. e.* if the pressure $P$ acts in the middle of the beam, $P$ becomes a minimum, because, [if, ~~we put~~ $\cancel{l_1 = \frac{l}{2} + x}$ ~~and~~ $\cancel{l_2} = \frac{l}{2} - x$, the l. product forming the denominator $l_1 l_2 = \frac{l^2}{4} - x^2$ is always less than $\frac{l^2}{4}$, whether $\frac{l}{2}$ be made somewhat ($x$) greater or less.] A beam, therefore, supported at its extremities, sustains least when the load is applied at its middle, and one so much the greater the nearer the load approaches one of the points of support.

If a load $Q$ be uniformly distributed over the length $c$, the centre of which is $l_1$ and $l_2$ distant from the points of support $A$ and $B$, Fig. 226, we shall then have to take the difference $\frac{Q l_1 l_2}{l} - \frac{Q}{2} \,.\, \frac{c}{4}$ for the moment of rupture, because the pressure $Q_1 = \frac{Q l_2}{l}$ at the arm $l_1$, and half the weight $\frac{Q}{2}$ acting at the arm $\frac{c}{4}$ is opposed to it. Therefore

FIG. 226.



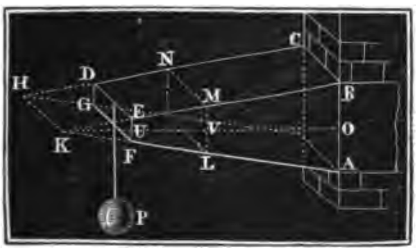

$$Q \left( \frac{l_1 l_2}{l} - \frac{c}{8} \right) = \frac{K}{6} \,.\, bh^2.$$

*Example.* What load does a hollow cast iron beam sustain, if its outer depth and breadth amount to 8 inches and 4 inches, and inner breadth and depth 6 inches and 2 inches; and if further, the middle of the load, uniformly distributed over 3 feet in length, is distant from one point of support 4, and from the other 2 feet? It is $\frac{b_1h_1^3 - b_2h_2^3}{h_1} = \frac{4 \,.\, 512 - 2 \,.\, 216}{8} = 202$; further, $\frac{l_1 l_2}{l} - \frac{c}{8} = \left(\frac{4 \,.\, 2}{6} - \frac{3}{8}\right) 12 = \frac{23}{2}$ inches; hence, $\frac{23}{2} Q = 1000 \,.\, 202$; and consequently, $Q = 17565$ lbs.

§. 203. *Plane of rupture.*—If the beams are not prismatic, if they have different transverse sections at different places, the plane of rupture, *i. e.* the plane in which rupture will ensue, will no longer be the same as prismatic bodies, because this place is not only dependent on the arm $x$, but also on the transverse section. If we suppose a rectangular section of variable breadth $w$, and height $z$, and assume the beam to be fixed at one extremity, and at the other acted upon by a pressure $P$, and the distance of the transverse section $wz$ from the extremity where the pressure acts $= x$, we must then put $P = \frac{K}{6} \cdot \frac{wz^2}{x}$, and find the minimum value of $\frac{wz^2}{x}$ in order to determine the weakest part, or plane of rupture of the beam.

Here many cases present themselves; let us consider only the following. Let the body $ABEG$, Fig. 227, be a truncated wedge, or have the form of a prism with a trapezoidal base, let the breadth $DE = FG$ at the extremity $= b$, the depth $EF = DG = h$, and the distance $UK$ of the edge cut off from the terminating surface $EG$, $= c$. Let us now assume that the plane of rupture $NL$ is distant $UV = x$ from the terminating surface, we shall then obtain for it the depth $ML = z = h + \frac{x}{c} h = h\left(1 + \frac{x}{c}\right)$, whilst the uniform breadth is $MN = w = b$. The value $\frac{wz^2}{x} = \frac{bh^2}{x}\left(1 + \frac{x}{c}\right)^2 = bh^2\left(\frac{1}{x} + \frac{2}{c} + \frac{x}{c^2}\right)$ increases and diminishes simultaneously with $\frac{1}{x} + \frac{x}{c^2}$, and is, therefore, also a minimum, when this latter ~~term~~ is, of the last value. But if in

FIG. 227.



place of $x$, we put $c + u$, where $u$ is a small number, we shall then obtain for it:

$$\frac{1}{c+u} + \frac{c+u}{c^2} = \frac{1}{c\left(1+\frac{u}{c}\right)} + \frac{1}{c} + \frac{u}{c^2}$$

$= \frac{1}{c}\left(2 - \frac{u}{c} + \frac{u^2}{c^2} - \ldots\right) + \frac{u}{c^2} = \frac{2}{c} + \frac{u^2}{c^3}$. As now in this last expression $u$ appears only as a square, it follows that every other value, which is obtained when the distance $x$ is assumed greater or less then $c$, gives a greater value than for $x = c$, that consequently for $x = c$, $\frac{1}{x} + \frac{x}{c^2}$, and, therefore, also $\frac{wz^2}{x} = bh^2$ $\left(\frac{1}{c} + \frac{2}{c} + \frac{1}{c}\right) = \frac{4bh^2}{c}$ is a minimum. From hence it follows, that the magnitude of the surface of rupture $= b \,.\, 2h = 2bh$, and is distant from the terminating surface $EG = bh$ as much again as the edge $HK$ of the portion cut off.

In a similar manner, the distance of the plane of rupture from the terminating surface of a truncated pyramid or truncated cone is equal to half the height of the supplementary pyramid or supplementary cone.

204. *Beams of the strongest form.*—A beam, which opposes an equal resistance to rupture throughout all its sections, of which, therefore, each may be considered as a plane of rupture, is called a *beam of the strongest form.* Of all beams of equal strength, the body of equal resistance at each point of its length has the least quantity of material, and is, therefore, the most suitable, and that which should be selected for architectural construction, and for machines, not only out of regard to economy, but also, that the weight may not be increased unnecessarily.

If we put the distance of a plane of rupture from the further extremity $= x$, and the measure of the moment of ~~flexure~~ for that section $= W$, we then have the pressure requisite for rupture $P = \frac{WK}{ex}$. As $K$ is a constant factor, a beam of the strongest form $\frac{W}{ex}$ must be constant also, *i. e.* it must be of the same value for every possible section. If for a beam of a rectangular section the variable breadth $= u$, and the depth $= v$; but the breadth at the origin, or end supposed fixed $= b$, and the depth there $= h$,

we have generally $W=\frac{uv^3}{12}$, and $e=\frac{v}{2}$, hence $P=\frac{uv^2}{x}\cdot\frac{K}{6}$, and for the origin, for which $x$ has become $l$, $P=\frac{bh^2}{l}\cdot\frac{K}{6}$.

If we make these two values of $P$ equal, we obtain the equation $\frac{uv^2}{x}=\frac{bh^2}{l}$ for the beam of the strongest form. In a beam of equal breadth $u=b$, is $\frac{v^2}{x}=\frac{h^2}{l}$, therefore, $\frac{v^2}{h^2}=\frac{x}{l}$, which is the equation to the parabola (§ 35, *Remark*), and points out that the longitudinal section $ABE$, Fig. 228, must have the form of a parabola whose vertex is the extremity or point of suspension $E$ of the load. If the beam $AB$, Fig. 229, rests upon its extremities, and sustains a load in its middle $C$, or if a beam $AB$, Fig. 230, is supported in its middle $C$, and two pressures, balancing each other, are applied at the extremities $A$ and $B$, then

FIG. 228.

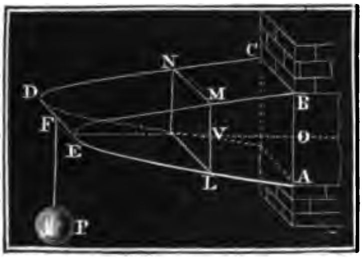



FIG. 229.

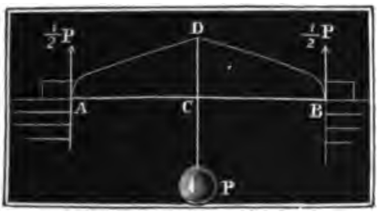
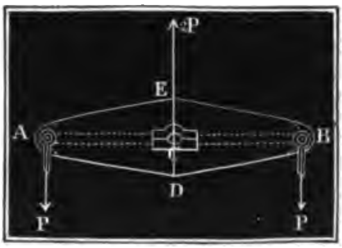



FIG. 230.




the longitudinal profile has the form of two parabolas meeting in the middle. The last case occurs in balances, which as they are weakened by the holes at the points $A$, $C$, $B$, are provided with ribs, or have a middle piece $AB$ given to them. If the depth $v=h$ is constant $\frac{u}{x}=\frac{b}{l}$ or $\frac{u}{b}=\frac{x}{l}$, for the breadth $u$ is proportional to its distance from the extremity, the horizontal projection, therefore, of the beam $ABD$, Fig. 231, forms a triangle $BCD$, and the whole beam a wedge with a vertical edge coinciding with the direction of force. If the body $ABD$, Fig. 232, is to have similar transverse sections, we

shall then have $\frac{v}{h} = \frac{u}{b}$, hence $\frac{u \cdot u^2 h^2}{b^2 x} = \frac{bh^2}{l}$, *i. e.* $\frac{u^3}{b^3} = \frac{x}{l}$, therefore, the breadth and the depth increase as the cube roots of the corresponding arms. For example, a section eight times further from the outer end than a given section, would only have double the height and breadth of that of the given section.

FIG. 231.

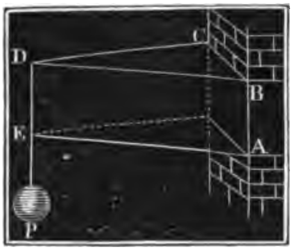
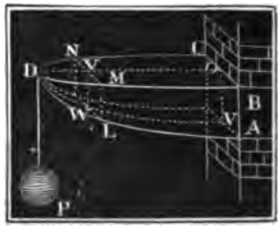



FIG. 232.




If a beam be uniformly loaded, we have the variable load $Q = qx$, and its arm $= \frac{x}{2}$, hence, instead of $Px$, we must put $xq \cdot \frac{x}{2} = \frac{x^2 q}{2}$, whence $\frac{x^2 q}{2} = uv^2 \cdot \frac{K}{6}$ and also $\frac{l^2 q}{2}$ must be taken $= bh^2 \frac{K}{6}$, and consequently $\frac{uv^2}{bh^2} = \frac{x^2}{l^2}$. Were the breadth invariable, that is $u = b$, we should have $\frac{v^2}{h^2} = \frac{x^2}{l^2}$, therefore, also $\frac{v}{h} = \frac{x}{l}$, and therefore, a triangle *ABE* for the longitudinal section, and a wedge *ABED*, Fig. 233, for the body of the strongest form. If we take a uniform depth $v = h$, we then obtain $\frac{u}{b} = \frac{x^2}{l^2}$, and, there-

FIG. 233.

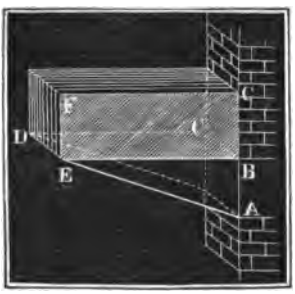
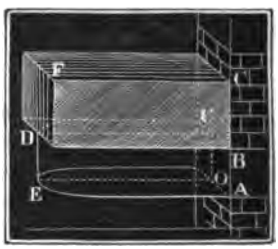



FIG. 234.

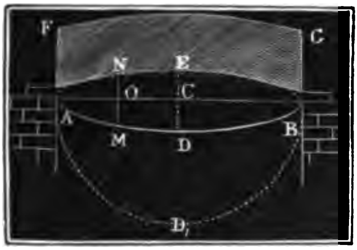



fore, for the plane a surface *BDC*, bounded by a parabolic arc, as in Fig. 234. If we again make similar transverse sections, then $\frac{u^3}{b^3} = \frac{x^2}{l^2}$, so that we have both in the vertical as in the horizontal

profile, a cubic parabola, in which the cubes of the ordinates increase, as the squares of the abscisses.

If a body $AB$ supported at both extremities, Fig. 235, is uniformly loaded over its whole length, we have for the moment of rupture at a distance from a point of support $AO = x$:

FIG. 235.




$$\frac{Q}{2}\cdot x - qx\cdot\frac{x}{2} = \frac{q}{2}(lx - x^2),$$

on the other hand for the middle point:

$$\frac{Q}{2}\cdot\frac{l}{2} - \frac{Q}{2}\cdot\frac{l}{4} = \frac{Ql}{8} = \frac{ql^2}{8}.$$

If we suppose the body to be of uniform breadth, we have to put $\frac{q}{2}(lx - x^2) = bv^2\cdot\frac{K}{6}$ and $\frac{ql^2}{8} = bh^2\cdot\frac{K}{6}$, and by division $\frac{v^2}{h^2} = \frac{lx - x^2}{\frac{1}{4}l^2}$,

or $v^2 = \left(\frac{h}{\frac{1}{2}l}\right)^2(lx - x^2)$. Were $h = \frac{1}{2}l$, $v^2$ would be $= lx - x^2$, and therefore the longitudinal section would be a circle $AD_1B$ described with $\frac{1}{2}l$ as a radius, but because $lx - x^2$ must still be multiplied by $\left(\frac{h}{\frac{1}{2}l}\right)^2$ in order to obtain the square $v^2$ of every ordinate $OM$, this circle passes into an ellipse $ADB$, whose semi-axes are $CA = a = \frac{1}{2}l$ and $CD = \beta = h$.

The same relations exist for bodies with circular sections as for those with similar rectangular sections. In the case of a beam embedded in a wall at one extremity, and loaded at the other $\frac{u^3}{r^3} = \frac{x}{l}$, *i. e.* the radii increase as the cube~~s~~ roots of the distances from the point of application.

§ 205. *The thickness of axles.*—In the parts of machines, as the shafts, axles, &c., flexures may prejudicially affect the working of machines, by giving rise to vibrations and shocks; and it is here, therefore, often more desirable to determine the sections, not according to their strength, but according to their degree of flexure. *Gerstner* and *Tredgold* maintain that a beam of wood, supported at both extremities and loaded in the middle, may suffer a deflexion $a = \frac{1}{288}\cdot l$ without disadvantage, and that such a beam of cast or wrought iron can only undergo a deflexion or height of

arc $a=\frac{1}{480} \cdot l$. But now from § 190: $a=\frac{Pl^3}{48\,WE}$, and from § 191: $W=\frac{bh^3}{12}$, hence follows the height of arc: $a=\frac{Pl^3}{4\,bh^3\,E}$ and $\frac{a}{l}=\frac{Pl^2}{4\,bh^3 E}$. If now we put $\frac{a}{l}=\frac{1}{288}$, and $E=1800000$, we obtain for wooden beams the tenacity or strength at the middle:

$$P=\frac{a}{l} \cdot \frac{4\,bh^3\,E}{l^2}=\frac{1}{288} \cdot \frac{4\,bh^3 \,.\, 1800000}{l^2}=25000 \,.\, \frac{bh^3}{l^2}.$$

For cast iron $\frac{a}{l}=\frac{1}{480}$, and $E=17000000$, hence:

$$P=\frac{1}{480} \cdot \frac{4\,bh^3}{l^2} \,.\, 17000000=142000 \,.\, \frac{bh^3}{l^2}.$$

If further we take for cast iron $\frac{a}{l}=\frac{1}{480}$, and $E=29000000$ lbs., we obtain for a rectangular beam of this material:

$$P=242000 \,.\, \frac{bh^3}{l^2}.$$

The co-efficients 25000, 142000, 242000 must be multiplied by $3\,\pi=9{,}42$, and $h$ and $b$ be replaced by $r$, for cylindrical beams as round axles, &c. The following table gives the dimensions of the transverse sections, $l$ being expressed in feet, $b$, $h$, $r$ in inches, and $P$ in pounds.

| Substances. | Rectangular section. | Circular section. |
|---|---|---|
| Wood . . . . | $bh^3=\frac{Pl^2}{170}$ | $r^4=\frac{Pl^2}{1600}$ |
| Cast iron . . . | $bh^3=\frac{Pl^2}{980}$ | $r^4=\frac{Pl^2}{9250}$ |
| Wrought iron . . | $bh^3=\frac{Pl^2}{1680}$ | $r^4=\frac{Pl^2}{15800}$ |

If the load $Q$ be uniformly distributed over the beam, $P$ must be replaced by $\frac{1}{2}\,Q$, § 190, and if the weight of the beam be taken into account by $P$, $P+\frac{1}{2}\,G$. If it be the case of a beam which is fixed at one extremity and loaded at the other, $P$ and $l$ must then be doubled, therefore, $Pl^2$ to be multiplied by eight; if, lastly, the beam fixed at one extremity sustains a load $Q$ uniformly distributed, for $Pl^2$, we must substitute $\frac{3}{8} \,.\, 8\,Ql^2=3\,Ql^2$ for $Pl^2$.

*Examples.*—1. What load will a wooden beam, 20 feet long, 7 inches thick and 9 deep, reposing on both its extremities, sustain for a length of time? This load is $P = 170 \cdot \frac{b\,h^3}{l^2} = 170 \cdot \frac{7 \cdot 9^3}{20^2} = 1190 \cdot \frac{729}{400} = 2170$ lbs.

In § 198 $P$ was found = 1890 lbs.—2. What thickness must an iron axle, 12 feet long, be ~~cast~~, if the same has to sustain a uniformly distributed load $Q = 40000$ lbs., without any detrimental flexure? $r^4 = \frac{\frac{5}{8}\,Q l^2}{15800}$, therefore here $r^4 = \frac{5}{8} \cdot \frac{40000 \cdot 12^2}{15800}$ $= 228$, and $r = \sqrt[4]{228} = 3{,}89$ inches; consequently, the thickness of the axle $2\,r = 7{,}78$, or about $7\frac{3}{4}$ inches. By the formula for strength, if the modulus of tenacity of wrought iron be taken at $\frac{14}{9}$ times that of cast iron: $r^3 = \frac{Q\,l}{8 \cdot \frac{14}{9} \cdot 4700}$ $= \frac{40000 \cdot 144 \cdot 9}{8 \cdot 14 \cdot 4700} = 98{,}5$; hence, $r = \sqrt[3]{98{,}5} = 4{,}62$ inches, and $2\,r = 9{,}24$ inches.

§ 206. *Rupture by compression.*—If prismatic bodies are so strongly compressed in the direction of their axes, as to amount to rupture, their resistance to compression has to be overcome. This rupture may take place in two ways. If the body be short, if it approximates to a cube, it will fall to pieces without undergoing flexion, but if the body is longer than it is broad and thick, Flexure ~~flexion similar to that which takes place will precede the rupture.~~ The one kind of rupture consists in a crushing, bruising, transverse strain, or splitting asunder of the body or its parts; the other, in a fracture or destruction of a section of the body. Hence *a distinction is made between the crushing strength and strength of rupture under compression.*

*The resistance to crushing is, for similar sections, proportional to their areas;* for regular sections, however, somewhat greater than for irregular, and greatest of all for circular sections. It is besides independant for the most part of the length of the body. Short wooden prisms split asunder in the direction of their length, or form a bulge; stones break into several pieces or separate along an inclined plane. Ten times the absolute strength is given to wood and stones; to iron, only one of five times; and to walls of rough stones, twenty times. If $K$ be the modulus of resistance to crushing, and $F$ the transverse section of the bodies, the working load will be

$P = F K_1$ and $F = \frac{P}{K_1}$, where for $K_1$, $\frac{1}{5}K$ to $\frac{1}{20}K$ must be substituted.

TABLE

OF THE MODULUS OF RESISTANCE TO CRUSHING.

| Names of substances. | Modulus $K$. | Names of substances. | Modulus $K$. |
|---|---|---|---|
| Basalt . . . . . | 27000 | Brick . . . . . | 580 to 2200 |
| Gneiss . . . . . | 5100 | Oak . . . . . | 2800 „ 6800 |
| Granite . . . . . | 6000 to 11000 | Pine . . . . . | 6800 „ 8000 |
| Limestone . . . . | 1500 „ 6000 | Fir . . . . . . | 2000 |
| Marble . . . . . | 3200 „ 12000 | Cast iron . . . . | 146000 |
| Mortar . . . . . | 450 „ 900 | Wrought iron . . | 72000 |
| Sandstone . . . . | 1400 „ 13000 | Copper . . . . | 60000 |

The values of $K$ contained in the preceding table are not unfrequently, especially for wooden columns, applicable even when the bodies are very long, only it has been found necessary to diminish these values by one, two, or three sixths, when the columns are twelve, twenty-four or forty-eight times as long as they are thick. Accordingly, for a column of oak, one foot thick and twenty-four long, $K$ must be taken at from $2800 (1 - \frac{2}{6}) = 1900$ lbs. to $6800 . \frac{4}{6} = 4500$ lbs. The formulæ developed in § 185 for the transverse section of bodies of considerable weight and of bodies of the strongest form here find their application.*

*Examples.*—1. What load can a round column of fir, 12 feet long and 11 inches diameter, sustain? $F = \frac{\pi . 11^2}{4} = 95$ square inches; if we now take for $K$ a mean value $= \frac{6800 + 8000}{2} = 7400$, and diminish the value one-sixth, because the length is 13 times that of the thickness, and therefore put $K = 7400 . \frac{5}{6} = 6200$ lbs.; and give a ten-times security, we shall then have $P = \frac{6200 . F}{10} = 620 . 95 = 58900$ lbs. —2. How thick must be the foundation walls of a massive building of 20000000 lbs. weight, 60 feet outer length, and 40 feet breadth, if for this purpose we use well finished blocks of gneiss? Let $x$ be the requisite thickness, $60 - x$ is the mean length, and $40 - x$ the breadth; therefore, the mean perimeter $2 (60 - x + 40 - x) = 200 - 4x$; if we multiply this by $x$, we obtain the base of the walls $(200 - 4x) x$ square feet $= 144 (200 - 4x) x = 576 (50 - x) x$ square inches. For a twenty-fold security, a square inch of gneiss sustains a pressure $= \frac{5100}{20} = 255$ lbs.; hence we have to put $255 . 576 (50 - x) x = 20000000$, or $50 x - x^2 = \frac{20000000}{146880} = 136$. Now $x = \frac{136 + x^2}{50}$, or about $x = \frac{136}{50} = 2,7$ feet. Now $x^2$ being put $= 2,7^2 = 7$,

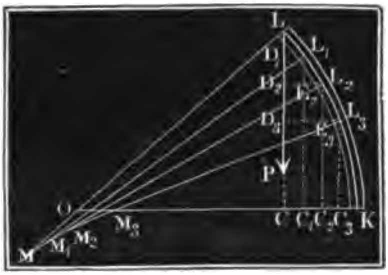
* See Appendix.

more accurately $x = \frac{136+7}{50} = \frac{143}{50} = 2{,}86$ feet, for which we may take 2,9 feet, $= 35$ inches.

§ 207. *Rupture under compression.*—If a prismatic body $ABCD$, Fig. 236, be fixed at one extremity, and at the other be acted on by a pressure $P$, which acts in the direction of the axis of the body, the relations of deflexion will come out otherwise than when the pressure acts perpendicular to the axis. The neutral axis $KL$ assumes another form, because the arms of the pressure $P$ are not formed by the abscisses, but by the ordinates, as $HK$. From § 188, we have for the angles of curvature $LML_1$, $L_1M_1L_2$, &c., of the

FIG. 236.


FIG. 237.




neutral axis $KL$, Fig. 237, $\phi_1 = \frac{M_1 \,.\, \overline{LL_1}}{WE}$, $\phi_2 = \frac{M_2 \,.\, \overline{L_1L_2}}{WE}$, &c.

but here the moments are $M_1 = P \,.\, \overline{D_1L_1}$, $M_2 = P \,.\, \overline{D_2L_2}$, &c., hence we have the measures of the angles: $\phi_1 = \frac{P \,.\, \overline{D_1L_1} \,.\, \overline{LL_1}}{WE}$,

$\phi_2 = \frac{P \,.\, \overline{D_2L_2} \,.\, \overline{L_1L_2}}{WE}$, &c. If we introduce the tangential angles $L_1LD_1 = KOL = a$, $L_2L_1E_2$, $= a_1 = a - \phi_1$, $L_3L_2E_3 = a_2 = a_1 - \phi_2$ $= a - \phi_1 - \phi_2$, &c., and if we suppose only a small curvature, we may then write: $LL_1 = \frac{D_1L_1}{a}$, $L_1L_2 = \frac{E_2L_2}{a_1} = \frac{E_2L_2}{a - \phi_1}$, $L_2L_3 = \frac{E_3L_3}{a_2}$ &c.; if further we divide the entire height of the arc $CK = a$ into $n$ equal parts, we may then put: $D_1L_1 = CC_1 = E_2L_2 = C_1C_2$, &c.

$=\frac{a}{n}$, but $D_2L_2=\frac{2\,a}{n}$, $D_3L_3=\frac{3\,a}{n}$, &c., and by the substitution of these values it follows that:

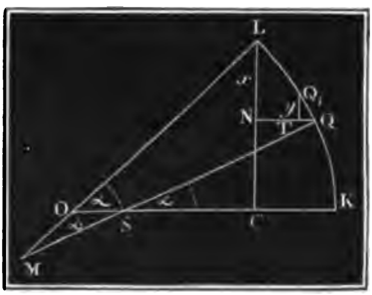
$$\phi_1=\frac{P\,.\,\frac{a}{n}.\frac{a}{n\,a}}{WE}=\frac{Pa^2}{WE\;n^2a},\quad \phi^2=\frac{P\,.\,\frac{2\,a}{n}.\frac{a}{n\,a_1}}{WE}=\frac{2\;Pa^2}{WE\,n^2a_1},$$

$$\phi_3=\frac{3\;Pa^2}{WE\,n^2a_2},\ \text{\&c., or}\ \phi_1a=\frac{Pa^2}{WE\;n^2},\quad \phi_2a_1=\frac{2\;Pa^2}{WE\;n^2},$$

$$\phi_3a_2=\frac{3\;Pa^2}{WE\;n^2},\ \text{\&c.}$$

The sum $\phi_1a+\phi_2a_1+\phi_3a_2+\ldots=\frac{Pa^2}{WE\;n^2}(1+2+3+\ldots+n)$ $=\frac{Pa^2}{WE\,.\,n^2}.\frac{n^2}{2}=\frac{Pa^2}{2\;WE}$, and may be also found, if $a$ be divided into $m$ equal parts, and any such part $\frac{a}{m}$, be put $=\phi_1=\phi_2=\phi_3$, &c. We shall then obtain $\phi_1a+\phi_2a_1+\phi_3a_2+\ldots=\frac{a}{m}\,.\,a+\frac{a}{m}\left(a-\frac{a}{m}\right)$ $+\frac{a}{m}\left(a-\frac{2\,a}{m}\right)+\ldots+\frac{a}{m}.\frac{a}{m}$, by taking out the common factor $\left(\frac{a}{m}\right)^2$, and writing it in an inverse order $=\left(\frac{a}{m}\right)^2(1+2+\ldots+m)$ $=\frac{a^2}{m^2}\,.\,\frac{m^2}{2}=\frac{a^2}{2}$, and by making these two sums equal to each other $a^2=\frac{Pa^2}{WE}$, an equation between the angle of curvature $LOK$ $=a^0$, and the height of the arc $CK=a$.

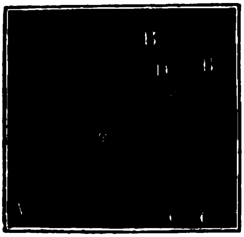
FIG. 238.




For the equation of the elastic line $LK$, Fig. 238, let us take $LN=x$ and $NQ=y$ as co-ordinates, and put the corresponding angle of curvature $LMQ=a_1$. In the last equation, if we put $a$ for $y$, and $y$ for $a$, we then have to replace the sum $\phi_1\,a+\phi_2\,a_1+\phi_3\,a_2+\ldots$ by $\frac{a^2-(a-a_1)^2}{2}$; hence, if we represent the supplementary angle $QSK=a-a_1$ by $a_2$, we

afterwards obtain $a^2 - a_2^2 = \frac{Py^2}{WE}$, or $a_2^2 = a^2 - \frac{Py^2}{WE}$, and so $a_2 = \sqrt{\frac{P}{WE}} \cdot \sqrt{a^2 - y^2}$. But since $a_2^0 = \angle QQ_1T$, and $tang.\ QQ_1T = \frac{TQ}{TQ_1} = \frac{\text{element } \delta \text{ of the ordinate } y}{\text{element } \epsilon \text{ of the absciss } x}$, we have $\frac{\delta}{\epsilon} = \sqrt{\frac{P}{WE}} \cdot \sqrt{a^2 - y^2}$.

FIG. 239.



If in a rectangular triangle $ABC$, Fig. 239, the angle $CAB$ increases with the hypothenuse $AB = a$, and cathetus $BC = y$ by a small amount $BAB_1 = \psi^0$, the cathetus $y$ decreases by the amount $DB_1 = \delta$, for which $\frac{DB_1}{BB_1} = \frac{AC}{AB}$, *i.e.* $\frac{\delta}{a\psi} = \frac{\sqrt{a^2 - y^2}}{a}$; and hence $\frac{\delta}{\psi} = \sqrt{a^2 - y^2}$.

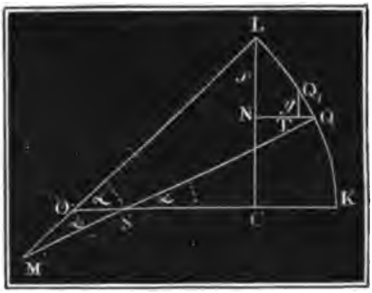
By comparing the two expressions $\frac{\delta}{\epsilon} = \sqrt{\frac{P}{WE}} \cdot \sqrt{a^2 - y^2}$ and $\frac{\delta}{\psi} = \sqrt{a^2 - y^2}$, we obtain the equation $\psi = \epsilon\sqrt{\frac{P}{WE}}$, or $\frac{\epsilon}{\psi} = \sqrt{\frac{WE}{P}}$. Therefore the ratio of the element $\epsilon$ of the absciss to the element of the arc $\psi$ is invariable, and $= \sqrt{\frac{WE}{P}}$, and hence, also, the ratio of the absciss $x$ to the whole arc $A = \sqrt{\frac{WE}{P}}$, *i. e.* $\frac{x}{A} = \sqrt{\frac{WE}{P}}$, and $A = x\sqrt{\frac{P}{WE}}$. If, finally, we substitute this value of $A$ in the equation $BC = AB\ sin.\ A$; *i. e.* $y = a\ sin.\ A$, we obtain the equation sought:

$$y = a\ sin.\left(x\sqrt{\frac{P}{WE}}\right).$$

With the assistance of this last formula we may find the ordinate $NQ = y$ corresponding to any absciss $LN = x$, Fig. 240. If in this we put $x = l$ and $y = a$, we then obtain $a = a\ sin.$

$\left(l\sqrt{\frac{P}{WE}}\right)$, *i. e.* $1 = sin. \left(l\sqrt{\frac{P}{WE}}\right)$, whence it follows that $l\sqrt{\frac{P}{WE}} = \frac{\pi}{2}$ and $P = \left(\frac{\pi}{2l}\right)^2 WE$.

FIG. 240.




As this formula does not contain the height of the arc $a$, it follows that the force $P$ is capable of maintaining equilibrium for every deflexion of the body. This remarkable circumstance is explained by the fact that an increase of the arm or of the statical moment is combined with an increase of the deflexion. Hence, therefore, the force for rupture is:

$$P = \left(\frac{\pi}{2l}\right)^2 WE.$$

§. 208. *Columns.*—If we put in the formula $P = \left(\frac{\pi}{2l}\right)^2 . WE$ for $W = \frac{bh^3}{12}$, we then obtain in $P$ the resisting strength of a rectangular column $P = \frac{\pi^2}{48} \cdot \frac{bh^3}{l^2} E$.

*The strength, therefore, of a parallelipiped increases as the breadth or greater dimension, and the cube of the thickness or less dimension of the transverse section, and inversely as the square of the length.*

If, on the other hand, we put $W = \frac{\pi}{4} r^4$, then for a cylindrical column we have $P = \frac{\pi^3}{16} \cdot \frac{r^4 E}{l^2}$.

*The strength of a cylinder increases, therefore, as the fourth power of the diameter, and inversely as the square of the length.*

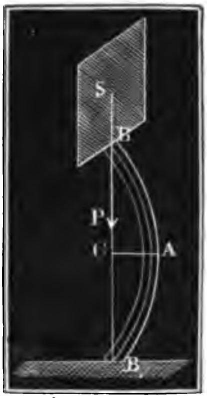
For a hollow column with the radii $r_1$, and $r_2$

$$P = \frac{\pi^3}{16} \cdot \frac{(r_1^4 - r_2^4) E}{l^2}.$$

If the column be not fixed at the lower extremity, it will assume a curvature $BAB_1$, Fig. 241, by which the lower half will be as strongly deflected as the upper, and the greatest curvature take

place in the middle. Therefore this beam must be regarded as the double of one imbedded in a wall, and for $l$, $\frac{l}{2}$ must be substituted, so that for the rectangular and for the cylindrical columns,

FIG. 241.




$$P = \frac{\pi^2}{12} \cdot \frac{bh^3}{l^2} E, \; P = \frac{\pi^3}{4} \cdot \frac{r^4}{l^2} E;$$

in both cases, however, there is a fourfold tenacity. These formulæ, when the columns are not very long, give generally a greater tenacity than the formula for the crushing strength, wherefore the ratios of the sections are often determined from the last. It is at least advisable only to make use of the formula for rupture under compression when the length is at least twenty times that of the thickness, and then further to allow a twenty-fold security.*

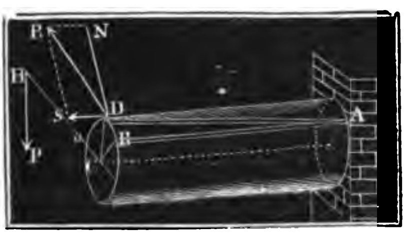
*Examples.*—1. For a column of fir, 12 feet long and 11 inches thick, the tenacity is $P = \frac{\pi^3}{16} \cdot \frac{r^4}{l^2} \cdot \frac{E}{20} = 31 \cdot \left(\frac{11}{24}\right)^4 \cdot \frac{1800000}{20} = 31 \cdot 0{,}044 \cdot 90000$ $= 123000$ lbs.; in Example No. 1, of § 206, 58900 lbs. only, therefore about $\frac{1}{2}$ of the above, was found.—2. How thick must a column of oak, 30 feet high be in order to be able to bear a load of 60000 lbs.?

$$\text{Here } r = \sqrt[4]{\frac{4 P l^2}{\pi^3 E}} = \sqrt[4]{\frac{4 \cdot 60000 \cdot (30 \cdot 12)^2}{31 \cdot \frac{1800000}{20}}} = \sqrt[4]{\frac{8 \cdot 6 \cdot 360^2}{31 \cdot 18}} = 10{,}3 \text{ inches};$$

consequently, the thickness of about 21 inches. The strength of crushing requires, if $K$ be put $= \frac{1}{10} \cdot \frac{4}{6} \cdot \frac{2800 + 6800}{2} = 320$ lbs., the transverse section $F = \frac{60000}{320}$ $= 188$ square inches; whence, $r = \sqrt{\frac{188}{\pi}} = 13{,}7 \cdot 0{,}564 = 7{,}7$ inches, and the thickness should be $15\frac{1}{2}$ inches. For this case the first value must be taken.

§ 209. *Torsion.*—When a body $A\,B\,C$, Fig. 242, fixed at one extremity, is acted upon by a force whose direction lies in the plane normal to the axis, and therefore endeavours to turn the body about the axis, or when two forces of revolution $P$ and

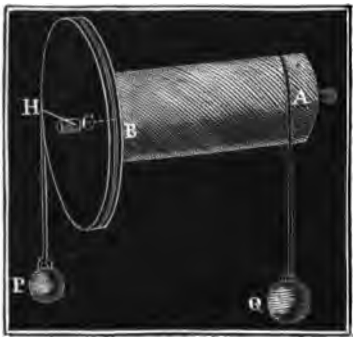
FIG. 242.




* See Appendix.

$Q$ act in different normal planes upon a body $A\ B$, fixed by its axis, Fig. 243, the fibres running parallel to the axis undergo a wrenching or torsion, the amount of which we wish to determine. Let

FIG. 243.




$A\ B$, Fig. 242, be a fibre before, and $A\ D$ the same fibre during the torsion, and therefore let the extremity of the fibre $B$ be advanced by the force of torsion to $D$. If now $l$ be the initial length $A\ B$, and $\lambda$ its extension, therefore $l + \lambda$ the length $A\ D$ during the torsion, and if $s$ be the corresponding torsion $B\ D$, we have after the Pythagorean law to put

$$\overline{A\ D}^2 = \overline{A\ B}^2 + \overline{B\ D}^2$$

$(l+\lambda)^2 = l^2 + s^2$, or $l^2 + 2\,l\lambda + \lambda^2 = l^2 + s^2$, may be put approximately $= \lambda = \frac{s^2}{2l}$. If further $F$ be the section of such a fibre, we then have for the force required to produce this extension in the direction of the fibre $S = \frac{s^2}{2\,l^2} \cdot F \cdot E$. But this force or ~~tension~~ to ($S$) of a fibre is only a component of the force of ~~tension~~ torsion $R$, which produces besides a further pressure $N$, normal to the fibres. From the similarity of the triangles $RDS$ and $BDA$, it follows that $S : R = s : l$, hence $S = \frac{Rs}{l}$, and by equating both values of $S$:

$$R = \frac{s}{2\,l} F \cdot E.$$

*Therefore the force of torsion of a fibre increases as the torsion* ($s$), *and the transverse section* $F$, *and inversely as the length* ($l$) *of the fibre.*

FIG. 244.


To find the force of torsion of a cylindrical axle $CBA$, Fig. 244, let us divide its radius $r$ into $n$ equal parts, and suppose concentric circles passing through the points of division, so that the transverse section becomes decomposed into annular elements of the

thickness $\frac{r}{n}$, and radii $\frac{r}{n}, \frac{2r}{n}, \frac{3r}{n} \dots \frac{nr}{n}$. The solid contents of these elements are $F_1 = 2\pi . \frac{r}{n} . \frac{r}{n} = 2\pi \left(\frac{r}{n}\right)^2$, $F_2 = 2\pi . \frac{2r}{n} . \frac{r}{n} = 4\pi \left(\frac{r}{n}\right)^2$, $F_3 = 2\pi . \frac{3r}{n} . \frac{r}{n} = 6\pi \left(\frac{r}{n}\right)^2$, &c. If all the fibres are twisted by the angle $BCD = a^0$, they have the corresponding torsions $s_1 = \frac{r}{n} a$, $s_2 = \frac{2r}{n} a$, $s_3 = \frac{3r}{n} a$, and hence the forces of torsion are: $R_1 = \frac{ra}{2\,nl} . 2\pi \left(\frac{r}{n}\right)^2 E = \frac{a\pi}{l} \left(\frac{r}{n}\right)^3 E$, $R_2 = \frac{4\,a\pi}{l} \left(\frac{r}{n}\right) E$, $P_3 = \frac{9a\pi}{l} \left(\frac{r}{n}\right)^3 E$, &c. If further we multiply these forces by the arms $\frac{r}{n}, \frac{2r}{n}, \frac{3r}{n}$, and add together the values so obtained, we have for the moments of torsion $Pa = \frac{a\pi}{l} \left(\frac{r}{n}\right)^4 (1^3 + 2^3 + 3^3 + \dots + n^3) E$, *i.e.* $Pa = \frac{a\pi}{l} \left(\frac{r}{n}\right)^4 . \frac{n^4}{4} = \frac{a\pi r^4}{4\,l} . E$, and inversely, the measure of the angle of torsion:

$$a = \frac{4\,l\,.\,Pa}{\pi r^4\,E}.$$

If the axle be hollow and have radii $r_1$ and $r_2$, we have then

$$Pa = \frac{a\pi}{4\,l} E\,(r_1^4 - r_2^4), \text{ therefore } a = \frac{4\,Pal}{\pi\,E\,(r_1^4 - r_2^4)}.$$

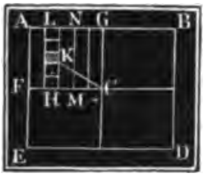
The application of hollow axles gives also with respect to torsion a saving in material, for if we put $r_2 = r$, and $r_1 = r\sqrt{2}$, we then obtain for the hollow axle, which has the same section as a solid one, the moment of torsion:

$$= \frac{a\pi\,E}{4\,l}\,(4r^4 - r^4) = 3\,.\,\frac{a\pi r^4\,E}{4\,l}, \text{ i. e.}$$

thrice as great as for the solid axle.

§ 210. For a shaft or axle of a rectangular section $ABDE$, Fig. 245, the moment of torsion is found in the following manner.

FIG. 245.




If we divide half the breadth $AG=b$ into $n$ equal parts, and carry through the points of division the parallel planes $HL$, $MN$, &c. we obtain elements of equal sections, each $=\frac{b}{n}\,.\,h$, where $h$ represents half the height $AF=GC$ of the section. If now we divide one of these elementary strips into $m$ equal parts, we have for its area $\frac{1}{m}\,.\,\frac{bh}{n}=\frac{bh}{mn}$. Let the normal distance $CH$ of the strip $HL$ from the centre $C$, $=c$, and the distance $KH$ of the element $K$ from the normal $CH$, $=e$, then the distance of the element from the axis is $CK=\sqrt{c^2+e^2}$, accordingly the arc of torsion $=\alpha\sqrt{c^2+e^2}$, and the moment of torsion

$$=\frac{a\sqrt{c^2+e^2}}{2\,l}\,.\,\frac{bh}{mn}\sqrt{c^2+e^2}\,.\,E=\frac{abh}{2mnl}\,(c^2+e^2)\,E.$$

If now we successively put $e=\frac{1}{m}\,h,\ \frac{2}{m}\,h,\ \frac{3}{m}\,h$, &c., and sum the results, we have the moment of the strip:

$$HL=\frac{abh}{2mnl}\,(c^2+\frac{h^2}{m^2}+c^2+\frac{4h^2}{m^2}+c^2+\frac{9h^2}{m^2}+\ldots)\,E$$

$$=\frac{abh}{2mnl}\left[mc^2+\frac{h^2}{m^2}\,(1+4+9+\ldots+m^2)\right]E.$$

But $1+4+9+\ldots+m^2=\frac{m^3}{3}$, hence the moment of the strip $=\frac{abh}{2nl}\left(c^2+\frac{h^2}{3}\right)E$. To obtain the moments of all the strips, let us again put $c=\frac{b}{n},\ \frac{2b}{n},\ \frac{3b}{n}$, &c., and again sum the results, we shall then have:

$$=\frac{abh}{2\,nl}\left[\left(\frac{b}{n}\right)^2+4\left(\frac{b}{n}\right)^2+9\left(\frac{b}{n}\right)^2+\ldots+\frac{nh^2}{3}\right]E=$$

$$\frac{abh}{2\,nl}\left[\left(\frac{b}{n}\right)^2(1+4+\ldots n^2)+\frac{nh^2}{3}\right]E=\frac{abh}{2\,nl}\left(\frac{b^2}{n^2}\cdot\frac{n^3}{3}+\frac{nh^2}{3}\right)E$$

$$=\frac{abh}{2\,l}\left(\frac{b^2+h^2}{3}\right)E.$$

Generally the sections are square, and therefore $b=h$. As we have only considered a fourth part of the shaft, it follows for the whole shaft that:

$$Pa=\frac{4\,ab^4}{3\,l}\,E.$$

For a cylindrical axle $P_1a_1=\frac{a\pi r^4}{4\,l}\,E$; if we put $b=r$ we then obtain $Pa=\frac{4}{3}\cdot\frac{4}{\pi}\,\frac{a\pi r^4}{4\,l}E=\frac{16}{3\pi}\,P_1a_1$, the moment of the square is, therefore, $=\frac{16}{3\pi}=1{,}756$ times as great as that of the round axle. But if we make $4b^2=\pi r^2$, and, therefore, both sections equal, we then obtain $Pa=\frac{a\,.\,\pi^2r^4}{4\,.\,3l}\,E=\frac{\pi^2}{4.3}\cdot\frac{4}{\pi}\cdot P_1a_1=\frac{\pi}{3}\;P_1a_1$, therefore the square shaft is only a little stronger than the cylindrical axle.

If the axle be hollow, and the outer and inner radii be $2b_1$ and $2b_2$, we then have:

$$Pa=\frac{4aE}{3\,l}\,(b_1^4-b_2^4).$$

§ 211. *Breaking Twist.*—When the torsion exceeds a certain limit, the fibres are torn asunder, and the cylindrical axle is *twisted asunder*. For the force ~~moment~~ of rupture of the fibre furthest from the axis, $\frac{\lambda}{l}=\frac{K}{E}$, but $\frac{\lambda}{l}$ is also $=\frac{s^2}{2\,l^2}=\frac{a^2r^2}{2\,l^2}$, hence it follows that $\frac{ar}{l}=\sqrt{\frac{2\,K}{E}}$. The statical moment of twisting for the round axle is: $\frac{a\pi r^4}{4l}E = \frac{ar}{l}\cdot\frac{\pi r^3}{4}E$

$$Pa=\sqrt{\frac{2\,K}{E}}\cdot\frac{\pi r^3}{4}E=\frac{\pi r^3}{2}\sqrt{\frac{KE}{2}};$$

but for the square shaft, where the greatest distance of a fibre is half the diagonal $b\sqrt{2}$, it follows that $\frac{4ab^4}{3l}E = \frac{ab}{l}\cdot\frac{4b^3}{3}$

$$\frac{K}{E}=\frac{2\,a^2b^2}{2\,l^2}\text{ hence }\frac{a\,b}{l}=\sqrt{\frac{K}{E}}\text{ and }Pa=\frac{4\,b^3}{3}\sqrt{KE}.$$

Since the fibres are not only extended by torsion, but also compressed, and as we have only had regard to extension in our development, so it may be expected that the formulæ found do not in their quantitative relations quite correspond with experiment

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and, therefore, it is necessary to take the constants $E$ and $\sqrt{KE}$ from experiments made especially to determine them.

If $\alpha$ be given in degrees, such observations admit of our putting for the torsion:

| Substances. | Circular section. | Square section. |
|---|---|---|
| Wood . . . . . | $Pa = 3500 \cdot \frac{\alpha^\circ r^4}{l}$. | $Pa = 5800 \frac{\alpha^\circ b^4}{l}$. |
| Cast iron . . . . | $Pa = 160000 \frac{\alpha^\circ r^4}{l}$. | $Pa = 280000 \frac{\alpha^\circ b^4}{l}$. |
| Steel and wrought iron . | $Pa = 280000 \frac{\alpha^\circ r^4}{l}$. | $Pa = 470000 \frac{\alpha^\circ b^4}{l}$. |

In what relates to the strength of torsion, numerous experiments made upon cast iron have given $\sqrt{\frac{KE}{2}} = 30000$ to $66000$ lbs., if therefore a five-fold security be taken, then is $\frac{\pi}{2}\sqrt{\frac{KE}{2}} = 12600$ lbs. therefore, for the round cast iron axle $Pa = 12600\ r^3$, and for the square $= 15000\ b^3$.

The same formulæ hold good for axles of wrought iron, but for wooden ones we may put $Pa = 1260\ r^3$ and $= 1500\ b^3$, *i. e.* the modulus of strength $= \frac{1}{10}$ that of iron axles. The modulus of strength for steel $\sqrt{\frac{KE}{2}}$ must be taken at twice that of iron, and gun metal at one half.*

*Examples.*—1. The iron upright axle of a turbine exerts at the circumference of a toothed wheel of 15 inches radius reposing upon it, a force of 2500 lbs.; what thickness must be given to it? $Pa = 2500 \cdot 15 = 37500$, and if we put $r^3 = \frac{Pa}{12600} = \frac{37500}{12600} = \frac{375}{126}$, we shall obtain $r = \sqrt[3]{\frac{375}{126}} = 1{,}44$ inches; hence, the thickness of the axle $2r = 2{,}88$ inches, for which 3 inches may be assumed. If the distance of the toothed wheel from the water wheel is 60 inches, the torsion of the axle $= \alpha^\circ = \frac{Pal}{160000\ r^4} = \frac{37500 \cdot 60}{160000 \cdot 1{,}44^4} = \frac{375 \cdot 6}{160 \cdot 4{,}28} = \frac{14{,}06}{4{,}28} = 3^\circ\ 3'$, therefore very considerable.—2. On a four-coined axle of fir, a force, $P = 500$ lbs., acts at an arm of 20 feet, whilst the load is applied at an arm of 2 feet, the distance measured in the direction of the axis $l = 10$ feet; how thick must this axle be made, and how great is the torsion? It is $Pa = Qb = 500 \cdot 2 \cdot 12 = 120000$ inch lbs.; but the load

* See Appendix.

$Q = \frac{a}{b} P = 5000$ lbs.; half the side $b$ of the axle is determined by $b^3 = \frac{Pa}{1500} = \frac{120000}{1500}$ $= 80$; hence $b = \sqrt[3]{80} = 4{,}31$ inches, and the whole side $= 8{,}62$ inches. The torsion amounts to $\alpha^o = \frac{Pal}{5800 \,.\, b^4} = \frac{120000 \,.\, 12 \,.\, 10}{5800 \,.\, 4{,}31^4} = \frac{144000}{58.345} = 7^o$; therefore, here very considerable. In general, less torsion is allowed, and therefore the axles are made much stronger. Generally, this angle does not amount to $\frac{1}{2}$ a degree. If, however, we put $\alpha^o = \frac{1}{2}^o$, for this case we shall obtain $b^4 = \frac{144000}{58 \,.\, \frac{1}{2}} = 4965$, hence $b = \sqrt[4]{4965} = 8{,}4$ inches, and $2b = 16{,}8$ inches. According to Gerstner, the angle of torsion of an axle ought not to amount to more than $0{,}1^o$.

*Remark.* If an axle has to sustain relative elasticity and that of torsion, we must make the calculation for both, and apply the greater ratio of the dimensions found.

# SECTION IV.

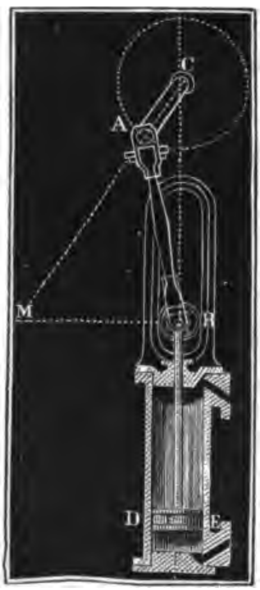
## DYNAMICS OF RIGID BODIES.

## CHAPTER I.

### DOCTRINE OF THE MOMENT OF INERTIA.

§ 212. *Kinds of motion.*—The motion of a rigid body is either one of *translation*, or one of *rotation*, or both. The spaces described by the particles of a body in motion of translation or progression are parallel and equal to each other; on the other hand, in motion of revolution or rotation, the particles of a body describe concentric circles about a certain straight line, which is called the *axis of revolution*. Compound motion may be regarded as a motion of revolution about a *moving axis*. The latter is either uniform or variable.

FIG. 246.




The piston $DE$, or the piston-rod $BF$ of a pump or steam-engine, Fig. 246, has a motion of translation, whilst the arm or the crank $AC$ has one of rotation, and the connecting rod $AB$ a compound motion; for one of its extremities $B$ has a progressive, the other $A$ a rotatory motion. The axis of revolution, in a rotating cylinder, is invariable; that of the connecting rod $AB$, on the other hand, is

variable; for it is the intersection $M$ of the perpendicular to the direction of the axis of the rod $BM$, and the prolongation of the crank $CA$.

§ 213. *Rectilinear motion.*—The laws of motion of a material point found in § 81, &c., have their direct application in rectilinear progressive motion. The particles of the mass $M_1$, $M_2$, $M_3$, &c., of a body, moving with the acceleration $p$ by their inertia, offer resistance to motion with the forces $M_1p$, $M_2p$, $M_3p$, &c. (§ 58); and since the motions of all these take place in lines parallel to each other, the directions, therefore, of these forces are parallel; hence, the resultant of all these forces arising from the inertia is equivalent to the sum $M_1p + M_2p + M_3p + \ldots = (M_1 + M_2 + M_3 + \ldots)\, p = Mp$, where $M$ represents the mass of the whole body, and its point of application coincides with the centre of gravity of the body. In order, therefore, to change the motion of an otherwise freely moving body of the mass $M$ or of the weight $G = Mg$ into one rectilinear and progressive, a force $P = Mp = \frac{Gp}{g}$, whose direction passes through the centre of gravity $S$ of the body, is requisite. If, in virtue of the action of the force $P$, the velocity $c$ acquired while the body describes the space $s$, ~~increases to the velocity $v$~~, then the mechanical effect accumulated in the mass (§ 71) is:

$$Ps = \left(\frac{v^2 - c^2}{2}\right) M = \left(\frac{v^2 - c^2}{2g}\right) G = (h - h_1)\, G.$$

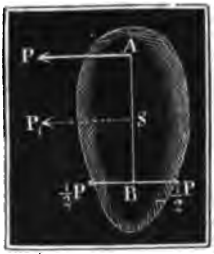
*Example.* The piston of a pump, with its rod, or that of a steam-engine or blowing machine, has a variable motion; it has no velocity at its highest and lowest positions, and at its mean position its velocity is a maximum. If the weight of the piston and rod be $= G$, and its maximum velocity at the middle of its ascent and descent $= v$, the effect which it will accumulate by virtue of its inertia in the first half of its path, and will give out again in the second half $= \frac{v^2}{2g} G$. For $G = 800$ lbs. and $v = 5$ ft. this effect $= 0{,}0155 \,.\, 5^2 \,.\, 800 = 310$ ft. lbs.; were half the path of the piston $s = 4$ feet, we should then have the mean force which would be requisite to accelerate the motion of the piston in the first half of this path, and which it would exert in the second half by its retardation:

$$P = \frac{v^2}{2gs} \,.\, G = \frac{310}{4} = 77 \text{ lbs.}$$

§ 214. *Motion of rotation.*—If the moving force $P$ of a body $AB$, Fig. 247, does not pass through the centre of gravity $S$, the body will take up a rotation about this point, and this ~~motion~~ will ~~go on~~ as if the force were directly applied to it, as may be

proved in the following manner. From the centre of gravity let a perpendicular $SA$ be drawn to the direction of the force; let this be prolonged backwards and the prolongation $SB$ be made equal to the perpendicular, and let two forces, balancing each other and acting parallel to $P$, the one $+\frac{1}{2}P$, and the other $-\frac{1}{2}P$, be applied to the point $B$. The force $+\frac{1}{2}P$ will give, when combined with half the force $P$ applied at $A$, the force applied at the centre of gravity $S$, $P_1 = \frac{1}{2}P + \frac{1}{2}P = P$, for which the force $-\frac{1}{2}P$ will form a couple with the second half of $P$ applied at $A$; there will result, therefore, from the excentrically acting force $P$, a force $P$ acting through the centre of gravity, which will move forward this point, together with the whole body, and a couple $(\frac{1}{2}P, -\frac{1}{2}P)$, which will cause the body to rotate about the centre of gravity without producing a pressure on it. The statical moment of this couple will be $= SA \cdot \frac{1}{2}P + SB \cdot \frac{1}{2}P = SA \cdot P$ equivalent to the statical moments of the force applied at $A$; consequently the resultant rotation will be the same as if the centre of gravity $S$ were fixed, and $P$ acted alone.

FIG. 247.




According to this, therefore, every arbitrarily directed force communicates two motions to a body; one of translation and one of rotation; and hence it is necessary to study the laws of this latter.

If, finally, a body be constrained by a path or directrix to assume a motion of translation, an excentric force will then produce the same effect as if it were applied to the centre of gravity, because the forces of rotation will be taken up by the directrix.

FIG. 248.


§ 215. *Moment of inertia.*—In the rotation of a body $AB$, Fig. 248, about a fixed axis $C$, all its points describe equal angles in equal times. If the body rotate in a certain time through the angle $\phi^0$ or arc $\phi = \frac{\phi^0}{180^0} \cdot \pi$, the elements of the body $M_1$, $M_2$ . . . will describe at the distances $CM_1 = y_1$, $CM_2 = y_2$, &c., from the axis, the spaces $\phi y_1$, $\phi y_2$, &c. If the angular velocity,

*i. e.* the velocity of those points of the body which are distant a unit of length (a foot) from the axis of revolution $= \omega$, then the simultaneous velocities of the elements of the mass at the distances $y_1$, $y_2$, &c., will be $= \omega y_1$, $\omega y_2$, &c.; hence their *vis viva* is $(\omega y_1)^2 M_1$, $(\omega y_2)^2 M_2$, &c., and the sum, or the *vis viva* of the whole body: $(\omega y_1)^2 M_1 + (\omega y_2)^2 M_2 + \ldots = \omega^2 (M_1 y_1^2 + M_1 y_2^2 + \ldots)$.

The sum of the product of the particles of the mass and the squares of their distances from the axis of revolution, is called the *moment of inertia* of the body, the *moment of rotation*, and the *moment of the mass*. If we represent this by $T$, and therefore put $T = M_1 y_1^2 + M_2 y_2^2 + \ldots$, we then obtain $\omega^2 T$ for the *vis viva* of a body revolving with the angular velocity $\omega$. Hence, to communicate to a body, already in a state of rest, an angular velocity $\omega$, a mechanical effect $Ps = \frac{1}{2} \omega^2 T$ must be expended; and inversely, a body produces this effect when it passes from this angular velocity into a state of rest. If, generally, the initial angular velocity of a rotating body $= \varepsilon$, and the terminal angular velocity $= \omega$, we then have for the mechanical effect produced, $Ps = \left(\frac{\omega^2 - \varepsilon^2}{2}\right) T$; and inversely, the terminal velocity corresponding to an expended or accumulated mechanical effect $Ps$:

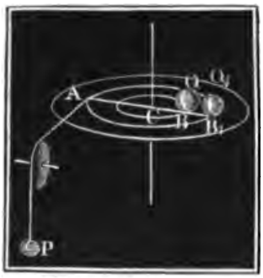
$$\omega = \sqrt{\varepsilon^2 + \frac{2\, Ps}{T}}.$$

*Example.* If a body, $AB$, Fig. 248, originally at rest, and capable of turning about a fixed axis $C$, possesses a moment of inertia of 50 ft. lbs., and by means of a cord lying over a pulley is made to rotate in a path $s = 5$ feet, the acquired angular velocity of this body will be $\omega = \sqrt{\frac{2\, Ps}{T}} = \sqrt{\frac{2 \,.\, 20 \,.\, 5}{50}} = \sqrt{4} = 2$ feet, *i. e.* each point at the distance of one foot from the axis of revolution will describe, after the accumulation of this mechanical effect, two feet in a second. The time of a revolution $t = \frac{2\pi}{\omega} = 3{,}1416$ seconds, and the number of revolutions per minute $u = \frac{60}{t} = \frac{60}{3{,}1416} = 19{,}1$. If the angular velocity found $\omega = 2$ feet, passes into the velocity $s = \frac{3}{4}$ feet, then this mass produces the mechanical effect $P_1 s_1 = \left[2^2 - \left(\frac{3}{4}\right)^2\right] = \frac{50}{2}\left(4 - \frac{9}{16}\right) . 25 = \frac{55}{16} . 25 = 85{,}93$ ft. lbs.; for instance, a weight $P_1$ of 10 lbs., raises 8,593 feet high.

§ 216. *Reduction of inert masses.*—If the angular velocities of two masses $M_1$ and $M_2$ be equal, if, for instance, these masses belong to one and the same rotatory body, their *vis viva* will be to each other as their moments of inertia, $T_1 = M_1 y_1^2$ and $T_2 = M_2 y_2^2$, and if these be equal to each other, the masses will

possess equal *vires vivæ*. Two masses have, therefore, an equal influence on the state of motion of a revolving body, and one may be replaced by the other without causing any change in the condition of motion, if they possess equal moments of inertia, $M_1 y_1^2$ and $M_2 y_2^2$, and are inversely as the squares of their distances from the axis of revolution. By help of the formula $M_1 y_1^2 = M_2 y_2^2$, a mass may be reduced from one distance to another; *i. e.*, a mass $M_2$ may be found, which at the distance $y_2$ has the same influence on the condition of motion of the revolving body, as a given mass $M_1$ at the distance $y_1$; namely, $M_2 = \frac{M_1 y_1^2}{y_2^2} = \frac{T_1}{y_2^2}$, *i. e. the mass reduced to the distance $y_2$ is the quotient of the moment of inertia of the mass and the square of that distance.*

FIG. 249.




Two weights, $Q$ and $Q_1$, attached to an axle, Fig. 249, with the arms $CB = b$ and $CB_1 = a$, have an equal influence, in virtue of their masses, on the motion of the wheel and axle, if $Q_1 a^2 = Qb^2$; therefore $Q_1 = \frac{Qb^2}{a^2}$. Hence, if a force $P$ act at the arm $CA = CB_1 = a$, and cause rotation in a mass of the weight $Q$ at the distance $CB = b$, we shall have then to reduce the latter to the arm $a$ of the force $P$; therefore, for $Q$ substitute $Q_1 = \frac{Qb^2}{a^2}$ and the mass set into motion by $P$ will be $= \left(P + \frac{Qb^2}{a^2}\right) \div g$; whence the accelerated motion of the weight $P$

$$p = \frac{\text{force}}{\text{mass}} = \frac{P}{P + \frac{Qb^2}{a^2}} \cdot g = \frac{Pa^2}{Pa^2 + Qb^2} \cdot g,$$

and the angular acceleration $\frac{p}{a} = \frac{Pa}{Pa^2 + Qb^2} \cdot g$.

*Example.* The weight of a rotating mass $Q = 360$ lbs., its distance from the axis of rotation $b = 2,5$ feet, the weight constituting the moving force $P = 24$ lbs., and its arm $a = 1,5$ feet, the inert mass put into accelerated motion by $P = \left[P + \left(\frac{2,5}{1,5}\right)^2 Q\right] : g = 0,031 \left(24 + \frac{25}{9} \cdot 360\right) = 0,032 \cdot 1024 = 31,75$ lbs.; and hence the accelerated motion of the weight: $p = \frac{24}{32,75} = 0,756$ feet; on the other hand the acceleration of the motion of the mass $Q$, $= \frac{b}{a} \cdot p = \frac{25}{15} p = \frac{5,756}{3} = 1,26$ft.

and the angular acceleration $= \frac{p}{a} = 0,488$ feet. After 4 seconds the acquired angular velocity will be $\omega = 0,488 \,.\, 4 = 1,952$ feet, and the corresponding distance $= \frac{1,952 \,.\, 4}{2} = 3,904$ feet; consequently, the angle of revolution $\phi^\circ = \frac{3,904}{\pi} \,.\, 180^\circ = 1,84 \,.\, 180^\circ = 233^\circ\, 7'$; consequently, the space described by the weight $P = \frac{pt^2}{2} = \frac{0,732 \,.\, 4^2}{2} = 5,86$ feet.

§ 217. *Reduction of the moment of inertia.*—When the moment of inertia of a body or system of bodies about an axis, passing through the centre of gravity $S$ of the body, is known, the moment of inertia about any other axis running parallel to it, may be easily found. Let Fig. 250 be the first axis of revolution, passing through the centre of gravity $S$, $D$ the second axis, for which the moment of inertia of the body is to be determined; further, let $SD = e$ be the distance of the two axes, and let $SN_1 = x_1$ and $N_1M_1 = y_1$ the rectangular co-ordinates of a particle of the mass $M_1$ of the body. Now the moment of inertia of this particle about $D = M_1 \,.\, \overline{DM_1}^2 = M_1\,(\overline{DN_1}^2 + \overline{N_1M_1}^2) = M_1\,[e + x_1)^2 + y_1^2]$, and about $S = M_1 \,.\, \overline{SM_1}^2 = M_1 \,.\, (\overline{SN_1}^2 + \overline{N_1M_1}^2) = M_1\,(x_1^2 + y_1^2)$; hence the difference of both moments:

FIG. 250.

$$= M_1\,(e^2 + 2\,ex_1 + x_1^2 + y_1^2) - M_1\,(x_1^2 + y_1^2) = M_1e^2 + 2\,M_1ex.$$

For another particle of the mass $M_2$, it is $= M_2e^2 + 2\,M_2\,ex_2$, for a third $= M_3e^2 + 2\,M_3\,ex_3$, &c., and for all the particles together:

$$= (M_1 + M_2 + M_3 + \ldots)\,e^2 + 2e\,(M_1x_1 + M_2x_2 + M_3x_3 + \ldots).$$

But $M_1 + M_2 + \ldots$ is the sum $M$ of all the masses, and $M_1x_1 + M_2x_2 + \ldots$ the sum $Mx$ of their statical moments; hence the difference between the moment of inertia $T_1$ of the whole body about the axis $D$ and the moment of inertia $T$ about $S$: $T_1 - T = Me^2 + 2\,e\,Mx$. But since, lastly, for every plane passing through the centre of gravity, the sum of the statical moments of the particles on the one side is as great as that of those on the other, the algebraical sum of all the particles therefore $= 0$; we have, also, $Mx = 0$, and, therefore, $T_1 - T = Me^2$; *i. e.* $T_1 = T + Me^2$.

*The moment of inertia of a body about an axis not passing through the centre is equivalent to its moment of inertia about an axis running parallel to it through the centre of gravity, increased*

*by the product of the mass of the body and the square of the distance of the two axes.*

It is also seen from this that of all the moments of inertia about parallel axes, that one is the least whose axis is a line of gravity of the body.

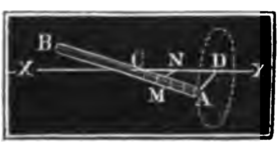
§ 218. It is necessary to know the moments of inertia of the principal geometric bodies, because they very often come into application in mechanical investigations. If these bodies are homogeneous, as in the following we will always suppose to be the case, the particles of the mass $M_1$, $M_2$, &c., are proportional to the corresponding particles of the volume $V_1$, $V_2$, &c., and hence the measure of the moment of inertia may be replaced by the sum of the particles of the volume, and the squares of their distances from the axis of revolution. In this sense the moments of inertia of lines and surfaces may also be found.

If the whole mass of a body be supposed to be collected into one point, its distance from the axis may be determined on the supposition, that the mass so concentrated possesses the same moment of inertia as if distributed over its space. This distance is called the *radius of gyration, or of inertia.* If $T$ be the moment of inertia, $M$ the mass, and $r$ the radius of gyration, we then have $Mr^2 = T$, and hence $r = \sqrt{\frac{T}{M}}$. We must bear in mind that this radius by no means gives a determinate point, but a circle only, within whose circumference the mass may be considered as arbitrarily distributed.

If into the formula $T_1 = T + Me^2$, we introduce $T = Mr^2$ and $T_1 = Mr_1^2$, we obtain $r_1^2 = r^2 + e^2$, *i. e.* the square of the radius of gyration referred to a given axis = the square of the radius of gyration referred to a parallel line of gravity, plus the square of the distance between the two axes.

§ 219. *The rod.*—The moment of inertia of a rod $AB$, Fig. 251, which turns about an axis $X\overline{X}$ through its middle point $C$, may be determined in the following manner. The cross section of the rod $= F$, its half length $CA = l$, and the angle which its axis includes with the axis of rotation $ACX = a$. If we divide half the length into $n$ parts,

FIG. 251.




we then obtain $n$ portions each of the contents $\frac{Fl}{n}$; the distances

of these portions from the middle $C$ are $\frac{l}{n}, \frac{2l}{n}, \frac{3l}{n}$, &c., hence their distances from the axis $X\overline{X}$ are $MN_{,} = \frac{l}{n}$ *sin.* $a \frac{2l}{n}$ *sin.* $a \frac{3l}{n}$ *sin.* $a$, &c., and these squares $= \left(\frac{l \sin. a}{n}\right)^2, 4\left(\frac{l \sin. a}{n}\right)^2, 9\left(\frac{l \sin. a}{n}\right)^2$ &c. By the multiplication of these with the contents of an element $\frac{Fl}{n}$, and by the addition of all the products, we have the moment of half the rod:

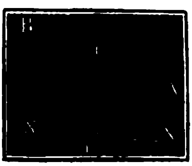
$$T = \frac{Fl}{n}\left[\left(\frac{l \sin. a}{n}\right)^2 + 4\left(\frac{l \sin. a}{n}\right)^2 + 9\left(\frac{l \sin. a}{n}\right)^2 + \ldots\right]$$

$= \frac{Fl^3 \sin. a^2}{n^3}(1^2 + 2^2 + 3^2 + \ldots + n^2)$, or, $1^2 + 2^2 + 3^2 + \ldots + n^2$ $= \frac{n^3}{3}$, $T = \frac{Fl^3 \sin. a^2}{3}$. But as the volume of half the rod $Fl$ is to be considered as the mass $M$, it follows finally that $T = \frac{1}{3} Ml^2 \sin. a^2$. The distance of the end of the rod from the axis $X\overline{X}$ is $AD = a = l \sin. a$, hence it follows more simply that $T = \frac{1}{3} Ma^2$, which formula is also to be applied to the whole rod, if $M$ be the mass of the whole. A mass $M_1$ at the extremity $A$ of the rod, has the moment of inertia $M_1 a^2$, hence if we make $M_1 = \frac{1}{3} M$, it has then the same moment of inertia as the rod. Whether, therefore, the mass be uniformly distributed over the rod, or its third part be collected at the extremity $A$, it comes to the same thing.

If we put $T = Mr^2$, we obtain $r^2 = \frac{1}{3}a^2$, and hence the radius of gyration of the rod: $r = a\sqrt{\frac{1}{3}} = 0{,}5773 \, . \, a$.

If the rod stands perpendicularly to the axis of rotation $a = l$, therefore, $T = \frac{1}{3} Ml^2$. If, lastly, the rod $AB$, Fig. 252, be not in the same plane with the axis of rotation, and if the shortest distance between the two axes be $CE = e$, we shall then have from § 217, the moment of inertia $T_1 = T + Me^2 = M(e^2 + \frac{1}{3}a^2)$.

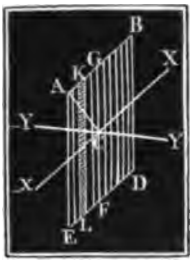
FIG. 252.



§ 220. *Rectangle and parallelipiped.*—The moment of inertia of a rectangular plate,

*ABDE*, Fig. 253, which turns about an axis $X\overline{X}$ passing through its middle $C$, and parallel to a side, is as for a rod $= \frac{1}{3} Ml^2$, but if the axis $Y\overline{Y}$ stand perpendicular to the plane, the moment of inertia is then determined from the former paragraph in the following manner: the half *AEFG* is divided by lines parallel to the side *AE* into strips of equal breadth, such as *KL*, the moments of these strips determine, and then added together. If the half length $FE = GA = l$, half the breadth $CF = CG = b$, and

FIG. 253.




the number of parts $= n$, the area of a part $= \frac{l}{n} \cdot 2b = \frac{2bl}{n}$.

The distances from $C$ of these strips are in the series $\frac{l}{n}, \frac{2l}{n}, \frac{3l}{n}$, &c.

therefore their squares $\left(\frac{l}{n}\right)^2$, $4\left(\frac{l}{n}\right)^2$, $9\left(\frac{l}{n}\right)^2$, &c., hence the moments of inertia are:

$$\frac{2bl}{n}\left[\left(\frac{l}{n}\right)^2 + \frac{b^2}{3}\right], \frac{2bl}{n}\left[4\left(\frac{l}{n}\right)^2 + \frac{b^2}{3}\right], \frac{2bl}{n}\left[9\left(\frac{l}{n}\right)^2 + \frac{b^2}{3}\right]$$

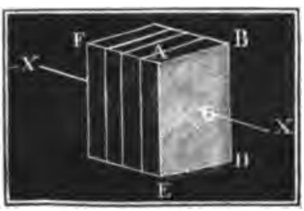
and the moment of inertia of half the plate:

$$T = \frac{2bl}{n}\left[\left(\frac{l}{n}\right)^2 (1+4+9+\ldots+n^2) + n \cdot \frac{b^2}{3}\right]$$

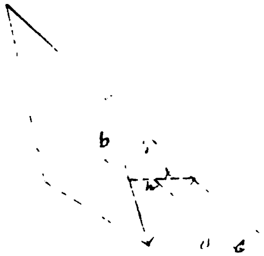
$$= \frac{2bl}{n}\left[\left(\frac{l}{n}\right)^2 \cdot \frac{n^3}{3} + \frac{nb^2}{3}\right] = \frac{2bl\,(l^2+b^2)}{3} = \tfrac{1}{3} M\,(l^2+b^2),$$

because $2bl$ must be considered as the mass of half the plate. As the semi-diagonal $CA = d = \sqrt{l^2 + b^2}$, we may put: $T = \frac{1}{3} Md^2$. If $M$ represent the whole mass, the formula holds good for the moment of inertia of the whole plate. Since further, a parallelipiped *BEF*, Fig. 254, may be decomposed by parallel planes into equal rectangular plates, the above formula is also applicable to this, if the axis of rotation pass through the middle points of any two opposite surfaces. It follows besides from this formula, that the

FIG. 254.

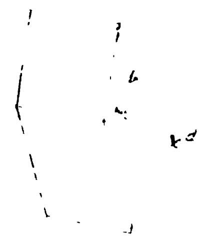


Triangular Prism.



$I_x$

Cone about the axis of Symmetry.



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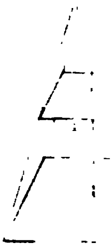

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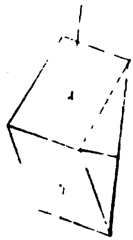


Segment of a Sphere

$\frac{bl - l x}{b^2}$

$= lb^2 - \frac{l b}{5} + \frac{1}{3} l\, b - \frac{1}{3} l^3 b + \frac{l^3 b}{5} - \frac{l^3 b}{5 \cdot 7}$

$= \frac{2}{3} l b \left( \ldots \right)$

§ 195

$\lambda \frac{r^4}{4} = M \frac{r^2}{4}$

axis perpendicular

$\int dy\, dy$

$= \frac{1}{4} r^4 2\pi = \frac{1}{2} \lambda r^2 \cdot r^2$

moment of inertia of the parallelipiped is equivalent to the moment of inertia of the third part of its mass applied at a corner $A$.

From the formula for the moment of inertia of a parallelipiped, that of a triangular prism may be also calculated. The diagonal plane $ADF$ divides the parallelipiped into two equal triangular prisms with rectangular and triangular bases $ABD$, Fig. 255; hence, for the rotation about an axis $\overline{XX}$, passing through the middle $C$ and $K$ of the hypothenuse, the moment of inertia $= \frac{1}{3} Md^2$. If now we make use of the proposition § 217, we obtain the moment of inertia about an axis $\overline{YY}$, passing through the centres of gravity $S$ and $S_1$ of the bases: $T = \frac{1}{3} Md^2 - M . \overline{CS}^2$

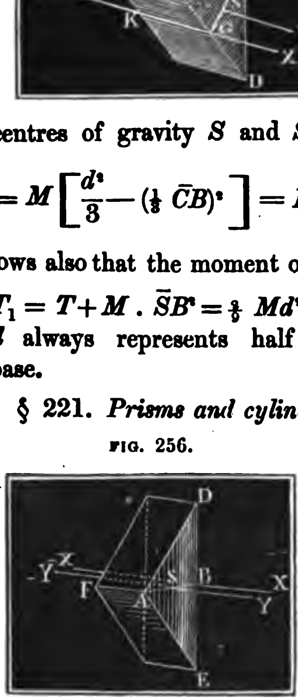
FIG. 255.

$$= M\left[\frac{d^2}{3} - (\tfrac{1}{3}\, \overline{CB})^2\right] = M\left[\frac{d^2}{3} - \left(\frac{d}{3}\right)^2\right] = \tfrac{2}{9}\, Md^2,$$ and it follows also that the moment of inertia about a side edge $BH$:

$T_1 = T + M . \overline{SB}^2 = \frac{2}{9} Md^2 + M(\frac{2}{3} d)^2 = \frac{6}{9} Md^2 = \frac{2}{3} Md^2$, where $d$ always represents half the hypothenuse of the triangular base.

§ 221. *Prisms and cylinders.*—For a prism $ADFE$, Fig. 256, with isoceles triangular bases the moment of inertia about an axis $\overline{XX}$, which connects the middle points of the bases, $T = \frac{2}{3} Md^2$, if $d$ represent half the side $AD$ of the surface of the base, because this surface may be decomposed by the line of the height $AB$ into two equal rectangular triangles. If now the height $AB$ of the isoceles triangular base $= h$, we have then for the moment of inertia of this prism about the axis $\overline{YY}$ passing through the centres of gravity of the base:

FIG. 256.



$$T = \tfrac{2}{3} M d^2 - M\left(\frac{h}{3}\right)^2 = M\,(\tfrac{2}{3}\, d^2 - \tfrac{1}{9}\, h^2) = \tfrac{1}{3}\, M\,(2\, d^2 - \tfrac{1}{3}\, h^2),$$

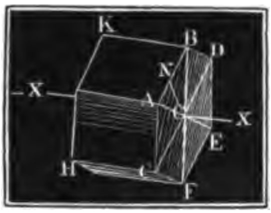
and finally, the moment of inertia about the edge passing through the points $A$ and $F$ of the bases:

$$T_1 = T + M\left(\tfrac{2}{3}h\right)^2 = M\left(\frac{2d^2}{3} - \frac{h^2}{9} + \frac{4h^2}{9}\right) = \tfrac{1}{3}M\left(2d^2 + h^2\right).$$

Hence the moment of inertia of a right and regular prism revolving about its geometric axis may be found. Let $h$ be the height $CA$, Fig. 257, of one of the supplementary triangles, $CA = CB = 2d = r$ the radius of the base or of a supplementary triangle, and $M$ the mass of the entire prism. We have then by the last formula:

$$T = \tfrac{1}{3}M\left(\frac{r^2}{2} + h^2\right).$$

FIG. 257.




The regular prism becomes a cylinder when $h = r$, hence the moment of inertia of a right cylinder about its geometric axis is:

$$T = \tfrac{1}{3}M\left(\frac{r^2}{2} + r^2\right) = \tfrac{1}{2}Mr^2.$$

The moment of inertia of a cylinder is, therefore, equivalent to the moment of inertia of half the mass of the cylinder collected at its circumference, or equivalent to the moment of inertia of the entire mass, at the distance $r\sqrt{\tfrac{1}{2}} = 0{,}7071 \,.\, r$.

FIG. 258.

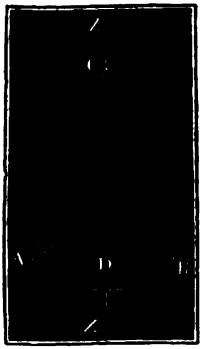

If the cylinder $ABDE$ be hollow, Fig. 258, the moment of inertia of the hollow space must be subtracted from that of the solid cylinder. If the outer radius $CA = r_1$, and the inner $CG = r_2$, we then have from what has preceded the moment of inertia of the hollow cylinder:

$$T = \tfrac{1}{2}\left(M_1 r_1^2 - M_2 r_2^2\right) = \tfrac{1}{2}\pi\left(r_1^2 \,.\, r_1^2 - r_2^2 \,.\, r_2^2\right) = \tfrac{1}{2}\pi\left(r_1^4 - r_2^4\right)$$
$$= \tfrac{1}{2}\pi\left(r_1^2 - r_2^2\right)\left(r_1^2 + r_2^2\right) = \tfrac{1}{2}M\left(r_1^2 + r_2^2\right),$$

because the volume considered as the mass $= \pi.\left(r_1^2 - r_2^2\right)$. If $r$ be the mean radius $\frac{r_1 + r_2}{2}$, and $b$ the breadth of the annular surface, we then have $T = M\left(r^2 + \frac{b^2}{4}\right)$.

§ 222. *Cone and sphere.*—The moment of inertia of a right cone, as well as that of a sphere, may be calculated from the formula for the moment of inertia of a cylinder. Let $ACB$, Fig. 259,

FIG. 259.



be a cone revolving about its geometric axis, $DA=DB=r$ the radius of its base, and $CD=h$ its height coinciding with the axis. If we make $n$ slices parallel to the base at equal distances, we then obtain thin discs of the radii $\frac{r}{n}, 2\frac{r}{n}, 3\frac{r}{n} \ldots n\frac{r}{n}$, and of the common height $\frac{h}{n}$. The half volumes of these discs are $\pi\left(\frac{r}{n}\right)^2 \cdot \frac{h}{2n}, \pi\left(\frac{2r}{n}\right)^2 \cdot \frac{h}{2n}$, $\pi \cdot \left(\frac{3r}{n}\right)^2 \cdot \frac{h}{2n}$, &c., and hence their moments of inertia:

$$\pi\left(\frac{r}{n}\right)^4 \cdot \frac{h}{2n}, \pi\left(\frac{2r}{n}\right)^4 \frac{h}{2n}, \pi\left(\frac{3r}{n}\right)^4 \frac{h}{2n}, \text{ \&c.};$$

the sum of these values gives, finally, the moment of inertia of the entire cone:

$$T=\frac{\pi r^4 h}{2n^5}(1^4+2^4+3^4+\ldots+n^4),$$

and as $1^4+2^4+3^4+\ldots+n^4 = \frac{n^5}{5}$, we have

$$T=\frac{\pi r^4 h}{10}=\frac{3}{10}\cdot\frac{\pi r^2 h}{3}\cdot r^2=\frac{3}{10}Mr^2.$$

FIG. 260.

For the right pyramid $ACE$, Fig. 260, with rectangular base, under the same circumstances $T=\frac{1}{5}Md^2$, if $d$ represent the semi-diagonal $DA$ of the base. Also by subtraction of the two moments of inertia, the moment of inertia of a right truncated cone with the radii $r_1$ and $r_2$ and the heights $h_1$ and $h_2$ may be obtained:

$$T=\frac{\pi}{10}(r_1{}^4h_1-r_2{}^4h_2)=\frac{\pi h_1}{10r_1}(r_1{}^5-r_2{}^5),$$

or, since the mass

$$M=\frac{\pi}{3}(r_1^2h_1-r_2^2h_2)=\frac{\pi h_1}{3r_1}(r_1^3-r_2^3),$$

$$T=\tfrac{3}{10}M\left(\frac{r_1^5-r_2^5}{r_1^3-r_2^3}\right).$$

In a similar manner, we find the moment of inertia of a sphere, revolving about one of its diameters $DE=2r$. Let us divide the hemisphere $ADB$, Fig. 261, by sections parallel to the base $ACB$, into $n$ equally thick circular slices, as $GKH$, &c., and determine their moments. The square $\overline{GK}^2$ of the radius of any such slice is:

FIG. 261.


$$=\overline{CG}^2-\overline{CK}^2=r^2-\overline{CK}^2,$$

hence its moment of inertia

$$=\tfrac{1}{2}\pi\cdot\frac{r}{n}(r^2-\overline{CK}^2)^2,$$

$$=\frac{\pi r}{2n}(r^4-2r^2\overline{CK}^2+\overline{CK}^4).$$

Let us put in succession for $CK\ \frac{r}{n}, \frac{2r}{n}, \frac{3r}{n}$, &c., to $\frac{nr}{n}$, and add the results, we shall then have the moment of inertia of the sphere:

$$T=\frac{\pi}{2n}\left[n\cdot r^4-2r^2\left(\frac{r}{n}\right)^2(1^2+2^2+3^2+..+n^2)+\right.$$

$$\left.\left(\frac{r}{n}\right)^4(1^4+2^4+3^4+..+n^4)\right]=\frac{\pi r}{2n}\left[nr^4-\frac{2r^4}{n^2}\cdot\frac{n^3}{3}+\right.$$

$$\left.\left(\frac{r}{n}\right)^4\cdot\frac{n^5}{5}\right]=\frac{\pi r^5}{2}(1-\tfrac{2}{3}+\tfrac{1}{5})=\frac{4\pi r^5}{15}.$$

Now the solid contents of a hemisphere $M=\frac{2}{3}\pi r^3$, hence we may put:

$$T=\tfrac{2}{5}\cdot\tfrac{2}{3}\pi r^3\cdot r^2=\tfrac{2}{5}Mr^2,$$

and if we take $M$ for the whole sphere, the formula will hold good for the case.

The formula $T=\frac{2}{5}Mr^2$ is true also for a spheroid whose equatorial radius $=r$ (§ 117).

If the sphere revolves about another axis at the distance $e$ from its centre, the moment of inertia is then

$$T=M(e^2+\tfrac{2}{5}r^2).$$

The radius of gyration $=r\sqrt{\tfrac{2}{5}}=0{,}6324\,.\,r$; two fifths of

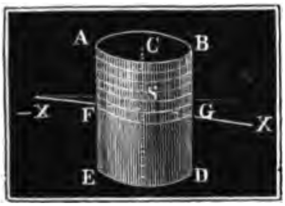
the mass of the sphere at a distance from the axis of rotation equal to the radius of the sphere, have the same moment of inertia as the whole sphere.

§ 223. The moment of inertia of a circular disc *ABDE*, Fig. 262, revolving about its diameter *BE*, is, as for the moment of flexure of a cylinder, (§ 195) $= \frac{\pi r^4}{4} = \frac{Mr^2}{4}$, consequently the radius of gyration $= r\sqrt{\frac{1}{4}} = \frac{1}{2}r$, *i. e.* half the radius of the circle.

FIG. 262.


From this we may now find the moment of inertia of a cylinder *ABDE*, Fig. 263, which revolves about a diameter *FG*, passing through the centre of gravity *S*. If $l$ be half the height and $r$ the radius of the cylinder, we then have the volume of half the cylinder $= \pi r^2 l$, and if we make equi-distant sections parallel to the base, we decompose this body into $n$ equal parts, each of which $= \frac{\pi r^2 l}{n}$, the first is distant $\frac{l}{n}$ from the centre of gravity *S*, the second $\frac{2l}{n}$, the third $\frac{3l}{n}$, &c.

FIG. 263.




In virtue of the formula § 217, the moments of inertia of these circular slices are:

$$\frac{\pi r^2 l}{n}\left[\frac{1}{4}r^2 + \left(\frac{l}{n}\right)^2\right], \frac{\pi r^2 l}{n}\left[\frac{1}{4}r^2 + \left(\frac{2l}{n}\right)^2\right],$$

$$\frac{\pi r^2 l}{n}\left[\frac{1}{4}r^2 + \left(\frac{3l}{n}\right)^2\right],$$

&c., their sum gives the moment of inertia of half the cylinder:

$$T = \frac{\pi r^2 l}{n}\left[\frac{nr^2}{4} + \left(\frac{l}{n}\right)^2 (1^2 + 2^2 + 3^2 + \ldots + n^2)\right] =$$

$$\pi r^2 l\left(\frac{r^2}{4} + \frac{l^2}{n^3}\cdot\frac{n^3}{3}\right) = M\left(\frac{r^2}{4} + \frac{l^2}{3}\right)$$

which holds good likewise for the whole cylinder, if $M$ represent its mass.

We find in like manner for the right cone *ABD*, Fig. 264,

FIG. 264.

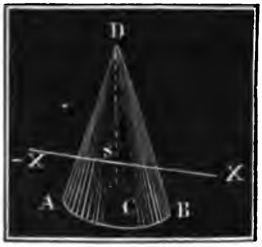



FIG. 265.


whose axis of revolution passes through its centre of gravity, and is perpendicular to the geometrical axis $CD$

$$T = \tfrac{3}{20} M \left(r^2 + \frac{h^2}{4}\right).$$

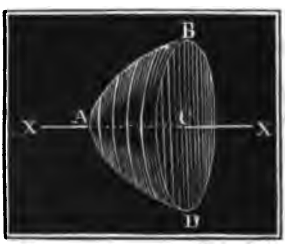
For a plate $ABC$, Fig. 265, in the form of a rectangular triangle, the moment of inertia about an axis passing through the centre of gravity $S$, and parallel to the cathetus $AC$, is according to § 193:

$$T = \frac{bh^3}{36} = \frac{bh}{2} \cdot \frac{h^2}{18} = \tfrac{1}{18} Mh^2,$$

if the breadth $b$ parallel to the axis of revolution, and $h$ the height perpendicular to it be given. This formula holds good even for an oblique angled triangle, if the axis runs parallel to the base, and $h$ represents the height of the triangle. From this the moment of inertia of a triangular prism $ADEF$, Fig. 266, may be found, if the axis of revolution $X\overline{X}$ passes through its centre of gravity $S$, and is parallel to the side $DE$ of the surface of the base, it follows from the same method as that adopted for the cylinder, that $T = M\left(\tfrac{1}{18} h^2 + \frac{l^2}{3}\right)$, where $l$ represents half the length of the prism.

FIG. 266.


FIG. 267.




§ 223. *Segments.*—The moment of inertia of a paraboloid of revolution, $BAD$, Fig. 267, which turns about its axis of rotation $AC$, is determined in a similar manner to that of a sphere. Let the radius of the base $CB = CD = a$, the height $CA = h$, and the body consist of

$n$ slices, each of the height $\frac{h}{n}$, we have then the contents of these

$$=\frac{h}{n}\cdot\pi\cdot\frac{1}{n}a^2,\ \frac{h}{n}\pi\cdot\frac{2}{n}a^2,\ \frac{h}{n}\cdot\pi\frac{3}{n}a^2,\ \&c.,$$

because the squares of the radii are as the heights. From this the moments of inertia are given

$$=\frac{h}{n}\cdot\frac{\pi}{2}\cdot\frac{a^4}{n^2}\cdot\frac{h}{n}\cdot\frac{\pi}{2}\cdot\frac{4\,a^4}{n^2},\ \frac{h}{n}\frac{\pi}{2}\cdot\frac{9\,a^4}{n^2},\ \&c.,$$

and hence, finally, it follows that the moment of inertia of the whole paraboloid is

$$T=\frac{\pi a^4 h}{2n^3}\,(1^2+2^2+3^2+\,..\,+n^2)=\frac{\pi a^4 h}{2n^3}\cdot\frac{n^3}{3}=\frac{\pi a^4 h}{6}$$
$$=\frac{\pi a^2 h}{2}\cdot\frac{a^2}{3}=\tfrac{1}{3}Ma^2,$$

because the volume of this body is $M=\frac{\pi a^2 h}{2}$.

The same formula is applicable to a small segment of a sphere, but if the height $h$ is not very small compared with $a$, we have to put the moment of inertia of slices

$$=\frac{\pi h}{2n}\cdot a^4=\frac{\pi h}{2n}\cdot h^2\,(2\,r-h)^2=\frac{\pi h}{2n}\cdot(4\,r^2\,h^2-4\,rh^3+h^4),$$

where $r$ represents the radius of the sphere. If now we take successively for $h$ the values $\frac{h}{n}, \frac{2h}{n}, \frac{3h}{n}$, &c. we then obtain for the moment of inertia of a segment of a sphere

$$T=\frac{\pi h}{2n}\left[4\,r^2\left(\frac{h}{n}\right)^2\cdot\frac{n^3}{3}-4r\left(\frac{h}{n}\right)^3\cdot\frac{n^4}{4}+\left(\frac{h}{n}\right)^4\cdot\frac{n^5}{5}\right]$$
$$=\frac{\pi h^3}{30}\,(20\,r^2-15\,rh+3\,h^2).$$

The contents of the segment are $M=\pi h^2\,(r-\frac{1}{3}\,h)$, hence

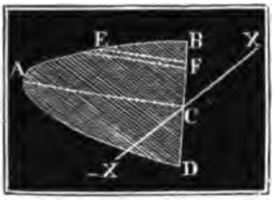
$$T=\pi h^2\,(r-\tfrac{1}{3}\,h)\cdot\frac{2h}{3}\left(r-\tfrac{5}{12}\,h+\tfrac{1}{60}\cdot\frac{h^2}{r-\frac{1}{3}\,h}\right)$$
$$=\tfrac{2}{3}\,Mh\left(r-\tfrac{5}{12}\,h+\tfrac{1}{60}\cdot\frac{h^2}{r-\frac{1}{3}\,h}\right).$$

Generally $T=\frac{2}{3}\,Mh\,(r-\frac{5}{12}\,h)$ is sufficiently correct. This formula finds its application in pendulum-bobs.

The moment of inertia of the surface of a parabola $ABD$,

Fig. 268, which revolves about an axis $X\overline{X}$, passing through the middle $C$ of the chord $BD$, is found if the surface be decomposed into equally broad stripes, such as $EF$, and their moments added together. Let $AC = l$ be the length, and $CB = b$ half the breadth of the surface, $CF = x$ the absciss, and $EF = y$ the ordinate or length of an element. Its moment of inertia is then $= \frac{b}{n} y \left(x^2 + \frac{y^2}{3}\right)$; but as $\frac{x^2}{b^2} = \frac{l-y}{l}$, therefore $y = l\left(1 - \frac{x^2}{b^2}\right)$, it follows that this moment $= \frac{b}{n}\left[lx^2\left(1 - \frac{x^2}{b^2}\right) + l^3\left(1 - \frac{x^2}{b^2}\right)^3\right]$. If now $x$ be successively put $= \frac{b}{n}, \frac{2b}{n}, \frac{3b}{n}$, &c., and the results added, we obtain the moment of inertia of half the surface of the parabola:

FIG. 268.




$$T = bl\left[\frac{l^2}{3n}\left(n - n + \tfrac{3}{5} n - \frac{n}{7}\right) + \frac{b^2}{3} - \frac{b^2}{5}\right]$$

$$= bl\left(\frac{16\, l^2}{3\,.\,35} + \frac{2\, b^2}{3\,.\,5}\right) = \tfrac{2}{3}\, bl\, \left(\tfrac{8}{35}\, l^2 + \tfrac{1}{5}\, b^2\right) = \tfrac{1}{5}\, M\, \left(\tfrac{8}{7}\, l^2 + b^2\right),$$

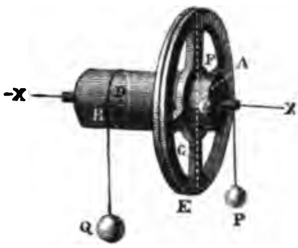
because the surface of the parabola is $M = \frac{2}{3}\, bl$.

This formula, which holds good for the entire surface of the parabola, is also applicable to a prism having a parabolic surface for the base, as in vibrating beams.

§ 225. *Wheel and axle.*—The theory of the moment of inertia finds its most frequent application in machines and instruments, because in these rotatory motions about a fixed axis are those which generally present themselves. Many applications of this doctrine will be met with in the sequel, hence it will suffice to treat of only a few simple cases for the present.

If two weights $P$ and $Q$, act on a wheel and axle $ACDB$, Fig. 269, with the arms $CA = a$ and $DB = b$ through the medium of a perfectly flexible string, and if the radius of the gudgeons be so small that their friction may be neglected, it will remain in equilibrium if the statical moments $P\,.\,CA$ and $Q\,.\,DB$ are equal, and

FIG. 269.




therefore $Pa = Qb$. But if the moment of the weight $P$ is greater than that of $Q$, therefore, $Pa > Qb$, $P$ will descend and $Q$ ascend; if $Pa < Qb$, $P$ will ascend and $Q$ descend. Let us now examine the conditions of motion in one of the latter cases. Let us suppose that $Pa > Qb$. The force corresponding to the weight $Q$ and acting at the arm $b$ generates at the arm $a$ a force $\frac{Qb}{a}$, which acts opposite to the force corresponding to the weight $P$, and hence there is a residuary moving force $P - \frac{Qb}{a}$ acting at $A$. The mass $\frac{Q}{g}$ is reduced by its transference from the distance $b$ to that of $a$ to $\frac{Qb^2}{ga^2}$, hence the mass moved by $P - \frac{Q\,b}{a}$ is $M = \left(P + \frac{Qb^2}{a^2}\right) \div g$, or, if the moment of inertia of the wheel and axle $= \frac{Gy^2}{g}$, and, therefore, its inert mass reduced to $A = \frac{Gy^2}{ga^2}$, we have more exactly:

$$M = \left(P + \frac{Qb^2}{a^2} + \frac{Gy^2}{a^2}\right) \div g = (Pa^2 + Qb^2 + Gy^2) \div ga^2.$$

From thence it follows that the accelerated motion of the weight $P$, together with that of the circumference of the wheel, namely

$$p = \frac{\text{moving force}}{\text{mass}} = \frac{P - \frac{Qb}{a}}{Pa^2 + Qb^2 + Gy^2} \cdot ga^2$$

$$= \frac{Pa - Qb}{Pa^2 + Qb^2 + Gy^2} \cdot ga;$$

on the other hand, the accelerated motion of the ascending weight $Q$, or of the circumference of the ~~wheel~~ is; axle

$$q = \frac{b}{a} p = \frac{Pa - Qb}{Pa^2 + Qb^2 + Gy^2} \cdot gb.$$

The tension of the string by $P$ is $S = P - \frac{Pp}{g} = P\left(1 - \frac{p}{g}\right)$ (§ 73), that of the string by $Q$: $T = Q + \frac{Qq}{g} = Q\left(1 + \frac{q}{g}\right)$, hence the pressure on the gudgeon is:

$$S+T=P+Q-\frac{Pp}{g}+\frac{Qq}{g}=P+Q-\frac{(Pa-Qb)^2}{Pa^2+Qb^2+Gy^2};$$

the pressure, therefore, on the gudgeons for a revolving wheel and axle is less than for one in a state of equilibrium. Lastly, from the accelerating forces $p$ and $q$, the rest of the relations of motion may be found, after $t$ seconds the velocity of $P$ is $v=pt$, of $Q$: $v_1=qt$, and the space described by $P$: $s=\frac{1}{2}pt^2$, by $Q$: $s_1=\frac{1}{2}qt^2$.

*Example.* Let the weight $P$ at the wheel be $= 60$ lbs., that at the axle $Q = 160$ lbs, the arm of the first $CA = a = 20$ inches, that of the second $DB = b = 6$ inches; further, let the axle consist of a solid cylinder of 10 lbs. weight, and the wheel of two iron rings and four arms, the rings of 40 and 12 lbs., the arms, together, of 15 lbs. weight; lastly, let the radii of the greater ring $AE = 20$ and 19 inches, that of the less $FG = 8$ and 6 inches. Required, the conditions of motion of this machine. The moving force at the circumference of the wheel is:

$$P-\frac{b}{a}Q = 60-\frac{6}{20}\,.\,160 = 60-48 = 12 \text{ lbs.},$$

the moment of inertia of the machine, neglecting the masses of the gudgeons and the strings, is equivalent to the moment of inertia of the axle $= \frac{Wb^2}{2} = \frac{10\,.\,6^2}{2}$ $= 180$, plus the moment of the smaller ring $= \frac{R_1\,(r_1^2+r_2^2)}{2} = \frac{12\,.\,(8^2+6^2)}{2}$ $= 600$, plus the moment of the larger ring $= \frac{40\,.\,(20^2+19^2)}{2} = 15220$, plus the moment of the arms, approximately $= \frac{A\,(\rho_1^3-\rho_2^3)}{3\,(\rho_1-\rho_2)} = \frac{A\,(\rho_1^2+\rho_1\rho_2+\rho_2^2)}{3}$ $= \frac{15\,.\,(19^2+19\,.\,8+8^2)}{3} = 2885$; hence, collectively, $Gy^2 = 180+600+15220+2885$ $= 18885$, or for foot measure $= \frac{18885}{144} = 131,14$. The collective mass, reduced to the circumference of the wheel is:

$$=\left(P+\frac{Qb^2+Gy^2}{a^2}\right):g = \left[60+160\left(\frac{6}{20}\right)^2+\frac{18885}{20^2}\right]:g$$

$$=\left(60+160\,.\,0,09+\frac{18885}{400}\right).\,0,031 = 121,61\times 0,031 = 377 \text{ lbs.}$$

Accordingly, the accelerated motion of the weight $P$, together with that of the circumference of the wheel, is:

$p = \dfrac{P-\frac{b}{a}Q}{\frac{Pa^2+Qb^2+Gy^2}{a^2}}\,.\,g = \frac{12}{3,77} = 3,183$ feet; on the other hand, that of $Q$: $q = \frac{b}{a}p = \frac{6}{20}\,.\,3,183 = 0,954$ feet; further, the tension of the string by $P$ is $=\left(1-\frac{p}{g}\right)P = \left(1-\frac{3,133}{32,2}\right).\,60 = 54,07$ lbs.; that by $Q$, on the other hand, $Q = \left(1+\frac{q}{g}\right).\,Q = (1+0,925\,.\,0,032)\,.\,160 = 1,030\,.\,160 = 164,8$ lbs.; and consequently the pressure on the gudgeons $S+T = 54,06+164,80 = 218,86$ lbs., or

inclusive of the weight of the machine = 218,86 + 77 = 295,86 lbs. After 10 seconds, $P$ has acquired the velocity $pt$ = 3,084 . 10 = 30,84 feet, and described the space $s = \frac{vt}{2}$ = 30,84 . 5 = 154,2 feet, and $Q$ has ascended a height $\frac{b}{a} s$ = 0,3 . 154,2 = 46,26 feet.

§ 226. The weight $P$ which communicates to the weight $Q$ the accelerated motion $q = \frac{Pab - Qb^2}{Pa^2 + Qb^2 + Gy^2} \cdot g$, may also be replaced by another weight $P_1$, without changing the acceleration of the motion $Q$, if it act at the arm $a_1$ for which:

$$\frac{P_1 a_1 - Qb}{P_1 a_1^2 + Qb^2 + Gy^2} = \frac{Pa - Qb}{Pa^2 + Qb^2 + Gy^2}.$$

The magnitude $\frac{Pa^2 + Qb^2 + Gy^2}{Pa - Qb}$, represented by $k$, and we obtain $a_1^2 - ka_1 = -\frac{Qb\,(b+k) + Gy^2}{P_1}$, and the arm in question:

$$a_1 = \tfrac{1}{2} k \pm \sqrt{\left(\frac{k}{2}\right)^2 - \frac{Qb\,(b+k) + Gy^2}{P_1}}.$$

We may also find by help of the differential calculus, that the motion of $Q$ is most accelerated by the weight $P$, when the arm of the latter corresponds to the equation $Pa^2 - 2\,Qab = Qb^2 + Gy^2$ therefore,

$$a = \frac{bQ}{P} + \sqrt{\left(\frac{bQ}{P}\right)^2 + \frac{Qb^2 + Gy^2}{P}}.$$

The formula found above assumes a complicated form if the friction of the gudgeons and the rigidity of the cord are taken into account. If we represent the statical moments of both resistances by $Fr$, we must then substitute for the moving force $P - \frac{b}{a} Q$, the value $P - \frac{Qb + Fr}{a}$, whence the acceleration of $Q$ comes out

$$q = \frac{(Pa - Fr)\,b - Qb^2}{Pa^2 + Qb^2 + Gy^2} \cdot g \text{ and}$$

$$a = \frac{Qb + Fr}{P} + \sqrt{\left(\frac{Qb + Fr}{P}\right)^2 + \frac{Qb^2 + Gy^2}{P}}.$$

*Examples.*—1. The weights $P$ = 30 lbs. $Q$ = 80 lbs. act at the arms $a$ = 2 feet, and $b = \frac{1}{2}$ foot of a wheel and axle, and their moments of inertia $Gy^2$ amount to 60 lbs.; then the accelerated motion of the ascending weight $Q$ is:

$$q = \frac{30 \,.\, 2 \,.\, \frac{1}{2} - 80 \,.\, (\frac{1}{2})^2}{30 \,.\, 2^2 + 80 \,.\, (\frac{1}{2})^2 + 60} \,.\, g = \frac{30 - 20}{120 + 20 + 60} \,.\, 32 \,.\, 2 = \frac{312{,}5}{200} = 1{,}6 \,.\, 15 \text{ feet.}$$

But if a weight $P_1 = 45$ lbs. generates the same acceleration in the motion of $Q$, the arm of $P_1$ is then:

$$a_1 = \frac{k}{2} \pm \sqrt{\left(\frac{k}{2}\right)^2 - \frac{80 \,.\, \frac{1}{2} (\frac{1}{2} + k) + 60}{45}}, \text{ or as } k = \frac{200}{60 - 40} = 10, \; a_1 \text{ is}$$

$$= 5 \pm \sqrt{25 - \frac{32}{3}} = 5 \pm \tfrac{1}{3} \sqrt{11{,}358} = 5 \pm 3{,}786 = 8{,}786 \text{ feet, or } 1{,}214 \text{ feet,}$$

—2. The accelerated motion of $Q$ comes out greatest if the arm of the force or radius of the wheel amount to:

$$a = \frac{\frac{1}{2} \,.\, 80}{30} + \sqrt{\left(\frac{40}{30}\right)^2 + \frac{20 + 60}{30}} = \frac{4}{3} + \sqrt{\frac{16}{9} + \frac{24}{9}} = \frac{4 + \sqrt{40}}{3} = 3{,}4415 \text{ feet,}$$

and $q$ is $= \left(\frac{30 \,.\, 1{,}7207 - 20}{30 \,.\, (3{,}4415)^2 + 80}\right) g = \frac{31{,}621}{435{,}32} \,.\, g = 2{,}339$ feet.—3. The statical moment of the friction, together with the rigidity of the string, is $Fr = 8$; then, instead of $Qb$, we must put $Qb + Fr = 40 + 8 = 48$; whence it follows that:

$$a = \frac{48}{30} + \sqrt{\left(\frac{40}{30}\right)^2 + \frac{8}{3}} = 1{,}6 + \sqrt{5{,}227} = 3{,}886, \text{ and the correspondent maximum}$$

accelerating force $q = \frac{30 \,.\, 1{,}943 - 8 \,.\, \frac{1}{2} - 20}{30 \,.\, (3{,}886)^2 + 80} \,.\, g = \frac{34{,}29}{533} \,.\, 32 \,.\, 2 = 2{,}071$ feet.

§ 227. *Attwood's machine.*—The formulæ found in § 225 for the wheel and axle hold good also for the simple fixed pulley, for if $b = a$, the wheel and axle becomes either a pulley or an axle. Retaining the other denominations of the paragraph mentioned, we have then for the accelerating motion with which $P$ descends and $Q$ ascends: $p = q = \frac{(P - Q)\, a^2}{(P + Q) a^2 + Gy^2} \,.\, g$, or having regard to friction:

$$p = q = \frac{(P - Q)\, a^2 - Far}{(P + Q)\, a^2 + Gy^2} \,.\, g.$$

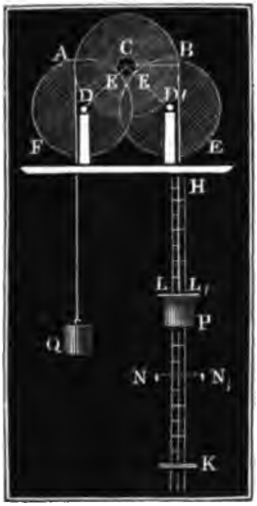
In order to diminish the friction of the gudgeons, the gudgeon $C$ of the pulley $AB$, Fig. 270, is placed upon the friction wheels $DEF$ and $D_1E_1F_1$. The moments of inertia of these are $G_1 y_1^2$, and their radii $DE = D_1E_1 = a_1$, we then have to put:

$$p = q = \frac{(P - Q)\, a^2 - Far}{(P + Q)\, a^2 + Gy^2 + G_1 \dfrac{y_1^2 r^2}{a_1^2}} \,.\, g,$$

because the inert masses of these wheels reduced to the circumference of the friction wheels, or to the gudgeon of the wheel, $= \frac{G_1 y_1^2}{a_1^2}$. By inversion we obtain the accelerating force of gravity:

$$g = \frac{(P+Q)\,a^2 + Gy^2 + G_1\,\frac{y_1^{\,2} r^2}{a_1^{\,2}}}{(P-Q)\,a^2 - Far}\,.\,p.$$

FIG. 270.




For a small difference of the two weights $P-Q$ the accelerating force $p$ comes out small, hence the motion goes on slowly, and if the resistance opposed to the weight by the air be inconsiderable, with the assistance of experiments upon the descent of weights in such an arrangement, the accelerating force of gravity may be measured with tolerable accuracy, which by a body falling freely it is impossible to do. Experiments of this kind were first instituted by Attwood,* whence this arrangement is known by the name of Attwood's machine. To determine the spaces fallen through there is a scale $HK$ along which the weight $P$ descends. From the space fallen through $s$, and the corresponding time $t$, it follows of course that $p = \frac{2\,.\,s}{t^2}$; if however during the descent the moving force be removed, while an equal weight $LL_1$ forming a hollow ring is taken up by a fixed narrower ring $NN_1$, the remaining part of the space $s_1$ will be described with a uniform motion, and having the time observed by a good clock, the velocity will be given $v = \frac{s_1}{t_1}$, and the accelerating force $p = \frac{v}{t} = \frac{s_1}{tt_1}$. If $t_1 = t = 1$, experiment gives directly $p = s_1$, and by putting the value found in the above formula, it will give the accelerating force of gravity.

FIG. 271.


§ 228. The acceleration of the motion of the weights $P$ and $Q$, which are suspended to a system consisting of a fixed pulley $AB$, and a moveable pulley $EG$ is given in the following manner. Let the weights of the pulleys $AB$ and $BC = G$ and $G_1$, their

* Attwood's Treatise on Rectilinear and Rotary Motion.

moments of inertia $Gy^2$ and $G_1 y_1^2$, and the radii $CA = a$ and $DE = a_1$, therefore the masses reduced to the circumference of the wheel

$$M = \frac{G}{g} \cdot \frac{y^2}{a^2}, \text{ and } M_1 = \frac{G_1}{g} \cdot \frac{y_1^2}{a_1^2}.$$

If the weight descend a certain space $s$, $Q + G_1$ will then ascend by $\frac{1}{2} s$ (§ 151), hence the mechanical effect produced will be $Ps - (Q + G_1)\frac{s}{2}$; if by this descent $P$ acquires the velocity $v$, then $Q + G_1$ takes the velocity $\frac{v}{2}$, and the pulley $AB$ at its circumference the velocity $v$, and the pulley $EG$, since in rolling motion the progressive and rotatory motion are equal to each other, at its circumference the velocity $\frac{v}{2}$. The sum of the *vis viva* corresponding to these masses and their velocities is $\frac{P}{g} \cdot v^2 + \frac{Q + G_1}{g} \cdot \left(\frac{v}{2}\right)^2 + \frac{Gy^2}{a^2} \cdot v^2 + \frac{G_1 y_1}{a_1^2} \cdot \left(\frac{v}{2}\right)^2$, and if their halves be equated to the mechanical effect expended, we shall obtain the equation:

$$\left(P - \frac{(Q + G_1)}{2}\right) s = \left(P + \frac{Q + G_1}{4} + \frac{Gy^2}{a^2} + \frac{G_1 y_1^2}{4 a_1^2}\right) \frac{v^2}{2g}.$$

Hence, the velocity corresponding to the space $s$ described by $P$:

$$v = \sqrt{\frac{2gs\left(P - \frac{Q + G_1}{2}\right)}{P + \frac{Q + G_1}{4} + \frac{Gy^2}{a^2} + \frac{G_1 y_1^2}{4 a_1^2}}}$$

For the acceleration $ps = \frac{v^2}{2}$; hence, here

$$p = \left(\frac{P - \frac{Q + G_1}{2}}{P + \frac{Q + G_1}{4} + \frac{Gy^2}{a^2} + \frac{G_1 y_1^2}{4 a_1^2}}\right) g.$$

The acceleration of $Q + G_1$ is $= \frac{p}{2}$, and the rotatory acceleration ~~rotatory~~ of $G_1$ is equal to it. moving ... inertia

The tension of the string $BE$, connecting both pulleys is $S = P - \left(P + \frac{Gy^2}{a^2}\right)\frac{p}{g}$, because the force $\left(P + \frac{Gy^2}{a^2}\right)\frac{p}{g}$ is

# Wheel & Axle.

[illegible] acceleration of [illegible] distance.

$Q$ applied at distance $a$ & $b$ [illegible]

[illegible] and moment [illegible] referred to

[illegible] $\therefore \; p = \dfrac{Pa - Qb}{Pa^2 + Qb^2 + G\gamma^2}\, g$

$\therefore \; p_a = \dfrac{Pa - Qb}{Pa^2 + Qb^2 + G\gamma^2}\, ga$

$\quad p_b = \dfrac{Pa - Qb}{Pa^2 + Qb^2 + G\gamma^2}\, gb$

Tension $S = P - \dfrac{Pf}{g}$ i.e. $\dfrac{S}{P} = 1 - \dfrac{f}{g}$

$\quad T = Q + \dfrac{Qf}{g}$ i.e. $\dfrac{T}{Q} = 1 + \dfrac{f}{g}$

$S + T = P + Q - \dfrac{Pa - Qb}{Pa^2 + Qb^2 + G\gamma^2}$ [illegible]

[illegible] on Gudgeons $S + T + G$

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226 Considering friction $p_1 = \dfrac{Pa - Qb - Fr}{Pa^2 + Qb^2 + G\gamma^2}\, g$

$\therefore \; p_1 b = \dfrac{(Pa - Fr)b - Qb^2}{Pa^2 + Qb^2 + G\gamma^2}$ differentiate w.r.t. [illegible]

[illegible]

[illegible] $a^2 - 2Fr + Qb$ [illegible]

$\therefore \; a = -\dfrac{Qb}{P} \cdot \dfrac{Fr}{P} \pm$ [illegible] $\dfrac{Fr}{P}$ [illegible] $\dfrac{Qb + G\gamma^2}{P}$

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Atwood's Machine.

As before let $p_1 = \frac{p}{a}$   b = a.

$\therefore p_1 = \frac{(P-Q)a - Fr}{(P+Q)a^2 + G_1 y^2} g$    $G_1 y_1 = M' \ldots$

of $DEF \frac{G_1 y_1^2}{j a_1^2}$ = mass reduced to circumference

$G_1 \ldots r' =$ " " " unit distance.

$\therefore p_1 = \frac{(P-Q)a - Fr}{(P+Q)a^2 + G_1 y^2 + G_1 \frac{\ldots}{a^2}} g$

by increment we obtain g

§229   $\frac{P}{p} = M$

I   $\frac{P-S}{p} = \frac{P + G\frac{y^2}{a^2}}{g}$

II   $\frac{S}{p-q} = G_1 \frac{\ldots}{a_1^2}$

III   $\frac{S-(Q+G_1)}{q} = \frac{G_1+Q}{g}$

from I   $p = \frac{P-S}{P+G\cdot\frac{y^2}{a^2}} g$ ; II $p - q = \frac{S a_1^2}{G_1 y_1^2} g$

III   $q = \frac{S-(Q+G_1)}{Q+G_1} g$   from which

$\frac{S a_1^2}{G_1 y_1^2} = \frac{Pa^2 - Sa^2}{Pa^2 + Gy^2} - \frac{S-(Q+G_1)}{Q+G_1}$   collecting S

$S\left(\frac{a_1^2}{G_1 y_1^2} + \frac{a^2}{Pa^2+Gy^2} + \frac{1}{Q+G_1}\right) = \frac{Pa^2}{Pa^2+Gy^2} + 1 = \frac{2Pa^2+Gy^2}{Pa^2+Gy^2}$

Multiplying both members by $Pa^2+Gy^2$

$S\left[\frac{a_1^2}{G_1 y_1^2} + \frac{1}{Q+G_1}\right]\cdot(Pa^2+Gy^2) + a^2 = 2Pa^2 + Gy^2$

$\therefore S = \frac{2Pa^2 + Gy^2}{\left(\frac{a_1^2}{G_1 y_1^2} + \frac{1}{Q+G_1}\right)\cdot(Pa^2+Gy^2) + a^2}$

From the value of $S$ we may find those of

$p$ & $q$. If we neglect $a$ & put $Q = 0$ [illegible]

(b) $S = \dfrac{2Pa^2 \cdot G_1 g_1^2}{(a^2+g^2) \cdot Pa^2 + G_1 g_1 a^2}$ [illegible] $\dfrac{2PG_1g_1^2}{(a^2+g^2)P + G_1 g_1^2}$

If $p = 0$ ∴ $p - q$ becomes $-q$

∴ $\dfrac{Sa_1^2}{G_1 g_1^2} q = -\dfrac{S - Q + G_1}{(Q + G_1)} q$ ∴ $S\left(\dfrac{a_1^2}{G_1 g_1^2} + \dfrac{1}{Q + G_1}\right) = 1$

∴ $S = \dfrac{(Q + G_1) \cdot G_1 g_1^2}{(Q + G_1) a^2 + G_1 g_1}$ If $Q = 0$

$S = \dfrac{G_1 g_1^2}{(a^2 + g_1^2)}$

If the body is a cylinder $\dfrac{g_1^2}{a^2} = \dfrac{1}{2}$

∴ $\dfrac{g^2}{a^2} = \dfrac{1}{2}$ ∴ $S = \dfrac{2PG_1 \cdot \frac{1}{2}}{(1+\frac{1}{2})P + \frac{1}{2}G_1} = \dfrac{2PG_1}{3P + G_1}$

[illegible] $S = \dfrac{G_1}{3}$

if $S > T$ ...... (b) $\dfrac{G_1}{P} > 1 + \dfrac{a_1^2}{g_1^2}$

if $S < G_1$ .. (b) $\dfrac{G_1}{P} > 1 - \dfrac{a_1^2}{g_1^2}$

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expended upon the acceleration of the motions of $P$ and $G$. The tension of the fixed string $GH$, on the other hand:

$$S_1 = S - \frac{G_1 y_1^2}{a_1^2} \cdot \frac{p}{2g}, \quad q = \frac{p}{2} : d$$

because the pulley $EG$ is put into rotation by the difference $S - S_1$ of the tensions of the strings.

*Example.* In a system of pulleys, Fig. 271, the weights $P = 40$ lbs. and $Q = 66$ lbs. are suspended, and each of the solid pulleys weighs 6 lbs.; required, the acceleration of the motions of these weights. The moving force is $P - \frac{Q + G_1}{2} = 40 - \frac{66 + 6}{2}$ $= 4$ feet, the mass of a pulley reduced to its circumference is: $\frac{Gy^2}{ga^2} = \frac{G_1 y_1^2}{ga_1^2}$ $= \frac{G}{2g} = \frac{6}{2g} = \frac{3}{g}$ (§ 221), and the aggregate of the inert masses:

$$= \left(P + \frac{Q + G_1}{4} + \frac{Gy^2}{a^2} + \frac{G_1 y_1^2}{4\, a^2}\right) : g = \left(40 + \frac{72}{4} + 3 + \frac{3}{4}\right) : g = \frac{247}{4\,g},$$

hence the accelerated motion of the descending weight is: $p = \frac{4}{247} \cdot 4\,g = \frac{16 \cdot g}{247}$ $= \frac{16 \cdot 32{,}2}{247} = \frac{515{,}2}{247} = 2{,}086$ feet; on the other hand, the accelerated motion of the ascending weight: $\frac{p}{2} = 1{,}043$ feet. The tension of the string $BB$ is:

$S = P - \left(P + \frac{G}{2}\right)\frac{p}{g} = 40 - 43 \cdot \frac{2{,}086}{32{,}2} = 40 - 2{,}782 = 37{,}218$ lbs.; that of the string $GH, = S - \frac{G}{2} \cdot \frac{p}{2g} = 37{,}218 - 3 \cdot \frac{1{,}043}{32{,}2} = 37{,}224$ lbs.

§ 229. The motion is more complicated, if the pulley $EG$, Fig. 272, be suspended only by a string passing round it.

FIG. 272.

Let us assume that $P$ descends with the accelerating force $p$, and $Q$ ascends with the accelerating force $q$, we then obtain the acceleration of the rotatory motion at the circumference of the loose pulley $q_1 = p - q$ (§ 42). Let us now put the tension of the strings at $AE = S$, we then obtain $P - S = \left(P + \frac{Gy^2}{a^2}\right)\frac{p}{g}$; further, $S - (Q + G_1) = (Q + G_1)\frac{q}{g}$, since from § 214, it may be assumed that $S$ acts at the centre of gravity $D$ of $EG$; and lastly, $S = \frac{G_1 y_1^2}{a_1^2} \cdot \frac{q_1}{g}$, since it may also be assumed that the centre of gravity $D$ is fixed, and the pulley put into rotation by $S$. The last three formulæ give the accelerating force

$$p = \frac{P - S}{P + \frac{Gy^2}{a^2}}\, g,\ q = \left(\frac{S - (Q + G_1)}{Q + G_1}\right) g \text{ and } q_1 = \frac{Sa_1^2}{G_1 y_1^2}\, g, \text{ and}$$

all three being put into the equation $q_1 = p - q$, we obtain

$$\frac{Sa_1^2}{G_1 y_1^2}\, g = \frac{P - S}{P + \frac{Gy^2}{a^2}}\, g - \frac{S - (Q + G_1)}{Q + G_1}\, g,$$

whence the tension of the string follows

$$S = \frac{2\, Pa^2 + Gy^2}{\left(\frac{a_1^2}{G_1 y_1^2} + \frac{1}{Q + G_1}\right)(Pa^2 + Gy^2) + a^2}.$$

The accelerating forces are given from the value of $S$ by the application of the above formulæ.

If we neglect the mass $G$ of the fixed pulley, and also put $Q = 0$, we obtain simply:

$$S = \frac{2\, Pa^2 \,.\, G_1 y_1^2}{P\,(a_1^2 + y_1^2)\, a^2 + Ga_1^2 y_1^2} = \frac{2\, PG_1 y_1^2}{G_1 y_1^2 + P\,(a_1^2 + y_1^2)}.$$

If the extremity of the string $AE$, instead of passing over the pulley $AB$, is fixed, we have then the accelerating force $p = 0$, hence $q_1 = -q$, and consequently the tension

$$S = \frac{(Q + G_1)\, G_1 y_1^2}{(Q + G_1)\, a_1^2 + G_1 y_1^2};\ \text{for } Q = 0$$

$$S = \frac{G_1 y_1^2}{a_1^2 + y_1^2}.$$

If the rolling body $G_1$ be a solid cylinder, we have then $\frac{G_1 y_1^2}{a_1^2} = \frac{1}{2}\, G_1$, and for the first the tension $S = \frac{2\, PG_1}{3P + G_1}$, and for the second $S = \frac{G_1}{3}$. If in the first case the weight $P$ descends, $p$ is then negative, therefore:

$$S > P;\ i.\ e.\ 2\, PG_1 y_1^2 > PG_1 y_1^2 + P^2\,(a_1^2 + y_1^2),$$

simply $\frac{G_1}{P} > 1 + \frac{a_1^2}{y_1^2}$; further, that $G_1$ may descend, it is necessary that $S < G_1$, therefore $\frac{G_1}{P} > 1 - \frac{a_1^2}{y_1^2}$.

*Example.* If when in a system of pulleys, Fig. 271, the string $GH$ suddenly breaks, the string $BE$ at the commencement becomes stretched by the force

$$S = \frac{2\,P + \frac{Gy^2}{a^2}}{\left(\frac{a_1^2}{G_1\,y_1^2} + \frac{1}{Q+G_1}\right)\left(P + \frac{Gy^2}{a^2}\right) + 1} = \frac{2\,.\,40+3}{\left(\frac{1}{3}+\frac{1}{72}\right)(40+3)\,+\,1}$$

$$= \frac{83\,.\,72}{25\,.\,43+72} = \frac{5976}{1147} = 5{,}210 \text{ lbs.}$$

The accelerated motion of the descending weight $P$ will be:

$$p = \left(\frac{P-S}{P+\frac{Gy^2}{a^2}}\right) g = \left(\frac{40-5{,}210}{40+3}\right)\,.\,32{,}2 = \frac{34{,}79}{43}\,.\,32{,}2 = 26{,}023 \text{ feet.}$$

Further, that of the descending pulley:

$$q = \left(\frac{Q+G_1-S}{Q+G_1}\right) g = \left(\frac{72-5{,}210}{72}\right) 32{,}2 = \frac{66{,}79}{72}\,.\,32{,}2 = 30{,}0 \text{ feet;}$$

and the acceleration of rotation of this pulley:

$$q_1 = \frac{Sa_1^{\,2}}{G_1\,y_1^{\,2}}\,.\,g = \frac{5{,}210}{3}\,.\,32{,}2 = 55{,}75 \text{ feet.}$$

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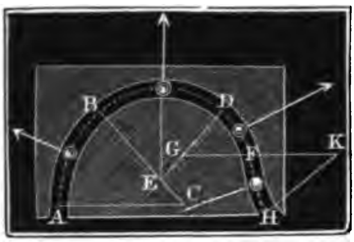
## CHAPTER II.

### CENTRIFUGAL FORCE.

§ 230. *Normal force.*—When a material point moves in a curved line, it has in every point of its path an accelerated motion, deviating from that of the direction of motion, which we have learned to know in phoronomics under the name of the *normal accelerating force.* If the radius of curvature at any place of the path of the moving point $= r$, and its velocity $= v$, we have then for the normal accelerating force $p = \frac{v^2}{r}$ (§ 41). Let now the mass of the point $= M$, the normal ~~accelerating~~ force will then be $Mp = \frac{Mv^2}{r}$, which we must regard as the first cause of the point changing its direction of motion at each position. If no other ~~tangential~~ force but the normal act upon the point, its velocity $v$ will be invariable and $= c$, and hence the normal force $P = \frac{Mc^2}{r}$ will be dependent only on the curvature at each moment and on the radius of curvature, and will be greater, the greater the curvature or less its radius; for double the radius of

FIG. 273.




curvature, for instance, the normal force is only half as great as for a single radius of curvature. If a material point $M$ is constrained by a horizontal path, Fig. 273, to describe a curve $ABDFH$, it will have, disregarding the friction at all places, the same velocity $c$, and will exert at each place a pressure equivalent to the normal force against the concave surface. During the description of the arc $AB$, the pressure $= \frac{Mc^2}{CA}$, during that of $BD = \frac{Mc^2}{EB}$, for the arc $DF$ it $= \frac{Mc^2}{GD}$, and for the arc $FH = \frac{Mc^2}{KF}$, if $CA$, $EB$, $GD$, and $KF$ are the radii of curvature of the portions of the path $AB$, $BD$, $DF$, and $FH$.

§ 231. *Centripetal and centrifugal force.*—When a material point or body moves in a circle, the normal force acts radially inwards, whence it is called the *centripetal force*; whilst the force which the body opposes, by virtue of its inertia, *i. e.*, which acts radially outwards, has received the name of *centrifugal force*. The centripetal force is that which acts directly upon the body; the centrifugal is the reacting force of the body. Each is equal in amount and opposite in direction to the other (§ 62).

In the revolution of the planets about the sun, the attractive force of the sun is the centripetal; but were the body constrained by a directrix, Fig. 273, to describe its circular orbit, this directrix would act, by its rigidity, as a centripetal force, and opposed to the centrifugal force of the body.

If, lastly, the revolving body be connected by a thread or by a rod with the centre of revolution, the elasticity of the rod will then be in equilibrium with the centrifugal force of the body, and thereby act as a centripetal force.

Let $G$ be the weight of the revolving body, therefore its mass $M = \frac{G}{g}$, the radius of the circle in which it revolves $= r$, and the velocity of revolution $= v$; from the last § we have the centrifugal force:

$$P = \frac{Mv^2}{r} = \frac{Gv^2}{gr} = 2 \cdot \frac{v^2}{2g} \cdot \frac{G}{r}, \text{ therefore also } P : G = 2 \cdot \frac{v^2}{2g} : r,$$

i. e. *the centrifugal force is to the weight of the body as double the height due to the velocity is to the radius of revolution.*

If the motion be uniform, which always takes place if no other force (tangential force) than the centripetal act upon the body, the velocity $v = c$ may be expressed by the time of revolution $T$, if we put $c = \frac{\text{path}}{\text{time}} = \frac{2\pi r}{T}$, and hence we shall obtain for the centrifugal force:

$$P = \left(\frac{2\pi r}{T}\right)^2 \frac{M}{r} = \frac{4\,\pi^2}{T^2} \cdot Mr = \frac{4\,\pi^2}{g\,T^2} \cdot Gr.$$

Since $4\,\pi^2 = 39{,}4784$, and for the measure in feet $\frac{1}{g} = 0{,}032$, we then have more conveniently for calculation

$$P = \frac{39{,}4784}{T^2} \cdot Mr = 1{,}224 \cdot \frac{Gr}{T^2}.$$

The number $n$ of revolutions per minute is often given, and therefore $T$ is replaced by $\frac{60''}{n}$, whence it follows:

$$P = \frac{39{,}4784}{3600}\, n^2\, Mr = 0{,}010966\, n^2\, Mr = 0{,}000331\, n^2\, Gr.$$

Hence, for equal times of revolution, or for an equal number of revolutions in a given time, the centrifugal force increases as the product of the mass and radius of revolution, and is inversely proportional, other circumstances being alike, to the squares of the times of revolution, or directly proportional to the squares of the number of revolutions. As $\frac{2\pi}{T}$ is the angular velocity $\omega$, we may finally put: $P = \omega^2 \cdot Mr$.

*Examples.*—1. If a body of 50 lbs. weight, describe a circle of 3 feet radius 400 times per minute, its centrifugal force will then be:

$$P = 0{,}000331 \cdot 400^2 \cdot 50 \cdot 3 = 52{,}96 \cdot 50 \cdot 3 = 7944 \text{ lbs.}$$

If this body be connected with an axis by a hempen cord, and the modulus of strength of the cord (§ 186) be 7000 lbs., it will follow that $7944 = 7000 \cdot F$; hence the section of this cord will be: $F = \frac{7944}{7000} = 1{,}1542$ square inches, and its radius

$D = \sqrt{\frac{4\,F}{\pi}} = \sqrt{\frac{4{,}6168}{3{,}1416}} = \sqrt{1{,}5} = 1{,}224$ inch. But for a threefold security $D$ must be taken $= 1{,}224\ \sqrt{3} = 1{,}224 \cdot 1{,}732 = 2{,}119$ inches.—2. From the earth's radius $r = 20\frac{1}{4}$ millions of feet, and the time of rotation or length of a day $T = 24$ h. $= 24 \cdot 60 \cdot 60 = 86400''$ the centrifugal force of a body at the earth's equator is

$P = 1{,}224 \cdot \frac{G \cdot 20{,}250000}{86400^2} = \frac{2478}{864^2} \cdot G = \frac{1}{300} \cdot G$ nearly; but were the length of the day 17 times less, therefore $\frac{24}{17} = 1^\circ\ 24'\ 42''$, this force would be $17^2 = 289$ times as great, therefore equal to about the weight of the body. At the equator, therefore, the centrifugal force would be equivalent to that of gravity, and the body would neither rise nor fall. In the revolution of the moon about the earth, its centrifugal force is counteracted by the attraction of the earth. Let $G$ be the weight of the moon, $r$ its distance from the earth, and $T$ its time of revolution; the centrifugal force of this heavenly body $= 1{,}224 \cdot \frac{Gr}{T^2}$. Let $a$ be the radius of the earth, and let us assume that the force of gravity at different distances from its centre increases inversely as a power of these distances; we have then the gravity of the moon or the attractive force of the earth $= G\left(\frac{a}{r}\right)^n$, and if we equate both forces to each other, we then obtain $\left(\frac{a}{r}\right)^n = 1{,}224 \cdot \frac{r}{T^2}$.

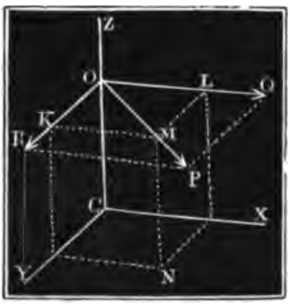
Now $\frac{a}{r} = \frac{1}{60}$, $r = 1250$ million feet, and $T = 27$ days, 7 hours, 42 minutes $= 39342$ minutes $= 39342 \cdot 60$ seconds; hence it follows:

$$\left(\frac{1}{60}\right)^n = \frac{1{,}229 \cdot 1250}{393{,}4^2 \cdot 36} = \frac{1}{3600}, \text{ nearly} = \left(\frac{1}{60}\right)^2$$

and hence, $n = 2$; *i. e.*, the gravitating force of the earth is in an inverse ratio to the square of the distance.

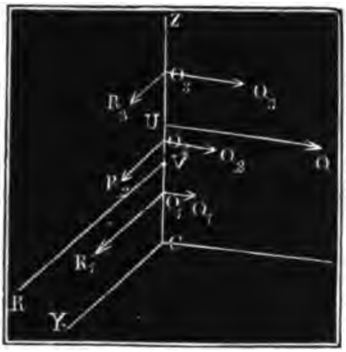
§ 232. *Centrifugal forces of extended masses.*—For any system of masses, or for a mass of finite extension, the formula above found for the centrifugal force is not directly applicable, because we know not beforehand what radius of gyration we have to introduce into the calculation. To find this we proceed in the following manner. In Fig. 274, let $CZ$ be the axis of revolution, $CX$ and $CY$ its two rectangular co-ordinate axes, further let $M$ be a particle $M_1$, and $MK = x$, $ML = y$, and $MN = z$, its distances from the co-ordinate planes $YZ$, $XZ$ and $XY$. As the centrifugal force $P$ acts radially, its point of application may be transferred to its point of intersection $O$ with the axis of revolution. If now we resolve this force in the direction of the axes $CX$ and $CY$, we shall obtain the component forces $OQ = Q$ and $OR = R$, for which

FIG. 274.



$OQ : OP = OL : OM$, and $OR : OP = OK : OM$, whence $Q = \frac{x}{r} P$ and $R = \frac{y}{r} P$, where $r$ represents the distance $OM$ of the particle from the axis of revolution. Let us proceed in a similar manner with all the particles, and we shall obtain two systems of parallel forces, one in the plane $XZ$, and the other in the plane $YZ$, but each acting perpendicularly to the axis $CZ$. For distinction, let us avail ourselves of the index numbers 1, 2, 3, &c., and therefore, put for the particles of the mass, $M_1$, $M_2$, $M_3$, and for their distances $x_1$, $x_2$, $x_3$, &c., we shall obtain the resultant of the one system, Fig. 275,

FIG. 275.




$$Q = Q_1 + Q_2 + Q_3 + \ldots$$
$$= \frac{P_1 x_1}{r_1} + \frac{P_2 x_2}{r_2} + \frac{P_3 x_3}{r_3} + \ldots =$$
$\omega^2 \cdot (M_1 x_1 + M_2 x_2 + \ldots)$ and that of the other $R = R_1 + R_2 + \ldots = \omega^2 \cdot (M_1 y_1 + M_2 y_2 + \ldots)$. Let us finally put the distances of the particles from the plane $XY$, $CO_1$, $CO_2$, &c., $= z_1$, $z_2$, &c., we shall obtain for the points of application of these resultants the distances $CU = u$, and $CV = v$ by the equations $(Q_1 + Q_2 + \ldots)\, u = Q_1 z_1 + Q_2 z_2 + \ldots$ and $(R + R_2 + \ldots)\, v = R_1 z_1 + R_2 z_2 + \ldots$, whence it follows:

$$u = \frac{Q_1 z_1 + Q_2 z_2 + \ldots}{Q_1 + Q_2 + \ldots} = \frac{M_1 x_1 z_1 + M_2 x_2 z_2 + \ldots}{M_1 x_1 + M_2 x_2 + \ldots}, \text{ and}$$

$$v = \frac{R_1 z_1 + R_2 z_2 + \ldots}{R_1 + R_2 + \ldots} = \frac{M_1 y_1 z_1 + M_2 y_2 z_2 + \ldots}{M_1 y_1 + M_2 y_2 + \ldots}.$$

Hence, therefore, in general the centrifugal forces of a system of bodies, or an extended body, may be reduced to two forces, which, so long as $w$ and $v$ are unequal, cannot be resolved into a single one.

*Example.* The masses of a system are

$M_1 = 10$ lbs., $M_2 = 15$ lbs., $M_3 = 18$ lbs., $M_4 = 12$ lbs.,
and their distances $x_1 = 0$ inches, $x_2 = 4$ inches, $x_3 = 2$ inches, $x_4 = 6$ inches,
$y_1 = 3$ „ $y_2 = 1$ „ $y_3 = 5$ „ $y_4 = 3$ „
$z_1 = 2$ „ $z_2 = 3$ „ $z_3 = 3$ „ $z_4 = 0$ „
we have then the following mean centrifugal forces
$Q = \omega^2 \cdot (10 \cdot 0 + 15 \cdot 4 + 18 \cdot 2 + 12 \cdot 6) = 168 \cdot \omega^2$, and

17*

$R = \omega^2 . (10 . 3 + 15 . 1 + 18 . 5 + 12 . 3) = 171 . \omega^2$, and the distances of their points of application from the origin $C$:

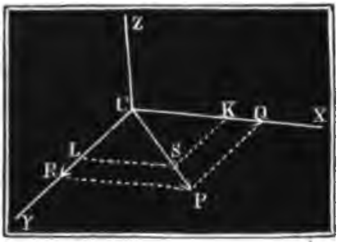
$$u = \frac{10 . 0 . 2 + 15 . 4 . 3 + 18 . 2 . 3 + 12 . 6 . 0}{10 . 0 + 15 . 4 + 18 . 2 + 12 . 6} = \frac{288}{168} = \frac{12}{7} = 1{,}714 \text{ inches, and}$$

$$v = \frac{10 . 3 . 2 + 15 . 1 . 3 + 18 . 5 . 3 + 12 . 3 . 0}{10 . 3 + 15 . 1 + 18 . 5 + 12 . 3} = \frac{375}{171} = \frac{125}{57} = 2{,}193 \text{ inches.}$$

The difference of these two values of $u$ and $v$ shows that the centrifugal forces cannot be replaced by a single force.

§ 233. If the particles of the mass lie in a plane at right angles to the axis of revolution, Fig. 276, their centrifugal forces may be reduced to a single force, because their directions intersect at a point in the axis. Retaining the denominations of the former §, we shall obtain the resultant centrifugal force in this case:

FIG. 276.




$$P = \sqrt{Q^2 + R^2} = \omega^2 \sqrt{[(M_1 x_1 + M_2 x_2 + \ldots)^2 + (M_1 y_1 + M_2 y_2 + \ldots)^2]}.$$

Now $CK = x$, and $CL = y$, are the co-ordinates of the centre of gravity of the system $M = M_1 + M_2 + \ldots$, we then have:

$$M_1 x_1 + M_2 x_2 + \ldots = Mx \text{ and } M_1 y_1 + M_2 y_2 + \ldots = My,$$

hence the centrifugal force:

$$P = \omega^2 \sqrt{M^2 x^2 + M^2 y^2} = \omega^2 M \sqrt{x^2 + y^2} = \omega^2 Mr,$$

provided further that $r = \sqrt{x^2 + y^2}$ represent the distance $CS$ of the centre of gravity from the axis of revolution $\overline{CZ}$.

For the angle $PCX = \alpha$, which this force includes with the axis $CX$, $tang. \alpha = \frac{R}{Q} = \frac{My}{Mx} = \frac{y}{x}$; *hence the direction of the centrifugal force passes through the centre of gravity of the system, and centrifugal force is exactly the same as if the collective masses were united at the centre of gravity.*

For a disc $AB$ at right angles to the axis of revolution $\overline{ZZ}$, Fig. 277, the centrifugal force is from this $= \omega^2 Mr$, if $M$ represents its mass, and $r$ the distance $CS$ of its centre of gravity $S$ from the axis. To find the centrifugal force of another body $ABDE$, Fig. 278, we must decompose it into elementary discs by

FIG. 277.

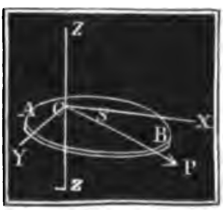
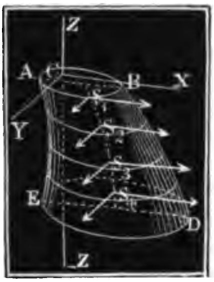


FIG. 278.




planes at right angles to the axis $\overline{ZZ}$, find their centres of gravity $S_1$, . $S_2$, &c., and determine the centrifugal forces by help of these last, decompose each of them in the direction of the axes $CX$ and $CY$ into component forces, and reduce the forces in the plane $ZCX$ to a resultant $Q$, and those in the plane $ZCY$ to a resultant $R$.

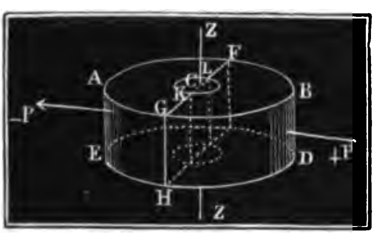
If the centres of gravity of the aggregate discs lie in a line parallel to the axis of revolution, $x = x_1 = x_2$, &c., and $y = y_1 = y_2$, &c., and therefore also $r = r_1 = r_2$, &c., hence the centrifugal force of the whole body $P = \omega^2 (M_1 r + M_2 r + \ldots) = \omega^2 \, Mr$, and the distance of its point of application from the plane $XY$:

$$z = \frac{(M_1 z_1 + M_2 z_2 + \ldots)\, r}{(M_1 + M_2 + \ldots)\, r} = \frac{M_1 z_1 + M_2 z_2 + \ldots}{M_1 + M_2 + \ldots}.$$

According to these equations, the centrifugal force of a body, whose elements are in a line parallel to the axis, is equivalent to the centrifugal force of the mass of this body, reduced to its centre of gravity, and its point of application and centre of gravity coincide. From this the centrifugal forces of all rotatory bodies, whose geometric axes run parallel with the axis of rotation may be found. If the geometric axis of any such body coincide with the axis of rotation, the centrifugal force is equal to nothing.

*Example.* The dimensions, the density, and strength of a millstone, $ABDE$, Fig. 279, are given; it is required to find the angular velocity $\omega$, in consequence of which rupture will take place in virtue of centrifugal force. If we put the radius of the millstone $= r_1$, the radius $CK$ of its eye $= r_2$, the height $AE = GH = l$, the density $= \gamma$, and the modulus of strength $= K$, we obtain the force required for rupture $= 2\,(r_1 - r_2)\; lK$, the weight of the stone $G = \pi\,(r_1^2 - r_2^2)\; l\gamma$, and the radius of gyration of each half of the stone, *i. e.* the distance of its centre of gravity from the axis of rotation (§ 109), $r = \frac{4}{3\pi} \cdot \frac{r_1^3 - r_2^3}{r_1^2 - r_2^2}$. At the moment of rupture, the centrifugal force of half the stone is equivalent to the strength; we hence obtain the equation of condition $\omega^2 . \frac{1}{2} \frac{Gr}{g} = 2\,(r_1 - r_2)\; lK$, *i. e.* $\omega^2 . \frac{2}{3}\,(r_1^3 - r_2^3)\frac{l\gamma}{g} = 2\,(r_1 - r_2)\; lK$, or leaving out $2\,l$ on both sides, it follows that

FIG. 279.



$$\omega = \sqrt{\frac{3g(r_1-r_2)K}{(r_1^3-r_2^3)\gamma}} = \sqrt{\frac{3gK}{(r_1^2+r_1r_2+r_2^2)\gamma}}.$$

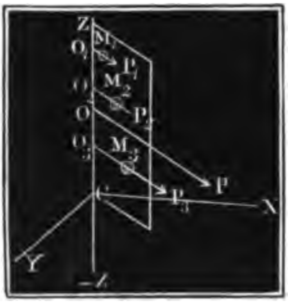
If $r_1$ = 2 feet = 24 inches, $r_2$ = 4 inches, $K$ = 750 lbs., and the specific gravity of the millstone = 2,5, therefore the weight of a cubic inch of its mass = $\frac{62,5 \times 2,5}{1728}$ = 0,0903 lbs., it follows that the angular velocity at the moment of rupture is:

$$\omega = \sqrt{\frac{3 \,.\, 12 \,.\, 32,2 \,.\, 750}{688 \,.\, 0,0903}} = \sqrt{\frac{869400}{62,1264}} = 112,1 \text{ inches.}$$

If the number of rotations per minute = $n$, we have then $\omega = \frac{2\pi n}{60}$; hence, inversely, $n = \frac{30\,\omega}{\pi}$, but here $= \frac{30 \,.\, 112,1}{\pi} = 1070$. The average number of rotations of such a millstone is only 120, therefore 9 times less.

§ 233. If the collective particles $M_1$, $M_2$ of a system of masses, Fig. 280, or the centres of gravity of the elements of a body lie in a plane passing through their axis of revolution, the centrifugal forces will then form a system of parallel forces, and these may be reduced according to the rule to a single force. The distances of the particles or the elements from the axis of revolution $\overline{ZZ}$, are $O_1M_1 = r_1$, $O_2M_2 = r_2$, &c., we obtain for their centrifugal forces:

FIG. 280.




$$P_1 = \omega^2 M_1 r_1,\ P_2 = \omega^2 M_2 r_2, \text{ \&c.,}$$

and hence the resultant centrifugal force:

$$P = \omega^2 (M_1 r_1 + M_2 r_2 + \ldots) = \omega^2 Mr,$$

$r$ representing the distance of the centre of gravity of the mass $M$ from the axis of revolution. Therefore, here also the distance of the centre of gravity from the axis of revolution must be regarded as the radius of ~~gyration~~. But to find the point of application $O$ of the resultant centrifugal force, let us put the distances of the particles of the mass from the normal plane: $CO_1 = z_1$, $CO_2 = z_2$, &c., into the formula:

$$CO = z = \frac{M_1 r_1 z_1 + M_2 r_2 z_2 + \ldots}{M_1 r_1 + M_2 r_2 + \ldots}.$$

By help of the formula $P = \omega^2 Mr$, the centrifugal forces of rotatory bodies and other geometric bodies may be found, when their axes and the axis of revolution lie in one plane. The centrifugal force of a right cone $ADB$, Fig. 281, may be found, if the

FIG. 281.


distance $SN$ of its centre of gravity $S$ from the axis of revolution $\overline{ZZ}$ be put as $r$ into the formula. If the height of the cone $CD = h$, the distance $DF$ of the point $D$ from the axis of revolution $= a$, and the angle $CGE$, by which the geometrical axis deviates from the axis of revolution $\overline{ZZ} = a$, we have then $r = a + \frac{3}{4} h \sin. a$. For a rod $AB$, Fig. 282, whose length $AB = l$, and angle of inclination $ABZ$ to the axis of revolution $BZ = a$, we have $r = SN = \frac{1}{2} l \sin. a$, therefore, the centrifugal force:

FIG. 282.


$P = \omega^2 \cdot \frac{1}{2} M l \sin. a$; but to find the point of application $O$ of this force, in the expression $\omega^2 \cdot \frac{M}{n} x \sin. a \cdot x \cos. a$ $= \omega^2 \cdot \frac{M}{n} x^2 \sin. a \cos. a$ for the moment of the element $\frac{M}{n}$ of the rod, let us put for $x$ successively the values $\frac{l}{n}, \frac{2l}{n}, \frac{3l}{n}$, &c., and add the results, in this manner we shall obtain the moment of the entire rod:

$$Pz = \omega^2 \frac{M}{n} \sin. a \cos. a \frac{l^2}{n^2} (1^2 + 2^2 + 3^2 + .. + n^2)$$
$$= \frac{1}{3} \omega^2 M l^2 \sin. a \cos. a,$$
$$z = \frac{1}{3} \omega^2 M l^2 \sin. a \cos. a : \frac{1}{2} \omega^2 M l \sin. a = \frac{2}{3} l \cos. a,$$ and

the distance of the point of application $O$ from the extremity of the rod $B$ lying in the axis, $BO = \frac{2}{3} l$.

FIG. 283.


If the rod $AB$, Fig. 283, does not reach the axis, we then have,

$P = \frac{1}{2} \omega^2 F l_1^2 \sin. a - \frac{1}{2} \omega^2 F l_2^2 \sin. a$ $= \frac{1}{2} \omega^2 F \sin. a (l_1^2 - l_2^2)$, and the moment:

$$Pz = \frac{1}{3} \omega^2 F \sin. a \cos. a \, (l_1^3 - l_2^3),$$

because the mass of $CA$, = the cross section into the length, $= Fl_1$, and the mass of

$CB = Fl_2$, hence it follows that the distance of the point of application $O$ from its intersection with the axis $C$ is:

$$CO = \frac{2}{3}\,\frac{l_1^3 - l_2^3}{l_1^2 - l_2^2} = l + \frac{(l_1 - l_2)^2}{12\,l},$$

FIG. 284.


where $l$ expresses the distance $CS$ of the centre of gravity, but $l_1 - l_2$ the length of the rod $AB$.

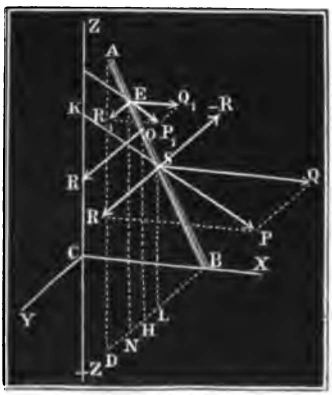
This formula is also applicable to a rectangular plate $ABDE$, Fig. 284, which is divided into two congruent right angles by the plane of the axis $COZ$, because the centrifugal force acts at the middle of each of the elements, which are obtained by sections normal to $CZ$. Therefore the distances $CF$ and $CG$ of the two bases $AB$ and $DE$ from the point $C$ of the axis, are $l_1$ and $l_2$, we have here also $CO = \frac{2}{3} \cdot \frac{l_1^3 - l_2^3}{l_1^2 - l_2^2}$.

§ 234. In the case where the particles of the body neither lie in a plane normal to, nor in a plane passing through the axis of revolution, the resultant centrifugal forces
$Q = \omega^2\,(M_1 x_1 + M_2 x_2 + \ldots)$ and $R = \omega^2\,(M_1 y_1 + M_2 y_2 + \ldots)$ cannot be reduced to a single force, nevertheless it is possible to replace these forces by a force acting at the centre of gravity:

$$P = \sqrt{Q^2 + R^2} = \omega^2\,Mr,$$

and by a couple composed of $Q$ and $R$. If, namely, we apply to the centre of gravity $S$, four forces $+Q$ and $-Q$, $+R$ and $-R$ balancing each other, the positive parts will give a resultant $P = \sqrt{Q^2 + R^2}$, and the negative parts on the other hand, $-Q$ and $-R$, will form the couples $(Q, -Q)$ and $(R, -R)$ with the centrifugal forces applied at $U$ and $V$, which may be reduced to a single couple. In order to make ourselves acquainted with this reduction of the centrifugal forces of a rotatory body, let us take

FIG. 285.



the following simple case. Let the bar $AB$, Fig. 285, which turns about the axis $\overline{Z}Z$, lie parallel to the plane $YZ$, and rest with its extremity $B$ on the axis $CX$. Let the length of this bar $= l$, its weight $= G$, the angle $BAD$ at which it is inclined to the axis of rotation $= a$, and its distance $CB$ from the plane $YZ$, which is also its shortest distance from the axis $Z\overline{Z} = a$. Let now $E$ be an element $\frac{M}{n}$ of the bar, and $BE = x$ its distance from the extremity $B$, we shall then have the projection $BN = x \sin. a$, and hence the components of the centrifugal force $P_1$ of this element:

$Q_1 = \omega^2 . \frac{M}{n} . CB = \omega^2 . \frac{M}{n} a$ and $R_1 = \omega^2 . \frac{M}{n} . BN = \omega^2 . \frac{M}{n} x \sin. a$,

and their moments about the principal plane $XCY$:

$Q_1 z_1 = \omega^2 . \frac{M}{n} . CB . EN = \omega^2 . \frac{M}{n} ax \cos. a$ and $R_1 z_1 = \omega^2 . \frac{M}{n} x^2 . \sin. a \cos. a$. The several components parallel to the plane $XZ$ give the resultant $Q = Q_1 + Q_2 + \ldots = n . \omega^2 . \frac{M}{n} a = \omega^2 . Ma$, and its moment $Qu = Q_1 z_1 + Q_2 z_2 + \ldots = \omega^2 . \frac{M}{n} a \cos. a \; (x_1 + x_2 + \ldots)$, or, as $x_1$ is to be taken $= \frac{l}{n}$, $x_2 = 2\frac{l}{n}$, $x_3 = \frac{3l}{n}$, &c.,

$Qu = \omega^2 . \frac{M}{n} a \cos. a . \frac{l}{n} (1 + 2 + 3 + .. + n) = \omega^2 . \frac{M}{n} a \cos. a . \frac{l}{n} . \frac{n^2}{2}$
$= \frac{1}{2} \omega^2 . Mal \cos. a$; the distance, therefore, of the point of application of this component from the plane $XY$ is:

$$LS = u = \frac{\frac{1}{2} \omega^2 Mal \cos. a}{\omega^2 Ma} = \frac{1}{2} l \cos. a,$$

*i. e.* it coincides with the centre of gravity of the bar. The components, which act parallel to $YZ$, give the resultant $R = R_1 + R_2 + \ldots = \omega^2 . \frac{M}{n} \sin. a \; (x_1 + x_2 + \ldots)$
$= \omega^2 . \frac{M}{n} \sin. a \frac{l}{n} . \frac{n^2}{2} = \frac{1}{2} \omega^2 Ml \sin. a$ with the moment
$\omega^2 . \frac{M}{n} \sin. a \cos. a (x_1^2 + x_2^2 + \ldots) = \omega^2 . \frac{M}{n} \sin. a \cos. a . \left(\frac{l^2}{n^2} + \frac{4l^2}{n^2} + \ldots\right)$
$= \omega^2 . \frac{M}{n} \frac{l^2}{n^2} \sin. a \cos. a (1 + 4 + 9 + .. + n^2) = \omega^2 . \frac{M}{n} . \frac{l^2}{n^3} \sin. a \cos. a . \frac{n^3}{3}$
$= \frac{1}{3} \omega^2 M l^2 \sin. a \cos. a$; the distance of the point of application of

the force from the plane $XY$ is: $HO = v = \dfrac{\frac{1}{3}\,\omega^2 M l^2\, sin.\, a\; cos.\, a}{\frac{1}{2}\,\omega^2 M l\, sin.\, a}$
$= \frac{2}{3}\; l\; cos.\; a$, *i. e.* this point lies about $(\frac{2}{3} - \frac{1}{2})\; l\; cos.\; a$
$= \frac{1}{6}\; l\; cos.\; a = \frac{1}{6}$ of the projection $AD$ parallel to the axis above the centre of gravity $S$ of the bar.

From the forces $Q = \omega^2\; M a$ and $R = \frac{1}{2}\;\omega^2\; M l\; sin.\; a$, the final resultant applied at the centre of gravity of the bar follows: $P = \sqrt{Q^2 + R^2} = \omega^2 M\; \sqrt{a^2 + \frac{1}{4}\, l^2\, sin.\, a^2}$, and the couple $(R_1 - R)$ with the moment

$$R\;.\;\overline{SO} = \tfrac{1}{2}\;\omega^2\; M\, l\; sin.\; a\;.\;\tfrac{1}{6}\; l = \tfrac{1}{12}\;\omega^2\; M l^2\; sin.\; a.$$

§ 235. *Free axes.*—In general, indeed, the centrifugal forces of a body revolving uniformly about an axis, exert a pressure upon the axis; it is, nevertheless, possible that these forces mutually counteract each other, and for this reason the axis will have no pressure to sustain. This case presents itself, for instance, in every solid of rotation revolving about its geometric axis, or its axis of symmetry, and especially in the wheel and axle, and in the water-wheel, &c. If under these circumstances no external forces act upon a rotatory body, or upon such a system, the body will remain for ever in this state of revolution, without its being necessary that the axis of revolution should be fixed. This axis is called, for this reason, a *free axis.* From the preceding, the conditions immediately follow by which an axis of revolution becomes a *free axis.* It is requisite not only that the resultants $P$ and $Q$ of the components of the centrifugal forces acting parallel to the planes of the axes $XZ$ and $YZ$, but that the sum of the statical moments of each of the two systems of forces should $= 0$. Hence, from this:

1. $M_1 x_1 + M_2 x_2 + \ldots = 0$,
2. $M_1 y_1 + M_2 y_2 + \ldots = 0$, further
3. $M_1 x_1 z_1 + M_2 x_2 z_2 + \ldots = 0$, and
4. $M_1 y_1 z_1 + M_2 y_2 z_2 + \ldots = 0$.

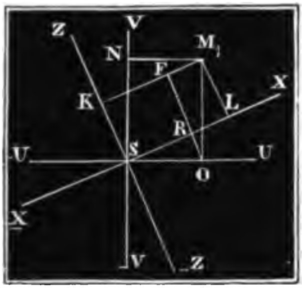
The two first equations require that the free axis pass through the centre of gravity of the body or system. The two last afford the elements for determining the position of this axis. It may, besides, be proved that every body or system has at least three *free axes*, and that these axes meet at right angles in the centre of gravity of the system.

The higher mechanics distinguishes other axes besides the free, which run parallel to these and intersect each other in a

point of the system, and are called the *principal axes*. It may be also proved that the moment of inertia of a body about one of the principal axes is a maximum, about a second axis a minimum, and about a third neither one nor the other.

§ 236. If the particles of a mass lie in one plane, for instance, if the mass forms a thin plate or plane figure, then the straight line passing through the centre of gravity of the entire mass, and normal to its plane, is a free axis of the mass; for in this case the mass has no ~~radius of gyration~~, and hence the only possible centrifugal force is $= 0$. To find the other two free axes, let us proceed in the following manner. Let $S$, Fig. 286, be the centre of gravity

FIG. 286.




of a mass, and let $U\overline{U}$ and $V\overline{V}$ be two co-ordinate axes in the plane of the mass, let us determine the molecules by co-ordinates parallel to these axes, viz. the molecules $M_1$ by the co-ordinates $M_1N = u_1$ and $M_1O = v_1$. Let $X\overline{X}$, on the other hand, be a free axis, $Z\overline{Z}$ an axis perpendicular to it; further, let the angle to be determined $XSU$, which the free axis makes with the co-ordinate axis $SU, = \phi$, and let the co-ordinates of the particles referred to the axes $X\overline{X}$ and $Z\overline{Z}$: be $x_1, x_2 \ldots, z_1, z_2 \ldots$, therefore for the particle $M_1$: $M_1K = x_1$ and $M_1L = z_1$. From this we easily obtain:

$x_1 = M_1K = SR + RL = SO \cos.\phi + OM_1 \sin.\phi = u_1 \cos.\phi + v_1 \sin.\phi$
$z_1 = M_1L = -OR + OF = -SO \sin.\phi + OM_1 \cos.\phi$
$= -u_1 \sin.\phi + v_1 \cos.\phi$; and hence the product:

$$x_1 z_1 = (u_1 \cos.\phi + v_1 \sin.\phi)(-u_1 \sin.\phi + v_1 \cos.\phi)$$
$$= -(u_1^2 - v_1^2) \sin.\phi \cos.\phi + u_1 v_1 (\cos.\phi^2 - \sin.\phi^2)$$

or, since $\sin.\phi \cos.\phi = \frac{1}{2} \sin. 2\phi$ and $\cos.\phi^2 - \sin.\phi^2 = \cos. 2\phi$, $x_1 z_1 = -\frac{1}{2}(u_1^2 - v_1^2) \sin. 2\phi + u_1 v_1 \cos. 2\phi$, and hence the moment of the particle $M_1$:

$$M_1 x_1 z_1 = -\frac{M_1}{2}(u_1^2 - v_1^2) \sin. 2\phi + M_1 u_1 v_1 \cos. 2\phi.$$ The

moment of the particle $M_2$:

$$M_2 x_2 z_2 = -\frac{M_2}{2}(u_2^2 - v_2^2) \sin. 2\phi + M_2 u_2 v_2 \cos. 2\phi, \text{ \&c.,}$$

and the sum of the moments of all the particles, or the moment of the entire mass:

$$M_1 x_1 z_1 + M_2 x_2 z_2 + \ldots = -\tfrac{1}{2} \sin.\, 2\phi\ [(M_1 u_1^2 + M_2 u_2 + ..) - (M_1 v_1^2 + M_2 v_2^2 + ..)] + \cos.\, 2\phi\ (M_1 u_1 v_1 + M_2 u_2 v_2^2 + \ldots).$$

That $X\overline{X}$ may become a free axis, its moment from the former paragraph must be $= 0$; hence we must put

$$\tfrac{1}{2} \sin.\, 2\phi\ [(M_1 u_1^2 + M_2 u_2^2 + \ldots) - (M_1 v_1^2 + M_2 v_2^2 + ..)] - \cos.\, 2\phi\ (M_1 u_1 v_1 + M_2 u_2 v_2 + ..) = 0,$$

and from this we obtain the equation of condition:

$$tang.\, 2\phi = \frac{\sin.\, 2\phi}{\cos.\, 2\phi} = \frac{2\,(M_1 u_1 v_1 + M_2 u_2 v_2 + ..)}{(M_1 u_1^2 + M_2 u_2^2 + \ldots) - (M_1 v_1^2 + M_2 v_2^2 + \ldots)}$$

$$= \frac{\text{twice the moment of the centrifugal force}}{\text{difference of the moments of inertia}}.$$

By this formula two values for $2\phi$ are given, which vary $180^0$ from each other, and therefore also two values of $\phi$, which vary $90^0$ from each other; on this account, not only is the axis $X\overline{X}$ determined by this angle $\phi$, a free axis, but also the axis $Z\overline{Z}$ perpendicular to it.

§. 237. The free axes of many surfaces and bodies are known without any calculation. In symmetrical figures, for instance, the axis of symmetry is a free axis, the perpendicular to this the centre of gravity is a second, and the axis perpendicular to the plane of the figure a third free axis. The axis of rotation $Z\overline{Z}$ of a rotatory body $AB$, Fig. 287, is a free axis, so is every normal $X\overline{X}$, $Y\overline{Y}$ . . to this, passing through the centre of gravity $S$. Every diameter of a sphere is a free axis, the axes $X\overline{X}$, $Y\overline{Y}$, $Z\overline{Z}$, of a right

FIG. 287.

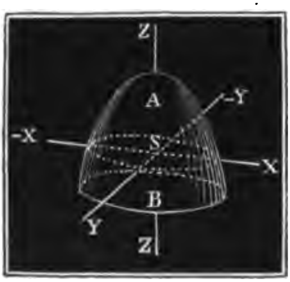
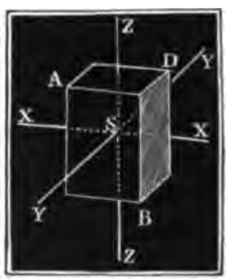


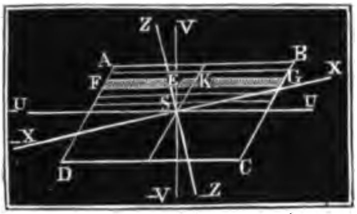

FIG. 288.




parallelepiped $ABD$, Fig. 288, bounded by six rectangles, passing through the centre of gravity $S$, and normal to the sides $BD$, $AB$ and $AD$, or running parallel with the edges, are free axes.

Let us now determine the free axes of an acute angled parallelogram $ABCD$, Fig. 289. Let us draw through its centre of gravity $S$, the co-ordinate axes $U\overline{U}$ and $V\overline{V}$ at right angles to each other, so that one of the sides $AB$ of the parallelogram may run parallel to it, and let us decompose the parallelogram by parallel lines into $2n$ equal strips, such as $FG$. If, now, one side $AB = 2a$, the other $AD = 2b$, and the angle $ADC$ between the two sides $= \alpha$, we then obtain for the strip $FG$, distant from $U\overline{U}$, $SE = x$, the length of one part:

FIG. 289.




$$EG = KG + EK = a + x\ cotg.\ a,$$

and that of the other $EF = a - x\ cotang.\ a$, and since $\frac{b}{n}\ sin.\ a$ is the breadth of both, the area of these strips $= \frac{b\ sin.\ a}{n}(a + x\ cotg. a)$ and $\frac{b\ sin.\ a}{n}\ (a - x\ cotg.\ a)$; the measure of the centrifugal forces about the axis $V\overline{V}$ is therefore:

$$= \frac{b\ sin.\ a}{n}\ (a + x\ cotg.\ a) . \tfrac{1}{2}\ (a + x\ cotg.\ a) = \frac{b\ sin.\ a}{2n}\ (a + x cotg. a)^2$$

and $\frac{b\ sin.\ a}{2n}\ (a - x\ cotg.\ a)^2$, and their moments about the axis $U\overline{U}$: $\frac{b\ sin.\ a}{2n}\ (a + x\ cotg.\ a)^2\ x$ and $\frac{b\ sin.\ a}{2n}\ (a - x\ cotg.\ a)^2\ x$.

As both the forces about $V\overline{V}$ act opposite to each other, the uniting of their moments gives the difference:

$$\frac{b\ x\ sin.\ a}{2n}\ [(a + x\ cotg.\ a)^2 - (a - x\ cotg.\ a)^2] = \frac{2}{n}\ abx^2\ cos.\ a.$$

If we substitute in this formula for $x$ the values:

$$\frac{b\ sin.\ a}{n},\ \frac{2\ b\ sin.\ a}{n},\ \frac{3\ b\ sin.\ a}{n},\ \&c.,$$

successively, and add the results, we shall obtain the measure of the moment of the centrifugal force of half the parallelogram:

$$\frac{2\ ab}{n} cos.\ a . \frac{b^2 sin. a^2}{n^2}\ (1^2 + 2^2 + 3^2 + .. + n^2) = 2\ ab^3\ sin.\ a^2\ cos.\ a . \frac{n^3}{3n^3}$$

$= \frac{2}{3} ab^3 \sin. a^2 \cos. a$, and, therefore, for the whole parallelogram, or $M_1 u_1 v_1 + M_2 u_2 v_2 + \ldots = \frac{4}{3} ab^3 \sin. a^2 \cos. a$. The moment of inertia of a strip $FG$ about the axis $V\overline{V}$ is:

$$= \frac{b \sin. a}{n} \left( \frac{(a + x \cot g. a)^3}{3} + \frac{(a - x \cot g. a)^3}{3} \right)$$

$$= \frac{2 b \sin. a}{3 n} (a^3 + 3 a x^2 \cot g. a^2) = \frac{2}{3} \frac{ab}{n} \sin. a \ (a^2 + 3 x^2 \cot g. a^2);$$

if now we substitute in succession for $x$:

$$\frac{b \sin. a}{n}, \ \frac{2 b \sin. a}{n}, \ \frac{3 b \sin. a}{n}, \ \&c.,$$

and sum the resulting values, we shall have the moment of inertia of a half $= \frac{2}{3} ab \sin. a \ (a^2 + b^2 \cos. a^2)$, and hence that of the whole $= \frac{4}{3} ab \sin. a \ (a^2 + b^2 \cos. a^2)$. On the other hand, the moment of inertia of the parallelogram about the axis of revolution $U\overline{U}$ is $= 4 ab \sin. a . \frac{b^2 \sin. a^2}{3} = \frac{4}{3} ab^3 \sin. a^3$ (§ 220); hence the difference of the moments of inertia sought, *i. e.*

$$(M_1 u_1^2 + M_2 u_2^2 + \ldots) - M_1 v_1^2 + M_2 v_2^2 + \ldots),$$

$$= \frac{4}{3} ab \sin. a \ (a^2 + b^2 \cos. a^2) - \frac{4}{3} ab^3 \sin. a^3$$

$$= \frac{4}{3} ab \sin. a \ [a^2 + b^2 (\cos. a^2 - \sin. a^2)]$$

$$= \frac{4}{3} ab \sin. a \ (a^2 + b^2 \cos. 2 a).$$

Lastly for the angle $USX = \phi$, which the *free axis* $X\overline{X}$ makes with the co-ordinate axis $U\overline{U}$ or the side $AB$ from § 237:

$$\tan g. 2 \phi = \frac{2 (M_1 u_1 v_1 + M_2 u_2 v_2 + \ldots)}{M_1 u_1^2 + M_2 u_2^2 + \ldots) - M_1 v_1^2 + M_2 v_2^2 + \ldots)}$$

$$= \frac{2 \frac{4}{3} ab^3 \sin. a^2 \cos. a}{\frac{4}{3} ab \sin. a \ (a^2 + b^2 \cos. 2 a)} = \frac{b^2 \sin. 2 a}{a^2 + b^2 \cos. 2 a}.$$

In the rhombus $a = b$, hence

$$\tan g. 2 \phi = \frac{\sin. 2 a}{1 + \cos. 2 a} = \frac{2 \sin. a \cos. a}{1 + \cos. a^2 - \sin. a^2}$$

$$= \frac{2 \sin. a \cos. a}{2 \cos. a^2} = \tan g. a,$$

therefore $2\phi = a$, and $\phi = \frac{a}{2}$. As this angle gives the direction of the diagonal, it follows that the diagonals are free axes of the rhombus.

*Example.* The sides of the acute angled parallelogram, *ABCD*, Fig. 289, $AB = 2a = 16$ inches, and $BC = 2b = 10$ inches, and the angle of the perimetre $ABC = a = 60°$, what directions have its free axes?

$$tang.\ 2\phi = \frac{5^2 \cdot sin.\ 120°}{8^2 + 5^2 \cdot cos.\ 120°} = \frac{25 \cdot sin.\ 60°}{64-25\ cos.\ 60°} = \frac{25 \cdot 0{,}86603}{64-25 \cdot 0{,}5} = 0{,}42040$$

$= tang.\ 22°\ 48'$, or $tang.\ 202°\ 48'$. From this it follows, that $\phi = 11°\ 24'$ and $101°\ 24'$ are the angles of inclination of the two free axes to the side *AB*. The third free axis stands at right angles to the plane of the parallelogram. These angles determine also the free axes of a right parallelepiped with rhomboidal bases.

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## CHAPTER III.

### OF THE ACTION OF GRAVITY ON MOTIONS ALONG CONSTRAINED PATHS.

§ 239. *Inclined plane.*—A heavy body may be impeded in various ways from falling freely, and in the following we shall consider only two cases, the one where a body is supported on an inclined plane, and the other where it revolves about a horizontal axis. In both cases the paths of the body are contained in a vertical plane. If the body rests on an inclined plane, its weight may be resolved into two components, of which the one is directed normal to the plane and taken up by it, and the other parallel to the plane, and acts upon the body as a moving force. If $G$ be the weight of the body *ABCD*, Fig. 290, and $a$ the inclination of the inclined plane *FHR* to the horizon, we shall then have from § 134, for the normal pressure; $N = G\ cos.\ a$, and this moving force $P = G\ sin.\ a$. The motion of the body may be either sliding or rolling, let us next consider the first only. In this case all the parts of the body equally participate in its motion, and hence have a common

FIG. 290.

motion of acceleration $p$, which is given by the known formula:

$$p = \frac{\text{force}}{\text{mass}}, = \frac{P}{M} = \frac{G\, sin.\, a}{G} \cdot g = g\ sin.\ a.$$

Therefore $p : g = sin.\ a : 1$, *i. e.* the *accelerated motion of a body on an inclined plane is to the accelerated motion of free descent as the sine of the angle of descent to unity.* In consequence of the friction which takes place, this formula is rarely sufficiently accurate, hence it is necessary in many cases of application to take this into account.

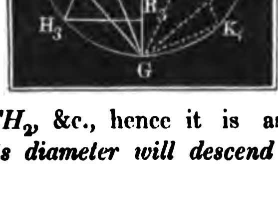
If the body moves on a curved surface, the accelerating force is variable and at each place equal to the accelerating force, which corresponds with the plane of contact to the curved surface.

§ 239. A body slides with the initial velocity 0 down an inclined plane, without friction, from § 10 the final velocity after $t$ seconds is: $v = g\ sin.\ a\ .\ t = 32{,}2\ sin.\ a\ .\ t$ ft., and the space described: $s = \frac{1}{2} g\ sin.\ a\ .\ t^2 + 16{,}1\ sin.\ a\ .\ t^2$ ft. In free descent $v_1 = gt$, and $s_1 = \frac{1}{2} gt^2$, hence we may put: $v : v_1 = s : s_1 = sin.\ a : 1$, *i. e.* the *final velocity and the space of descent down the inclined plane are to the final velocity and space of free descent as the sine of the angle of inclination of the inclined plane to unity.*

FIG. 291.


The cathetus $FH$ of a right-angled triangle, $FGH$, Fig. 291, with vertical hypothenuse $FG$, $= FG\ sin.\ FGH = FG\ sin.\ FHR = FG\ sin.\ a$, if $a$ is the angle of inclination of this cathetus to the horizon, hence $FH : FG = sin.\ a : 1$, and a body describes the vertical hypothenuse $FG$, and the inclined cathetus $FH$ in one and the same time. The space of free descent corresponding to the space of descent down the inclined plane may be found from this, and the latter from the former by construction. Since the angles of the periphery $FH_1G$, $FH_2G$, &c., on the diameter $FG$, Fig. 292, are right angles, the semi-circle on $FG$ cuts off from all the inclined planes, commencing at $F$, the spaces described with this diameter, and therefore, in equal times, $FH_1$, $FH_2$, &c., hence it is asserted, *the chords of a circle and its diameter will descend simultaneously or isochronously.* This

FIG. 292.



isochronism is besides true, not only for the chords $FH_1$, $FH_2$, &c., which have their origin at the highest point $F$ of the circle, but also for the chords $K_1G$, $K_2G$, &c., which commence at the lowest point $G$, for chords $FK_1$, $FK_2$, &c., may be drawn through $F$, which have like positions and equal lengths with the chords $GH_1$, $GH_2$, &c.

§ 240. From the equation $s = \frac{v^2}{2p} = \frac{v^2}{2\,g\,.\,sin.\,a}$ it follows that $s\ sin.\ a = \frac{v^2}{2g}$, and, inversely, $v = \sqrt{2\,gs\ sin.\ a}$. But now $s\ sin.\ a$ is

FIG. 293.

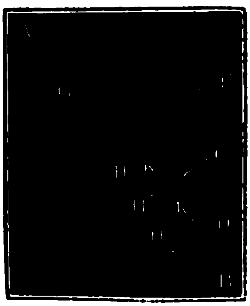
the height $FR$ of the inclined plane or the vertical projection $s_1$ of the space $FH = s$ upon it, hence the final velocities of bodies which descend with an initial velocity 0 down planes of equal heights $F_1H_1$, $F_2H_2$, &c., and of different inclination, Fig. 293, are equal, and also equal to the velocity which a body would acquire if it fell freely from the height $FR$ of these planes.

From the equation $s = \frac{1}{2}\,g\ sin.\ a\ .\ t^2$ follows the formula for the time:

$$t = \sqrt{\frac{2\,s}{g\ sin.\ a}} = \frac{1}{sin.\ a}\sqrt{\frac{2\,s\ sin.\ a}{g}} = \frac{1}{sin.\ a}\,.\,\sqrt{\frac{2\,.\,FR}{g}}.$$

But for a free descent through the height $FR$ the time is: $t_1 = \sqrt{\frac{2\ FR}{g}}$, it follows accordingly $t : t_1 = 1 : sin.\ a = FR : FH$, *the time of descent down the inclined plane is to the time of free descent from the height of this plane as the length of the inclined plane is to its height.*

FIG. 294.



*Examples.*—1. The initial point $F$ of an inclined plane $RH$, Fig. 294, is given, and the final point $H$ in a given line $AB$; required to determine the descent down the plane so that it may take place in the shortest time. If the horizontal line $FG$ be drawn through $F$ to its intersection with $AB$, and $GH$ be made $= GF$, we shall obtain in $H$ the point sought, and therefore in $FH$ the plane of quickest descent; for if through $F$ and $H$ a circle tangent to $FG$ and $FH$ be carried, its isochronously described chords $FK_1$, $FK_2$ &c. will be shorter than the lengths $FH_1$, $FH_2$, &c., of the corresponding inclined planes; consequently, therefore, the time of descent for these chords will be less than for these lengths, and the

time of descent for the inclined plane $FH$, which coincides with a chord, will be the shortest.

FIG. 295.


2. Required the inclination of that inclined plane $FH$, Fig. 295, down which a body would fall in the same time as if it originally fell freely from the height $FR$, and then proceeded with the acquired velocity horizontally to $H_1$. The time of falling down from the vertical height $FR = s_1$ is $t_1 = \sqrt{\frac{2 s_1}{g}}$, and the acquired velocity at $R$ : $v = \sqrt{2 g s_1}$. If now no loss of velocity ensue in transition from the vertical to the horizontal motion, which would follow if the corner $R$ were rounded, the space $RH_1 = s_1 \; cotg. \; a$ will be uniformly described, and in the time $t_2 = \frac{s_1 \; cotg. \; a}{v} = \frac{s_1 \; cotg. \; a}{\sqrt{2 g s_1}} = \frac{1}{2} \; cotg. \; a \sqrt{\frac{2 s_1}{g}}$. The time of descent down the inclined plane is $t = \frac{1}{sin. \; a} \sqrt{\frac{2 s_1}{g}}$; hence, if we put $t = t_1 + t_2$, we shall obtain the equation of condition $\frac{1}{sin. \; a} = 1 + \frac{1}{2} \; cotg. \; a$, whose solution will give $tang. \; a = \frac{3}{4}$. In the corresponding inclined plane, accordingly, the height is to the base and to the length as 3 is to 4 is to 5, and the angle of inclination is $a = 36^\circ \; 52' \; 11''$.—3. The time for sliding down an inclined plane of a given base $a$ is

$$t = \sqrt{\frac{2 s}{g \; sin. \; a}} = \sqrt{\frac{2 a}{g \; sin. \; a \; cos. \; a}} = \sqrt{\frac{4 a}{g \; sin. \; 2 a}};$$

hence the descent is quickest when $sin. \; 2 a$ is a maximum, *i. e.* $= 1$; therefore $2 a^\circ = 90^\circ$, or $a^\circ = 45^\circ$. Hence, water falls down in the shortest time from roofs of $45^\circ$ inclination.

§ 241. If the motion on an inclined plane proceeds with a certain initial velocity $c$, we shall then have to apply the formulæ found in § 13 and § 14. According to these the terminal velocity of a body ascending an inclined plane is $v = c - g \; sin. \; a. \; t$, and the space described $s = ct - \frac{1}{2} \; g \; sin. \; a, \; t^2$; on the other hand, for a body falling down the inclined plane:

$$v = c + g \; sin. \; a \, . \, t, \text{ and } s = ct + \tfrac{1}{2} \, g \; sin. \; a \, . \, t^2.$$

In both cases of motion the formula is true:

$$s = \frac{v^2 - c^2}{2 \, g \; sin. \; a}, \text{ or } s \; sin. \; a = \frac{v^2 - c^2}{2g} = \frac{v^2}{2g} - \frac{c^2}{2g}.$$

*The vertical projection, therefore, ($s$ sin. $a$) of the space ($s$) described along the inclined plane is always equal to the difference of the heights due to the velocity.*

FIG. 296.

If two inclined planes $FGQ$ and $GHR$, Fig. 296, meet each other in a rounded edge, no impulse will take place in the passage from one plane to the other, and for this reason no loss of velocity ensue; the rule for the descent of a body down this combination of two planes is also true, *the height of descent* ($FR$) *is equal to the difference of the heights due to the velocity.* It is easy to ascertain that this rule is correct also for the ascent or descent on any system of any number of planes, and for the ascent or descent on curved lines or surfaces. (Compare § 82.)

*Examples.*—1. A body ascends with a 21 feet initial velocity an inclined plane of 22° inclination, what is the amount of its velocity and its space described in $1\frac{1}{2}$ seconds? The velocity is:
$v = 21 - 32,2\ sin.\ 22° \,.\, 1,5 = 21 - 32,2 \,.\, 0,3746 \,.\, 1,5 = 2,906$ feet; and the space:

$$s = \frac{c+v}{2} \,.\, t = \frac{21 + 2,906}{2} \,.\, \frac{3}{2} = \frac{23,906 \,.\, 3}{4} = 17,928 \text{ feet.}$$

2. How high does a body, with an initial velocity of 36 feet, ascend an inclined plane of 48° acclivity? The vertical height is $s_1 = \frac{v^2}{2g} = 0,01550\ v^2 = 0,0155 \,.\, 36^2 = 21,638$ feet; hence the whole space up the inclined plane: $s = \frac{s_1}{sin.\ a} = \frac{21,638}{sin.\ 48°} = 28,494$ feet. The time required is:

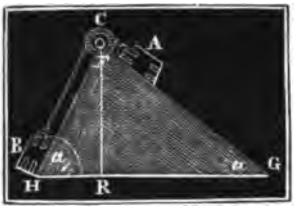
$$t = \frac{2 \,.\, s}{v} = \frac{2 \,.\, 28,499}{36} = \frac{28,494}{18} = 1,583 \text{ seconds.}$$

§ 242. Sliding friction exerts a considerable influence upon the ascent and descent of a body along an inclined plane. From the weight $G$ of the body, and from the angle of inclination $a$ of the inclined plane, the normal pressure follows, $N = G\ cos.\ a$, and again from this, the friction $F = f\,N = f\ G\ cos.\ a$. If we subtract this from the force $P = G\ sin.\ a$, with which the gravity urges the body down the plane, there then remains for the moving force $= G\ sin.\ a - f\ G\ cos.\ a$, and the accelerating force of the body sliding down the plane is known:

$$p = \frac{\text{the force}}{\text{the mass}} = \frac{G\ sin.\ a - f\ G\ cos.\ a}{G}\ g = (sin.\ a - f\ cos.\ a)\ g.$$

The moving force of the body ascending the inclined plane is negative and $= G\ sin.\ a + f \,.\, G\ cos.\ a$, hence also the accelerating force $p$ is negative and $= -(sin.\ a + f\ cos.\ a)\ g$.

FIG. 297.




If two bodies are supported on different planes $FG$ and $FH$, Fig. 297, by perfectly flexible strings connected with each other, passing over a roller $C$, it is then possible for one of the two bodies to descend and pull up the other. If we represent the weights of these bodies by $G$ and $G_1$, and the angles of inclination of the inclined planes along which they move by $a$ and $a_1$, and if we assume that $G$ descends and draws $G_1$ upwards, we shall then obtain as the moving force:

$G \sin. a - G_1 \sin. a_1 - f G \cos. a - f G_1 \cos. a_1 = G (\sin. a - f \cos. a)$
$- G_1 (\sin. a_1 + f \cos. a_1)$, and the mass moved $= \frac{G + G_1}{g}$, hence the accelerated motion with which $G$ descends and $G_1$ ascends:

$$p = \frac{G(\sin. a - f \cos. a) - G_1 (\sin. a_1 + f \cos. a_1)}{G + G_1} \cdot g.$$

Since friction as a resisting force can generate no motion, it is requisite for the fall of $G$ and the rise of $G_1$, that

$$G (\sin. a - f \cos. a) \text{ be } > G_1 (\sin. a_1 + f \cos. a_1), \text{ therefore}$$

$\frac{G}{G_1} > \frac{\sin. a_1 + f \cos. a_1}{\sin. a - f \cos. a}$. If, on the other hand, $G_1$ descend, and $G$ be drawn up, then must:

$$\frac{G_1}{G} \text{ be } > \frac{\sin. a + f \cos. a}{\sin. a_1 - f \cos. a_1}, \text{ or, } \frac{G}{G_1} < \frac{\sin. a_1 - f \cos. a_1}{\sin. a + f \cos. a}.$$

So long, however, as $\frac{G}{G_1}$ lies within the limits:

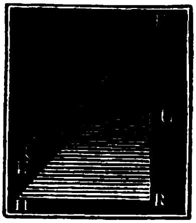
$$\frac{\sin. a_1 + f \cos. a_1}{\sin. a - f \cos. a}, \text{ and } \frac{\sin. a_1 - f \cos. a_1}{\sin. a + f \cos. a},$$

so long will the friction resist motion.

*Examples.*—1. A sledge moves down an inclined snow plane, 150 feet long and 20° inclination, and when arrived at the bottom, proceeds along a horizontal one until friction brings it to rest. If the co-efficient of friction between the snow and the sledge be taken $= 0{,}03$ feet, what space will the sledge describe along the horizontal plane, neglecting the resistance of the air? The accelerating force $p = (\sin. a - f \cos. a)\, g = (\sin. 20^0) \,.\, 32{,}2 = (0{,}3420 - 0{,}03 \,.\, 0{,}9397) \,.\, 32{,}2$ $= 0{,}3138 \,.\, 32{,}2 = 10{,}104$ feet; hence, the final velocity of descent is, $v = \sqrt{2ps} = \sqrt{2 \,.\, 10{,}104 \,.\, 150} = \sqrt{3031{,}2} = 55{,}54$ feet. On the horizontal plane the accelerating force is $p_1 = -fg = -0{,}03 \,.\, 32{,}2 = 0{,}966$ feet; hence, the

space $s_1 = \frac{v^2}{2fg} = \frac{3031,2}{1,932} = 1630$ feet. The time of descent is $t = \frac{2s}{v} = \frac{300}{55,54}$ $= 5,22$ seconds, and that for sliding onward $t_1 = \frac{2s_1}{v} = \frac{3260}{55,54} = 58,6$ seconds; hence the whole time of the course $t + t_1 = 63,82$ seconds $= 1' \, 3,82''$.

FIG. 298.

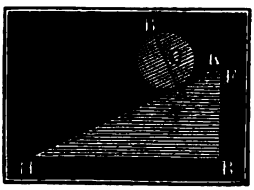


—2. A filled tub $K$, Fig. 298, of 250 lbs. clear weight, is drawn up an inclined plane $FH$, 70 feet long and of 50° inclination, by a descending weight $G$ of 260 lbs.; what will be the time required for this, if the co-efficient of friction of the tub along its path amount to 0,36. The moving force is $= G - (\sin. a + f \cos. a) \, K = 260 - (\sin. 50° + 0,36 \, . \, \cos. 50°) \, . \, 250 = 260 - 0,9974 \, . \, 250 = 10,6$ lbs.; hence, the accelerating force $p = \frac{10,6}{250 + 260} = \frac{10,6}{510}$ $= 0,0208$ feet; further, the time $t = \sqrt{\frac{2s}{p}} = \sqrt{\frac{140}{0,0208}} = \sqrt{6731} = 82,04$ seconds $= 1' \, 22''$, and the final velocity $v = \frac{2s}{t} = \frac{140}{82} = 1,70$ feet.

§ 243. *Rolling motion.*—When a carriage rolls down an inclined plane, the friction of the axle chiefly acts in opposition to the accelerating force; if $r$ be the radius of the axle, and $a$ that of the wheel, the friction will amount to $\frac{fr}{a} N = \frac{fr}{a} G \cos. a$, and hence the accelerating force $p = (\sin. a - \frac{fr}{a} \cos. a) \, g$

If a round body $AB$, a cylinder or sphere, for example, roll down an inclined plane $FH$, Fig. 299, we have to consider a progressive and a rotatory motion at the same time. Generally the acceleration of the progression is equal to that of the rotation (§ 156); since if we put the moment of inertia of the rolling body $= G y^2$, and the radius of the cylinder $= a$, we shall then obtain for the force $AK = K$, with which the cylinder is set into revolution by virtue of the penetration of its parts into those of the inclined plane: $K = p \, . \, \frac{G y^2}{g a^2}$. But the force $K$ acts opposed to the force for descent $G \sin. a$, hence it follows that the moving force for progressive motion $= G \sin. a - K$, and the accelerating force $p = \frac{G \sin. a - K}{G} \, . \, g$. If we eliminate

FIG. 299.



$K$ from both equations, we shall obtain $Gp = Gg \sin. a - \frac{Gy^2}{a^2} \cdot p$, consequently the accelerating force sought:

$$p = \frac{g \sin. a}{1 + \frac{y^2}{a^2}}.$$

For the case of a homogeneous rolling cylinder $y^2 = \frac{1}{2} a^2$ (§ 221), hence $p = \frac{g \sin. a}{1+\frac{1}{2}} = \frac{2}{3} g \sin. a$; but for a sphere $y^2 = \frac{2}{5} a^2$ (§ 222), hence $p = \frac{g \sin. a}{1+\frac{2}{5}} = \frac{5}{7} g \sin. a$; therefore, the accelerating force of the rolling cylinder is only $\frac{2}{3}$, that of a rolling sphere only $\frac{5}{7}$ that of a body sliding without friction.

The force of rotation is:

$$K = \frac{g \sin. a}{1 + \frac{y^2}{a^2}} \cdot \frac{Gy^2}{g a^2} = \frac{G y^2 \sin. a}{a^2 + y^2}.$$

As long as this is less than the sliding friction $f G \cos. a$, the body descends rolling perfectly down the plane. But if

$$K \text{ is } > f G \cos. a, \text{ i. e. } \tang. a > f\left(1 + \frac{a^2}{y^2}\right),$$

the friction is no longer sufficient to communicate to the body a velocity of rotation equal to its velocity of progression; hence the acceleration of progression, as for sliding friction, is:

$$p = \frac{G \sin. a - f G \cos. a}{G} \cdot g = (\sin. a - f \cos. a)\, g,$$

and that of rotation:

$$p_1 = \frac{f G \cos. a}{Gy^2 : a^2} \cdot g = f \frac{a^2}{y^2} g \cos. a.$$

For a carriage of the weight $G$ with wheels of the radius $a$, and with the moment of inertia $G_1 y^2$ we have:

$$K = p \frac{G_1 y^2}{g a^2} \text{ and } p = \frac{G \sin. a - f\frac{r}{a} G \cos. a - K}{G} \cdot g, \text{ i. e.}$$

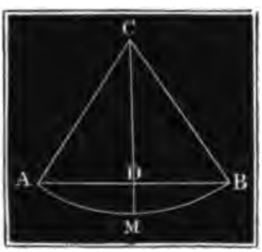
$$p = \frac{g\left(\sin. a - f\frac{r}{a} \cos. a\right)}{1 + \frac{G_1 y^2}{Ga^2}}.$$

*Examples.*—1. A loaded waggon of 3600 lbs. weight, with wheels 4 feet high, and moment of inertia 2000 ft. lbs. rolls down an inclined plane of 12° inclination, what will be its accelerated motion, if the co-efficient of axle friction = 0,15, and the thickness of the axles of the wheels amounts to 3 inches?

It is $\frac{G_1 y^2}{G a^2} = \frac{2000}{3600 \cdot 2^2} = \frac{5}{36} = 0{,}139$ and $f \frac{r}{a} = 0{,}15 \cdot \frac{1}{4 \cdot 4} = 0{,}0094$, hence the accelerating force sought is $p = \frac{32{,}2\,(\sin. 12° - 0{,}0094 \cdot \cos. 12°)}{1 + 0{,}139}$ $= \frac{32{,}2\,(0{,}2079 - 0{,}0094 \cdot 0{,}978)}{1{,}139} = \frac{32{,}2 \cdot 0{,}1987}{1{,}139} = 6{,}398$ feet.—2. What will be the accelerating forces of a olid cylinder rolling down an inclined plane of a 40° angle of descent? The co-efficient of the sliding friction of the cylinder on the plane = 0,24, we have then $f\left(1 + \frac{a^2}{y^2}\right) = 0{,}24\,(1 + 2) = 0{,}72$; but now the $tang.\ 40° = 0{,}839$, hence the $tang.\ \alpha$ is greater than $f\left(1 + \frac{a^2}{y^2}\right)$, and the acceleration of the rolling motion less than that of the progressive. The last is $p = (\sin. \alpha - f \cos. \alpha)\, g = (0{,}6428 - 0{,}24 \cdot 0{,}7660) \cdot 32{,}2 = 0{,}459 \cdot 32{,}2 = 14{,}78$ feet, but the first only $p_1 = 0{,}24 \cdot 2 \cdot 32{,}2 \cos. 40° = 11{,}85$ feet.

§ 244. *Circular pendulum.*—Equilibrium subsists in a body suspended to a horizontal axis so long as its centre of gravity lies vertically below the axis; but if its centre of gravity be drawn out of the vertical plane containing the axis, and the body be left to itself, it will take up an oscillatory motion; that is, it will move up and down in a circle. In general, however, a body oscillating about a horizontal axis is called a *pendulum*. If the oscillating body is a material point, and its connexion with the axis of revolution be made by a line devoid of weight, we then have the *mathematical* or *simple pendulum*; but if the pendulum consists of a body having dimensions, or of several bodies, we have then a *compound*, *physical*, or *material pendulum*. Such a pendulum may be regarded as a connexion of simple pendulums oscillating about a common axis. The simple pendulum is an imaginary one only, but its assumption possesses great advantage, because it is easy to reduce the theory of the motion of the compound to that of the simple pendulum.

FIG. 300.

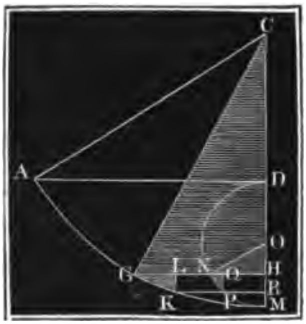



If the pendulum suspended at $C$, Fig. 300, be drawn out of its vertical position $CM$ into the position $CA$, and then left to itself, it will go back by virtue of its gravity with an accelerated motion towards $CM$, and its mass will arrive at its lowest point $M$ with a velocity $v$, whose height $\frac{v^2}{2g}$ is equal to the height of descent $DM$. In virtue

of this velocity it will now describe on the other side the arc $MB = MA$, and will thereby ascend to the height $DM$. From $B$ it will again fall back to $M$ and $A$, and so it will go on successively describing the circular arc $AB$. If the resistance of the air and friction were entirely set aside, this oscillation of the pendulum would go on indefinitely, but because these resistances can never be done away with, the amplitude of the oscillation will become smaller and smaller, and the pendulum come at last to a state of rest.

The motion of the pendulum from $A$ to $B$ is called an *oscillation*, the arc $AB$ the *amplitude*, the angle measuring half the amplitude by which the pendulum is distant from either side of the vertical $CM$, *the angle of elongation* or *angle of deviation*. Lastly, the time in which the pendulum makes an oscillation, is called the *time of oscillation*.

FIG. 301.




§ 245. On account of the frequent application of the pendulum to the purposes of life, to clocks namely, it is of consequence to know the times of oscillation, hence the determination of these is one of the principal problems in mechanics. With the view of solving this problem, let us put the length of the pendulum $AC = MC = r$, Fig. 301, and the height of ascent or descent corresponding to a complete oscillation $MD = h$. Let us assume that the pendulum has fallen from $A$ to $G$, and let the height of fall $DH = x$ correspond to this motion, we may then put the acquired velocity $v = \sqrt{2gx}$, and the particle of time in which the particle of space $GK$ is described, $\tau = \frac{GK}{v} = \frac{GK}{\sqrt{2gx}}$. If, now, from the centre $O$ of $MD = h$ and the radius $OM = OD = \frac{1}{2}h$, we describe the semicircle $MND$, we then have a portion of this arc $NP$ of the height $PQ = KL = RH$ equal to $GK$, which is in a simple ratio to this particle of space $GK$. From the similarity of the triangles $GKL$ and $CGH$, $\frac{GK}{KL} = \frac{CG}{GH}$, and from the similarity of the triangles $NPQ$ and $ONH$, $\frac{NP}{PQ} = \frac{ON}{NH}$; hence, if we divide

these two equations by each other, and bear in mind that $KL = PQ$, we then obtain the ratio of the said portion of arc: $\frac{GK}{NP} = \frac{CG \,.\, NH}{GH \,.\, ON}$. From the properties of the circle, and from the theory of the mean proportional, $\overline{GH}^2 = MH\,(2\,CM - MH)$ and $\overline{NH}^2 = MH \,.\, DH$; hence it follows:

$$\frac{GK}{NP} = \frac{CG \,.\, \sqrt{DH}}{ON \,.\, \sqrt{2\,CM - MH}} = \frac{r\sqrt{x}}{\frac{1}{2}h\sqrt{2r - (h - x)}},$$

and the time for describing an element of space is:

$$\tau = \frac{r\sqrt{x}}{\frac{1}{2}h\sqrt{2r - (h - x)}} \cdot \frac{NP}{\sqrt{2gx}} = \frac{2r}{h\sqrt{2g\,[2r - (h - x)]}} \,.\, NP$$

$$= \sqrt{\frac{r}{g}} \cdot \frac{NP}{h\sqrt{1 - \frac{h - x}{2r}}}.$$

In most cases of application, a small angle of deviation is given to the pendulum, and for this reason $\frac{h}{2r}$, as also $\frac{x}{2r}$, and therefore also $\frac{h - x}{2r}$ is so small a quantity that we may neglect it as well as its powers, and now put $\tau = \sqrt{\frac{r}{g}} \cdot \frac{NP}{h}$. The duration of a semi-oscillation, or the time in which the pendulum describes the arc $AM$, is equal to the sum of all the particles of time corresponding to the elements $GK$ or $NP$, or as $\frac{1}{h} \cdot \sqrt{\frac{r}{g}}$ is a constant factor, equal to $\frac{1}{h}\sqrt{\frac{r}{g}}$ times the sum of all the elements forming the semicircle $DNM$; *i. e.* $= \frac{1}{h}\sqrt{\frac{r}{g}}$ times the semicircle $\left(\frac{\pi h}{2}\right)$ itself, therefore

$$= \frac{1}{h}\sqrt{\frac{r}{g}} \cdot \frac{\pi h}{2} = \frac{\pi}{2}\sqrt{\frac{r}{g}}.$$

The pendulum, however, requires the same time for ascending, for here the velocities are the same, and only opposite in direction, and for this reason the duration of a complete oscillation is twice as great,

$$i.\ e.\ t = \pi\sqrt{\frac{r}{g}}.$$

§ 246. To determine the duration of an oscillation with greater

accuracy, which is necessary where the angles of oscillation are large, let us transform the expression:

$$\frac{1}{\sqrt{1-\frac{h-x}{2r}}} = \left(1-\frac{h-x}{2r}\right)^{-\frac{1}{2}} \text{ into the series}$$

$$1 + \tfrac{1}{2}\cdot\frac{h-x}{2r} + \tfrac{1}{8}\cdot\left(\frac{h-x}{2r}\right)^2 + \ldots,$$

and we shall obtain the time for an element of space

$$\tau = \left[1 + \tfrac{1}{2}\cdot\frac{h-x}{2r} + \tfrac{3}{8}\left(\frac{h-x}{2r}\right)^2 + \ldots)\right]\sqrt{\frac{r}{g}}\cdot\frac{NP}{h}.$$

If we put the angle $NOM$, subtended at the centre by $NM$, $= \phi$, we shall then also obtain

$$MH = h - x = NO\,(1-\cos.\phi) = \tfrac{1}{2}\,h\,(1-\cos.\phi);\text{ hence:}$$

$$\tau = \left(1+\tfrac{1}{2}\cdot\frac{h\,(1-\cos.\phi)}{4r} + \ldots\right)\sqrt{\frac{r}{g}}\cdot\frac{NP}{h}.$$

If we divide the semicircle $DNM$ into $n$ equal parts, and if we put each $= NP = \frac{\pi h}{2n}$, we shall obtain

$$\tau = \left(1 + \tfrac{1}{2}\cdot\frac{h\,(1-\cos.\phi)}{4r} + \ldots\right)\sqrt{\frac{r}{g}}\cdot\frac{\pi}{2n};$$

by substituting successively for $\phi = \frac{\pi}{n}, \frac{2\pi}{n}, \frac{3\pi}{n} \ldots$ to $\frac{n\pi}{n}$, and adding the results, we shall then obtain half the time of an oscillation:

$$t = \left(n + \frac{h}{8r}\,(n - \text{the sum of all the cosines}) + \ldots\right)\sqrt{\frac{r}{g}}\cdot\frac{\pi}{2n}.$$

But the sum of the cosines of all the angles from $\phi = o$ to $\phi = \pi$ is $=0$; hence, we have more correctly: $t = \left(1+\frac{h}{8r}\right)\cdot\pi\sqrt{\frac{r}{g}}$.

If we have regard to more members of the series, we shall obtain:

$$t = \left[1 + (\tfrac{1}{2})^2\cdot\frac{h}{2r} + \left(\frac{1.3}{2.4}\right)^2\left(\frac{h}{2r}\right)^2 + \left(\frac{1.3.5}{2.4\,6}\right)^2\cdot\left(\frac{h}{2r}\right)^3 + \ldots\right]\cdot\pi\sqrt{\frac{r}{g}},$$

the last formula but one is, however, generally sufficient. If the pendulum oscillates in a semicircle, we then have $h = r$; hence the duration of an oscillation:

$$t = \left(1 + \tfrac{1}{8} + \frac{9}{256} + \frac{225}{18432} + \ldots\right)\pi\sqrt{\frac{r}{g}} = 1{,}180\ldots\pi\sqrt{\frac{r}{g}}.$$

From the angle of elongation $a$, it follows that

$cos.\, a = \frac{r-h}{r} = 1 - \frac{h}{r}$, therefore, $\frac{h}{r} = 1 - cos.\, a$; and hence,

$\frac{h}{8r} = \frac{1}{4} \cdot \frac{1 - cos.\, a}{2} = \frac{1}{4}\left(sin.\, \frac{a}{2}\right)^2$; from this, consequently, the correction for the time of oscillation corresponding to a given angle of elongation may be found. If, for example, this angle $= 15^0$, we have $\frac{h}{8r} = \frac{1}{4}\left(sin.\, \frac{15^0}{2}\right)^2 = 0{,}00426$; on the other hand, for $a = 5^0 : \frac{h}{8r} = 0{,}00047$; for the latter angle of elongation, therefore, the time of oscillation is $t = 1{,}00047 \,.\, \pi \sqrt{\frac{r}{g}}$.

We may, therefore, for a deviation under $5^0$ put tolerably accurately the time of oscillation:

$$t = \pi \sqrt{\frac{r}{g}} = \frac{\pi}{\sqrt{g}} \sqrt{r} = 0{,}562 \sqrt{r}.$$

§ 247. As the angle of deviation does not appear in the formula $t = \pi \sqrt{\frac{r}{g}}$, it follows that the small times of oscillation of pendulums are independent of this angle, and therefore that pendulums of equal length, but of different angles of deviation, vibrate isochronously, or perform their oscillations in equal times. A pendulum deviating $4^0$ has the same time of oscillation as a pendulum deviating $1^0$.

If we compare the time of oscillation $t$ with the time of free descent, we shall then arrive at the following. The time of free descent from the height $r$ will be $t_1$

$$= \sqrt{\frac{2r}{g}} = \sqrt{2} \cdot \sqrt{\frac{r}{g}}, \text{ hence } t : t_1 = \pi : \sqrt{2};$$

the time of an oscillation is, therefore, to the time in which a body of the length of the pendulum ~~freely descends~~, as $\pi$ to the square root of 2, or since $t_1$ is also $= \sqrt{\frac{4 \,.\, \frac{1}{2} r}{g}} = 2 \sqrt{\frac{\frac{1}{2} r}{g}}$, the *time of ~~oscillation is to the time of descent of half the length of the pendulum as π is to 2~~.*

If we put the times of oscillation $t$ and $t_1$, corresponding to

the lengths of the pendulum $r$ and $r_1$, we then obtain $t : t_1 = \sqrt{r} : \sqrt{r_1}$; therefore, *for one and the same acceleration of gravity, the times of oscillation are as the square roots of the lengths of the pendulum.* On the other hand, if $n$ be the number of oscillations which a pendulum makes in a certain time, one minute, and $n_1$ the number which another pendulum makes in the same time, we then have $t : t_1 = \frac{1}{n} : \frac{1}{n_1}$, hence, inversely, $n : n_1 = \sqrt{r_1} : \sqrt{r}$, i. e. *the number of oscillations is in an inverse ratio to the square roots of the pendulum.* A pendulum four times the length gives, therefore, half the number of oscillations.

A pendulum is called a *seconds pendulum,* when its time of oscillation is one second. If we put $t = 1$ into the formula $t = \pi\sqrt{\frac{r}{g}}$, we obtain the length of the seconds pendulum

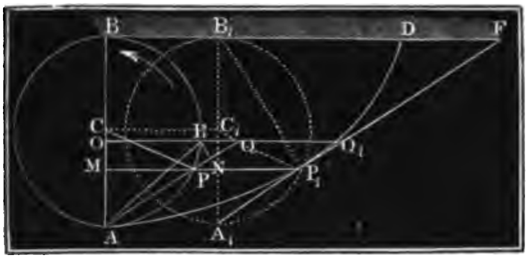
$$r = \frac{g}{\pi^2} = 39{,}13929 \text{ inches} = 0{,}9938 \text{ meters}.$$

From the formula $t = \pi\sqrt{\frac{r}{g}}$ it follows by inversion that $g = \left(\frac{\pi}{t}\right)^2 r$; the acceleration of gravity may be found, therefore, from the length of a pendulum, and from its time of oscillation $t$. This method is both simpler and safer than that of Attwood's machine.

*Remark.* The diminution of gravity from the poles to the equator has been proved by pendulum observations, and its quantity determined. This diminution is due to the effect of the centrifugal force, which is generated by the diurnal rotation of the earth about its axis, and to the increase of the earth's radius from the poles to the equator. The centrifugal force at the equator diminishes by $\frac{1}{290}$ of its value (§ 231), whilst at the poles it is null. If $\beta$ be the latitude of the place of observation, the accelerating force of gravity from pendulum observations will be $g' = 32{,}2\ (1 - 0{,}00259\ \cos.\ 2\ \beta)$, therefore at the equator where $\beta = 0$; therefore $\cos.\ 2\ \beta = 1$, $g = 32{,}2\ (1 - 0{,}00259) = 32{,}18$ feet, and at the poles, where $\beta = 90°$; therefore $\cos.\ 2\ \beta = \cos.\ 180° = -1$; $g'' = 32{,}2\ .\ 1{,}00259 = 32{,}283$ ft. For the rest $g$ is less on mountains and in mines than at the level of the sea.

§ 248. *Cycloid.*—We may in an infinite number of ways set a body into vibration, or into an oscillating motion, and we call every body in this condition of motion a *pendulum,* and distinguish accordingly several kinds of pendulums, for example, the *circular pendulum,* which we have already considered, and the *cycloidal,*

FIG. 302.

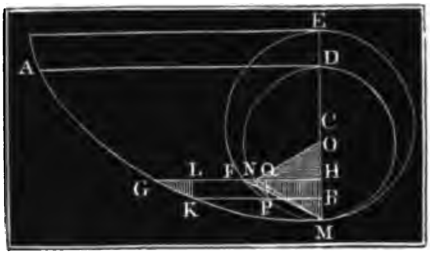



where the body, by virtue of gravity, oscillates to and fro in a cycloidal arc, and the *torsion pendulum*, where the body vibrates by virtue of the torsion of a thread, or wire, &c. We shall here speak only of the *cycloidal pendulum*.

The *cycloid AD*, Fig. 302, is a curved line described by a point $A$ of a circle $APB$ which rolls along a straight line $BD$. If this generating circle has rolled forward $BB_1 = CC_1$, and, therefore, come into the position $A_1B_1$, it has then also revolved through the arc $AP = A_1P_1 = BB_1 = PP_1$, consequently the ordinate corresponding to any absciss $MP_1$ = ordinate $MP$ of the circle plus the arc of revolution $AP$. In this rolling the generating circle revolves about the point of contact at each instant with the base line, if, therefore, it be in $A_1B_1$, it will then revolve about $B_1$, and describe thereby the elementary arc $P_1Q_1$ of the cycloid; consequently the chord $B_1P_1$ will be the direction of the normal, and the chord $A_1P_1$ that of the tangent to the cycloid at the point $P_1$. The prolongation $PQ$ of the chord $AP$ reaching to the ordinate $OQ_1$ is, therefore, equal to the element of the cycloid $P_1Q_1$, as further the space of revolution is equal to the space $RQ$ of progression, $PQ$ is then the base line of an isosceles triangle $PRQ$, and equal to double the line $PN$, which the perpendicular $RN$ cuts off, but $PN$ is the difference of the two contiguous chords $AP$, $AR$, and consequently the element of the cycloid $P_1Q_1$ = twice the difference of the chords ($AR$—$AP$).

As the continuous elementary arcs make up together the whole arc $AP_1$, and likewise the aggregate of the differences of the chords, the whole chord $AP$, the length of the cycloidal arc $AP_1$ is from this equal to double the chord of the circle $AP$, appertaining to it. To the semi-cycloid $AP_1D$, corresponds the diameter as a chord of a circle, hence the length of the half of the cycloid is equal to double the diameter of the generating circle.

FIG. 303.




§ 249. ***Cycloidal pendulum.***—From the above known properties of the cycloid, the theory of the cycloidal pendulum, or the formula for the time of oscillation of a body vibrating in a cycloidal arc, may be easily developed. Let $AKM$, Fig. 303, be the half of the cycloidal arc in which a body ascends or descends, or oscillates, and $ME$ the generating circle, therefore, $CE = CM = r$ its radius. If the body has described the arc $AG$, it has, therefore, fallen from the height $DH = x$ (§ 246), it has then acquired the velocity $v = \sqrt{2\,gx}$, with which it describes the elementary arc $CK$ in the time $\tau = \frac{GK}{v} = \frac{GK}{\sqrt{2\,gx}}$.

But from the similarity of the triangles $GLK$ and $FHM$, $\frac{GK}{KL} = \frac{FM}{MH}$, or, as $\overline{FM}^2 = MH\,.\,ME$, $\frac{GK}{KL} = \frac{\sqrt{MH\,.\,ME}}{MH}$ $= \frac{\sqrt{ME}}{\sqrt{MH}}$; from the similarity of the triangles $NPQ$ and $ONH$, $\frac{NP}{PQ} = \frac{ON}{NH}$, or, since $\overline{NH}^2 = MH\,.\,DH$, $\frac{NP}{PQ} = \frac{ON}{\sqrt{MH\,.\,DH}}$.

Now $KL = PQ$, hence it follows by division:

$$\frac{GK}{NP} = \frac{\sqrt{ME}}{\sqrt{MH}} \cdot \frac{\sqrt{MH\,.\,DH}}{ON} = \frac{\sqrt{ME\,.\,DH}}{ON},$$

or, since $ON$ is half the height of descent $= \frac{h}{2}$, $ME = 2r$, and $DH = x$:

$$\frac{GK}{NP} = \frac{\sqrt{2\,rx}}{\frac{1}{2}\,h} = \frac{2\sqrt{2\,rx}}{h}.$$

By putting $GK = \frac{2\sqrt{2\,rx}}{h}\,.\,NP$ into the formula $\tau = \frac{GK}{\sqrt{2\,gx}}$ we obtain:

$$\tau = \frac{2\sqrt{2\,rx}}{\sqrt{2\,gx}\,.\,h}\,.\,NP = \frac{2}{h}\sqrt{\frac{r}{g}}\,.\,NP.$$

The time of falling from $A$ to $M$ is the sum of all the values of $\tau$, which are obtained; if for $NP$ all the particles of the semi-circle $DNM$ be successively substituted, therefore, $= \frac{2}{h}\sqrt{\frac{r}{g}}$ times the semi-circle $DNM\left(\frac{\pi}{2}h\right)$. In this manner we obtain the time for falling through the arc $AM$,

$$= \frac{\pi}{2} h \cdot \frac{2}{h}\sqrt{\frac{r}{g}} = \pi\sqrt{\frac{r}{g}},$$

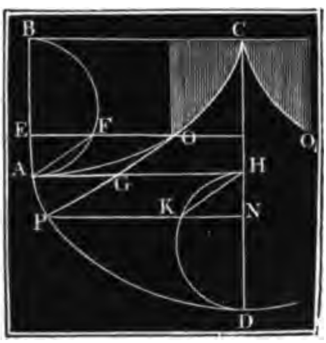
and as the time for ascending the arc $MB$ is likewise as great, the time of oscillation, or the time of describing the whole arc $AMB$:

$$t = 2\pi\sqrt{\frac{r}{g}} = \pi\sqrt{\frac{4r}{g}}.$$

As this quantity is quite independent of the length of the arc, it follows that, mathematically speaking, the times of oscillation for all arcs of one and the same cycloid are equal, the oscillations of the cycloidal pendulum are, therefore, perfectly isochronous. If we compare this formula with the formula for the time of oscillation of a circular pendulum, it follows that the times of oscillation for both kinds of pendulums are equal, if the length of the circular pendulum is equal to four times the radius of the generating circle of the cycloidal pendulum.

*Remark* 1. It may be proved by the higher calculus that the cycloid has, besides the property of *isochronism* or *tautochronism*, also that of *brachistochronism*, which is that line between two given points in which a body falls in the shortest time from one point to the other.

FIG. 304.




*Remark* 2. In order to make a body, suspended to a perfectly flexible thread, vibrate in a cycloidal arc, and thereby represent the cycloidal pendulum, we suspend the body between two cycloidal arcs $CO$ and $CO_1$, Fig. 304, so that the thread for every deviation unwinds from the one arc and winds round the other. By this winding and unwinding of the thread $COP$, its extremity $P$ describes a curve similar to the given cycloid, and it may be similarly represented that the evolute of the cycloid is a similar cycloid in an inverse position. As the length of half the cycloid $COA = CD = 2\,AB$, we have likewise the arc = to the straight line evolved $OP$; but the arc $OA = 2$ chord $AF = 2\,GO$, hence also $PG = GO = AF$, and $HN = AE$. If now we describe upon $DH$ a semicircle $DKH$ and draw the ordinate $NP$, we then have $KH = PG$; and hence

also $PK = GH = AH - AG = AH - FO =$ arc $AFB -$ arc $AF =$ arc $BF$ $=$ arc $DK$; and lastly, the ordinate $NP =$ the ordinate $NK$ of the circle, plus the corresponding arc $DK$; therefore $NP$ is the ordinate of a cycloid, and $DPA$ the cycloid corresponding to the generating circle $DKH$.

For the application of the cycloidal pendulum to clocks, see *Jahrbücher des polytech.* Institutes in Wien. vol. xx. art. 2.

FIG. 305.


§ 250. *Compound pendulum.*—To find the time of oscillation of the compound pendulum, or that of any other body $AB$, Fig. 305, oscillating about a horizontal axis $C$, let us first seek the centre of oscillation, *i. e.* that point $K$ of the body, which if it oscillates of itself about $C$, or forms a mathematical pendulum, has the same time of oscillation as the whole body. It is easily seen from this explanation that there are several centres of oscillation in a body, but in general, that point only is meant which lies with the centre of gravity, in one and the same perpendicular to the axis of revolution.

From the variable angle of deviation $KCF = \phi$, the accelerating force of the isolated point $K$, $= g \sin. \phi$, because we may suppose that it slides down an inclined plane of the inclination $KHR = KCF$. But if $My^2$ be the moment of inertia of the entire body or set of bodies $AB$, $Ms$ will be its statical moment, *i. e.* the product of the mass, and the distance $CS = s$ of its centre of gravity $S$ from the axis of revolution $C$, and $r$ the distance $CK$ of the centre of oscillation $K$ from the axis of revolution, or the length of the simple pendulum which vibrates isochronously with the material pendulum $AB$, we have then the mass reduced to $K = \frac{My^2}{r^2}$, and the force of revolution reduced to this $\frac{s}{r} M g \sin. \phi$; consequently the accelerating force $= \frac{\text{force}}{\text{mass}} = \frac{s}{r} M g \sin. \phi \div \frac{My^2}{r^2} = \frac{Msr}{My^2} \cdot g \sin. \phi$. That this pendulum may have the same time of oscillation as a mathematical one, it is requisite that both should have their motion in every position equally accelerated, that therefore, $\frac{M s r}{M y^2} \cdot g \sin. \phi = g \sin. \phi$. Now this equation gives:

$$r = \frac{My^2}{Ms} = \frac{\text{moment of inertia}}{\text{statical moment}}.$$

*We therefore find the distance of the centre of oscillation from the centre of gyration, or the length of the simple pendulum, which has a time of oscillation equal to that of the compound one, if we divide the moment of inertia of the compound pendulum by its statical moment.*

If we substitute this value in the formula $t = \pi \sqrt{\frac{r}{g}}$, we obtain for the time of oscillation of the compound pendulum the formula $t = \pi \sqrt{\frac{My^2}{Mgs}} = \pi \sqrt{\frac{y^2}{gs}}$, or more accurately $= \left(1 + \frac{h}{8r}\right) \pi \sqrt{\frac{y^2}{gs}}$. Inversely, the moment of inertia may be found from the time of oscillation of a suspended body, if we put:

$$My^2 = \left(\frac{t}{\pi}\right)^2 . Mgs \text{ or } y^2 = \left(\frac{t}{\pi}\right)^2 gs.$$

FIG. 306.

*Examples.*—1. For a uniform prismatic rod $AB$, Fig. 306, whose centre of oscillation is distant $CA = l_1$ and $CB = l_2$ from the extremities $A$ and $B$, we have for (§ 219) the moment of inertia: $My^2 = \frac{1}{3} F(l_1^3 + l_2^3)$, and the statical moment $Ms = \frac{1}{2} F(l_1^2 - l_2^2)$; hence, the length of the mathematical pendulum which vibrates isochronously with this rod is $r = \frac{My^2}{Ms} = \frac{2}{3} . \frac{l_1^3 + l_2^3}{l_1^2 - l_2^2} = \frac{l^2 + 3d^2}{6d}$, if $l$ represent the sum $l_1 + l_2$, and $d$ the difference $l_1 - l_2$. If this rod beats half seconds, we have $r = \frac{1}{4} . \frac{g}{\pi^2} = \frac{1}{4} . 10132 . 32{,}2 = 8156$ feet $= 9{,}737$ inches, but if the entire length $l$ of the rod amount to 12 inches, we must then put: $9{,}737 = \frac{144 + 3d^2}{6d}$ or $d^2 - 19d = -48$ nearly; hence it follows:

FIG. 307. $d = \frac{19 - \sqrt{169}}{2} = 3$ nearly; and from this

$l_1 = \frac{l+d}{2} = \frac{15}{2} = 7\frac{1}{2}$ inches, and $l_2 = \frac{l-d}{2} = \frac{9}{2} = 4\frac{1}{2}$ inches.

—2. For a pendulum with a spherical lenticular bob $AB$, Fig. 307, if $G$ be the weight and $l$ the length $CA$ of the rod or thread; $K$, on the other hand, the weight of the bob, and $\rho$ its radius $MA = MB$:

$$r = \frac{\frac{1}{3} Gl^2 + K[(l + \rho)^2 + \frac{2}{5} \rho^2]}{\frac{1}{2} Gl + K(l + \rho)}.$$

If, now, the wire is 0,05 lbs., the bob 1,5 lbs., further, the length of the rod 1 foot, and the radius of the bob 1,15 inches, we then have the distance of the centre of oscillation of this pendulum from the axis of rotation:

$$r = \frac{\frac{1}{3} . 0{,}05 . 12^2 + 1{,}5 . (13{,}15^2 + \frac{2}{5} . 1{,}15^2)}{\frac{1}{2} . 0{,}05 . 12 + 1{,}5 . 13{,}15} = \frac{2{,}4 + 260{,}177}{0{,}3 + 19{,}725} = \frac{262{,}577}{20{,}025}$$

$= 13{,}112$ inches. Disregarding the rod, $r$ would $= \frac{262{,}577}{19{,}725} = 13{,}312$ inches; and

the inert mass of the bob being reduced to its centre, $r$ would $=$ 13,15 inches. The time of oscillation of this bob is:

$$t = \pi \sqrt{\frac{r}{g}} = 0{,}562 \sqrt{\frac{13{,}112}{12}} = 0{,}562 \sqrt{1{,}0926\ldots} = 0{,}5874 \text{ seconds.}$$

§ 251. The centre of suspension and centre of oscillation of a material pendulum are reciprocal, *i. e.* the one may be interchanged with the other, and the pendulum may be suspended at the centre of oscillation, without the time of oscillation being altered. The proof of this proposition may be given by aid of § 217, in the following manner. If $T$ be the moment of inertia of the compound pendulum $AB$, Fig. 308, oscillating about the centre of gravity $S$, we have then for a revolution about the axis $C$, distant $CS = s$ from the centre of gravity $S$, $T_1 = T + Ms^2$, hence the distance of the centre of oscillation $K$ from the axis of revolution $C$:

FIG. 308.

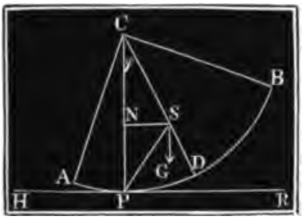

$$r = \frac{T_1}{Ms} = \frac{T + Ms^2}{Ms} = \frac{T}{Ms} + s.$$

If now we represent the distance $KS = r - s$ of the centre of oscillation from the centre of gravity by $s_1$, we then obtain the simple equation $ss_1 = \frac{T}{M}$, in which $s$ and $s_1$ appear in a similar manner, and hence may be substituted one for the other. This formula is not only true for the descent, if $s$ represents the distance of the centre of oscillation from the centre of gravity, but also inversely, if $s$ expresses the distance of the centre of oscillation, and $s_1$ that of the centre of gyration from the centre of gravity, and $C$ will therefore serve for the centre of oscillation if $K$ serve for the centre of suspension. We avail ourselves of this property in the so-called *convertible pendulum* $AB$, Fig. 309, first proposed by Bohnenberger, and afterwards applied by Kater, which is furnished with two knife edges $C$ and $K$, which are so situated with regard to each other, that the times of oscillation remain the same whether the pendulum oscillates about one or the other axis. In order that the axes may not be displaced with regard to each other, two sliding weights $P$ and $Q$ are applied, the smallest of which is

FIG. 309.

attached by a fine screw. If by the shifting or adjustment of these weights, the time of oscillation comes to be the same, the pendulum may be suspended at $C$ or at $K$, we shall then obtain in the distance $CK$ of the two edges, the length $r$ of the simple pendulum which vibrates synchronously with the convertible pendulum, and we shall now obtain the time of oscillation by the formula $t = \pi \sqrt{\frac{r}{g}}$.

FIG. 310.




§ 252. The swinging or rocking of a body with cylindrical base may be compared with the oscillations of a pendulum. This rocking, like every other rolling motion, is composed of a progressive and a rotatory motion, but it may be assumed that it consists of a simple rotatory motion with a variable axis of rotation. This axis of rotation is the point of support $P$, by which the vibrating body $ABC$, Fig. 310, rests on the horizontal base $HR$. If $CD = CP$ is the ~~diameter~~ radius of the rolling base $ADB = r$, and the distance $CS$ of the centre of gravity of the entire body from the centre $C$ of this base $= s$, we have then for the distance corresponding to the angle of rotation $SCP = \phi$, $SP = y$ of the centre of gravity from the centre of gyration:

$$y^2 = r^2 + s^2 - 2\,r\,s\,cos.\,\phi = (r - s)^2 + 4\,r\,s\left(sin.\,\frac{\phi}{2}\right)^2;$$

hence, if further we represent the moment of inertia of the entire body about the centre of gravity $S$ by $Mk^2$, we shall obtain the moment of inertia about the point of support $P$:

$$T = M\,(k^2 + y^2) = M\,[k^2 + (r - s)^2 + 4\,r\,s\left(sin.\,\frac{\phi}{2}\right)^2],$$

which for small angles of vibration may be put

$$= M\,[k^2 + (r - s)^2 + r\,s\,\phi^2],$$

or only $M\,[k^2 + (r - s)^2]$. Since now the moment of force $= G\,.\,SN = Mg\,.\,CS\,sin.\,\phi = M\,g\,s\,sin.\,\phi$, it follows that the angular acceleration for the rotation about $P$:

$$\pi = \frac{\text{moment of force}}{\text{moment of inertia}} = \frac{M\,g\,s\,sin.\,\phi}{M\,[k^2 + (r - s)^2]} = \frac{g\,s\,sin.\,\phi}{k^2 + (r - s)^2}.$$

For the simple pendulum it is $= \frac{g \sin. \phi}{r_1}$, if $r_1$ represent its length; if both are to vibrate isochronously, it is necessary that:

$$\frac{g s \sin. \phi}{k^2 + (r - s)^2} = \frac{g \sin. \phi}{r_1}; \; i.\,e.\; r_1 = \frac{k^2 + (r - s)^2}{s}.$$

The time of the vibration of the balance is from this:

$$t = \pi \sqrt{\frac{r_1}{g}} = \pi \sqrt{\frac{k^2 + (r - s)^2}{g s}}.$$

FIG. 311.


This theory may also be applied to a pendulum $AB$, Fig. 311, with a rounded axis of rotation $CM$, if for $r$ the radius of curvature $CM$ of the axis be substituted. If instead of the rounded axis a knife edge $D$ be applied, the time of vibration will then be

$$t_1 = \pi \sqrt{\frac{k^2 + \overline{DS}^2}{g\,DS}} = \pi \sqrt{\frac{k^2 + (s - x)^2}{g\,(s - x)}},$$

the distance $CD$ of the edge from the centre of the round axis being represented by $x$. Both pendulums have equal times of vibration if

$$\frac{k^2 + (s - x)^2}{s - x} = \frac{k^2 + (r - s)^2}{s}, \text{ or } \frac{k^2}{s - x} - x = \frac{k^2 + r^2}{s} - 2r.$$

If we write $\frac{k^2}{s - x} = \frac{k^2}{s} + \frac{k^2 x}{s^2}$ approximately, and neglect $r^2$, we shall then obtain $x = \frac{2 r s^2}{s^2 - k^2}$.

FIG. 312.


*Remark* 1. In the Second Part, under the article "Regulator," the conical pendulum will be mentioned.

*Remark* 2. *Elastic pendulum.*—Bodies are likewise very often set into vibratory motion by elasticity. A string, or fine wire, $AB$, Fig. 312, is stretched by a weight $G = Mg$. If this weight is carried from the point of repose $C$ to $D$, the string is thereby stretched $CD = r$; and if the weight be afterwards left to itself, it will, by virtue of the elasticity of the string, be raised again to $C$; it will arrive there with a certain velocity, and ascend by its *vis viva* to $E$, from whence it will again fall to $D$ and $C$. In this manner the weight will oscillate a certain time in the space $DE = 2\,CD = 2\,r$ to and fro, and the question now is, as to its duration of oscillation. From the length $AB = l$, transverse section $F$ and modulus of elasticity $E$ of the string, it follows, § 183, that the force to extend it a length $CM = x$ is $P = \frac{x}{l} \cdot FE$; hence, the mechanical effect required to extend it the length $\frac{r}{n}$ is $= \frac{P r}{n} = \frac{x}{l} \cdot \frac{r}{n} FE$.

Let us now put successively $x = \frac{r}{n}, \frac{2r}{n}, \frac{3r}{n}$, &c., and add the corresponding mechanical effects, we shall then obtain the whole mechanical effect for the extension of the string

$$CD = r : L = \frac{r}{n\,l} \cdot FE\left(\frac{r}{n} + \frac{2r}{n} + \dots\right) = \frac{r^2}{n^2\,l}\, FE\,(1 + 2 + \dots + n)$$

$= \frac{r^2}{n^2\,l} FE . \frac{n^2}{2} = \frac{r^2}{2\,l} . FE$; and for the extension $CM = x : L_1 = \frac{x^2}{2\,l} . FE$. If now, inversely, the string be contracted by $DM$, therefore the weight $D$ ascend from $D$ to $M$, *i. e.* $r - x$, it will give the mechanical effect $L - L_1 = \left(\frac{r^2 - x^2}{2\,l}\right) FE$, and communicate to the weight $G$ a velocity $v$ corresponding to the *vis viva* $v^2 M = \frac{v^2}{g} . G$; whence we shall have to put $\frac{v^2}{2g} G = \left(\frac{r^2 - x^2}{2\,l}\right) FE$, and the variable velocity of oscillation will be $v = \sqrt{\frac{FE}{Ml}} \sqrt{r^2 - x^2}$. But now $\sqrt{r^2 - x^2}$ may be put equal to the ordinate $MK = y$ of a semicircle described upon $DE$; hence it follows, more simply, that $v = \sqrt{\frac{FE}{Ml}} . y$, and the instant for describing the particle of space $MN$: $\tau = \frac{MN}{y} \cdot \sqrt{\frac{Ml}{FE}}$. From the similarity of the triangles $KLH$ and $KCM$. $\frac{KH}{KL} = \frac{KM}{KC}$, *i. e.* $\frac{MN}{KL} = \frac{y}{r}$, or $\frac{MN}{y} = \frac{KL}{r}$; hence, it follows that $\tau = \frac{KL}{r}\sqrt{\frac{Ml}{FE}}$; and lastly, the whole time of oscillation, or the time of describing the space $DE$ is : $t = \frac{1}{r}\sqrt{\frac{Ml}{FE}}$ times the sum of all the elements of the semicircle $= \frac{1}{r}\sqrt{\frac{Ml}{FE}}$ times the semicircle

$$\pi r = \frac{\pi r}{r}\sqrt{\frac{Ml}{FE}} = \pi\sqrt{\frac{Ml}{FE}} = \frac{\pi}{\sqrt{g}}\sqrt{\frac{Gl}{FE}}\ .$$

FIG. 313.


If, for example, an iron wire, 20 feet long and 0,1 inch thick, be stretched by a weight $G = 100$ lbs. and set into longitudinal vibration, the duration of the oscillations will then be, since from § 186 $E = 26000000$,

$$t = \frac{\pi}{\sqrt{g}} \cdot \sqrt{\frac{100 . 20}{(0,1)^2 . \frac{\pi}{4} . 26000000}}$$

$$= 0,553 \sqrt{\frac{2}{65 . \pi}} = 0,05464 \text{ seconds.}$$

*Remark* 3. We have a torsion pendulum if a string or wire $CD$, Fig. 313, turns about an arm $AB$ and is brought out of its natural position $MN$ into the position $AB$, and then left to itself. The rod or arm $AB$ is set into vibration by virtue of the torsion of the string, which extends to an equal distance on both sides of $MN$, so that $AM = A_1M$.

If we put the force of torsion for the distance (1) and for the arc of vibration (1) = $K$, it will then be, for the angle of vibration $MCP = \phi^\circ$, $= K\phi$, and the corresponding mechanical effect $\frac{K\phi^2}{2}$; on the other hand, for the entire angle of elongation $MCA = \alpha$, $= L_1 = \frac{K\alpha^2}{2}$. If now the inert mass of the entire pendulum $= M$ be reduced to the distance ($r$), and the angular velocity with which it passes from the position $AB$ into that of $PQ = \omega$, we shall then have $\frac{M\omega^2}{2} = \frac{K(\alpha^2 - \phi^2)}{2}$; and hence, $\omega = \sqrt{\frac{K}{M}(\alpha^2 - \phi^2)}$; and finally, the time of oscillation $t = \pi\sqrt{\frac{M}{K}}$.

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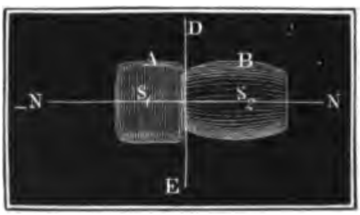
# CHAPTER IV.

## THE DOCTRINE OF IMPACT.

§ 253. *Impact in particular.*—In virtue of the impenetrability of matter, two bodies cannot simultaneously occupy one and the same position. But when two bodies in motion come into contact with one another, so that the one strives to penetrate the space occupied by the other, a reciprocal action takes place, producing a consequent change in the conditions of motion of the two bodies. This reciprocal action is what is called *impact* or *collision*.

The relations of impact depend upon *the law of equality of action and re-action* (§ 62); during impact, the one body presses exactly as forcibly on the other as does this latter in an opposite direction on the former. The straight line, perpendicular to the surfaces in which the two bodies touch, and passing through the point of contact, is the direction of the impact. If the centres of gravity of the two bodies lie within this line, the impact is then called a *centric*; but if without, an *excentric impact*. The bodies $A$ and $B$, in Fig. 314, give a centric impact, because their centres of gravity $S_1$ and $S_2$ lie in the normal $N\overline{N}$ to the plane of contact $DE$;

FIG. 314.



of the bodies $A$ and $B$, Fig. 315, $A$ thrusts centrically, and $B$ excentrically, because $S_1$ lies within, and $S_2$ without the normal line $N\overline{N}$.

With respect to the direction of motion, we distinguish between *direct impact* and *oblique impact*. The direction of motion, in

FIG. 315.

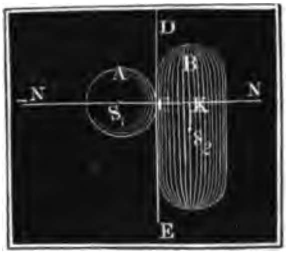
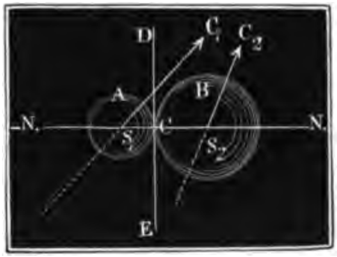



FIG. 316.




the case of direct impact, lies in the line of impact; but in that of oblique impact, there is a deviation between the two directions. If, for example, the bodies $A$ and $B$, Fig. 316, move in the directions $S_1C_1$ and $S_2C_2$, which deviate from the normal or line of impact $N\overline{N}$, an oblique impact will take place; whilst, if the directions coincided with the normal, it would be direct.

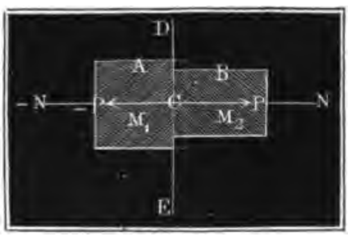
We make the further distinction of *the impact of free bodies* and *the impact of bodies entirely or partially supported.*

§ 254. The time occupied in the communication or change of motion by impact is indeed very small, but by no means indefinitely small; it depends, as well as the impact itself, upon the mass, velocity, and elasticity of the impinging bodies. We may regard this time as consisting of two periods. In the first period the bodies become mutually compressed, and in the second they again partially or entirely extend themselves. Elasticity is brought into action by this compression, and puts itself into equilibrium with the inertia, and thereby alters the state of motion of the impinging bodies. If the limit of elasticity is not exceeded by the compression, the body at the end of the impact perfectly recovers its former figure, and we then call it a *perfectly elastic* body; but if at the end of the impact a disfigurement takes place, we then call it an *imperfectly elastic body;* and lastly, if the body retains its original form, at a maximum pressure, and therefore has no tendency to expansion, we call it an *inelastic body*. At any rate, however, the distinction must only be taken as

correct relatively to a certain strength of impact, for it is possible that one and the same body may show itself elastic to a weak, and inelastic to a stronger impact. Strictly speaking, no body is perfectly elastic or perfectly inelastic; yet we shall, in the sequel, call bodies elastic which nearly recover their form after impact, and those inelastic which undergo a considerable and permanent disfigurement by impact (compare § 181).

In practical mechanics, impinging bodies, such as wood, iron, &c., are generally considered inelastic bodies, because they possess but little elasticity, and by repetition of the blows, lose still further that elasticity. It is a most important rule, moreover, to avoid as far as possible, in machines and constructions, all jars or impacts, or so to moderate their effects as to convert them into elastic ones; because shocks and abrasions would be thereby produced, and a part of the mechanical effect consumed.

FIG. 317.




§ 255. *Inelastic impact.*—Let us in the first place develope the laws of the direct central impact of freely moving bodies. Let us suppose the time of impact to be made up of equal parts $\tau$, and let us assume that the pressure during the first instant is $P_1$, during the second $P_2$, during the third $P_3$, and so on. Let, now, the mass of the one body be $A = M_1$ Fig. 317, we shall then have the corresponding accelerating force:

$$p_1 = \frac{P_1}{M_1},\ p_2 = \frac{P_2}{M_1},\ p_3 = \frac{P_3}{M_1},\ \&c.;$$

but from § 19, the change of velocity due to the accelerating force $p$ and particle of time $\tau$ is $\kappa = p\,\tau$; hence, for the ensuing fall, we shall have the elementary increment or decrement: $\kappa_1 = \frac{P_1 \tau}{M_1}$, $\kappa_2 = \frac{P_2 \tau}{M_1}$, $\kappa_3 = \frac{P_3 \tau}{M_1}$, &c., and the consequent increment or decrement of velocity of the mass $M_1$ in a given finite time $\kappa_1 + \kappa_1 + \kappa_2 + \ldots = (P_1 + P_2 + P_3 + \ldots)\frac{\tau}{M_1}$, as also the consequent change of velocity of the mass $B$ of the magnitude $M_2 :\ = (P_1 + P_2 + P_3 + \ldots)\frac{\tau}{M_2}$.

In the following or impinging body $A$, the pressure acts opposite to the velocity $c_1$, consequently here a decrement of velocity takes place; and after a certain time, the residuary velocity of the body is: $v_1 = c_1 - (P_1 + P_2 + ..)\frac{\tau}{M_1}$; in the preceding or impinged body $B$, on the other hand, the pressure acts in the direction of motion; hence there is an increment of velocity $c_2$, and it is converted into

$$v_2 = c_2 + (P_1 + P_2 + ..)\frac{\tau}{M_2}.$$

If we eleminate from both equations $(P_1 + P_2 + ..)\,\tau$, there will then remain the general formula:

I. $M_1 (c_1 - v_1) = M_2 (v_2 - c_2)$, or $M_1 v_1 + M_2 v_2 = M_1 c_1 + M_2 c_2$.

The product of the mass and velocity of a body is called the *momentum of the body*, and it may therefore be enunciated, *that for each instant of the time of impact, the aggregate of the momenta of the two bodies is as great as before impact.*

At the instant of maximum compression, both bodies have an equal velocity; hence, instead of $v_1$ and $v_2$, we may put this value into the equation found; then $M_1 v + M_2 v$ will remain $= M_1 c_1 + M_2 c_2$, and the velocity of the two bodies at the instant of maximum compression will be:

$$v = \frac{M_1 c_1 + M_2 c_2}{M_1 + M_2}.$$

If the two bodies $A$ and $B$ are inelastic, they exert therefore no power after compression to re-expand themselves, and the communication of a change of motion will then cease, if both bodies are compressed to a maximum; and hence the two will go on after impact with a common velocity:

$$v = \frac{M_1 c_1 + M_2 c_2}{M_1 + M_2}.$$

*Examples.*—1. An inelastic body $B$ of 30 lbs. weight, moves with a 3 feet velocity, and is struck by another inelastic body $A$ having a 7 feet velocity, the two will then proceed, after the blow, with the velocity

$v = \frac{50 \,.\, 7 + 30 \,.\, 3}{50 + 30} = \frac{350 + 90}{80} = \frac{44}{8} = \frac{11}{2} = 5\frac{1}{2}$ feet.—2. To cause a body of 120 lbs. weight to pass from a velocity $c_2 = 1\frac{1}{2}$ feet into a 2 feet velocity $v$, it is struck by a heavy body of 50 lbs., what velocity will the body acquire? Here

$$c_1 = v + \frac{(v - c_2)\, M_2}{M_1} = 2 + \frac{(2 - 1{,}5) \,.\, 120}{50} = 2 + \frac{6}{5} = 3{,}2 \text{ feet.}$$

§ 256. *Elastic impact.*—If the impinging bodies are perfectly elastic, they will then expand themselves after compression in the first period, gradually again in the second period of the time of impact, and when they have resumed the former shape, they will proceed in their motions with different velocities. But, since the mechanical effect which is expended on the compression of an elastic body is equal to the effect which the same gives out again by its expansion, no loss in *vis viva* will take place from the collision of elastic bodies, and hence the second following equation will be also true for this case:

$$\text{II. } M_1 v_1^2 + M_2 v_2^2 = M_1 c_1^2 + M_2 c_2^2, \text{ or}$$
$$M_1 (c_1^2 - v_1^2) = M_2 (v_2^2 - c_2^2).$$

From the equations I and II, the velocities $v_1$ and $v_2$ of the bodies after impact may be found. First, it follows by division that

$$\frac{c_1^2 - v_1^2}{c_1 - v_1} = \frac{v_2^2 - c_2^2}{v_2 - c_2}, \text{ i. e. } c_1 + v_1 = v_2 + c_2, \text{ or } v_2 - v_1 = c_1 - c_2;$$

if now we put the resulting value of $v_2 = c_1 + v_1 - c_2$, into the equation I:

$$M_1 v_1 + M_2 v_1 + M_2 (c_1 - c_2) = M_1 c_1 + M_2 c_2, \text{ or,}$$

$(M_1 + M_2)\, v_1 = (M_1 + M_2)\, c_1 - 2\, M_2 (c_1 - c_2)$, from which we have the value:

$$v_1 = c_1 - \frac{2\, M_2}{M_1 + M_2} (c_1 - c_2), \text{ and}$$

$$v_2 = c_1 - c_2 + c_1 - \frac{2\, M_2 (c_1 - c_2)}{M_1 + M_2} = c_2 + \frac{2\, M_1 (c_1 - c_2)}{M_1 + M_2}.$$

Whilst for inelastic bodies the loss in velocity of the one body is

$$c_1 - v = c_1 - \frac{M_1 c_1 + M_2 c_2}{M_1 + M_2} = \frac{M_2 (c_1 - c_2)}{M_1 + M_2},$$

for elastic bodies it comes out twice as great, namely:

$$c_1 - v_1 = \frac{2\, M_2 (c_1 - c_2)}{M_1 + M_2},$$

and while the gain in velocity of the other body for inelastic bodies is

$$v - c_2 = \frac{M_1 c_1 + M_2 c_2}{M_1 + M_2} - c_2 = \frac{M_1 (c_1 - c_2)}{M_1 + M_2},$$

for elastic bodies it is

$$v_2 - c_2 = \frac{2\, M_1 (c_1 - c_2)}{M_1 + M_2}, \text{ likewise twice as great.}$$

*Example.* Two perfectly elastic spheres, the one of 10 lbs. the other of 16 lbs. weight, impinge with the velocities 12 and 6 feet against each other, what will be their velocities after impact? Here $M_1 = 10$ and $c_1 = 12$ feet, but $M_2 = 16$ and $c_2 = -6$ feet, hence the loss of velocity of the first body will be

$$c_1 - v_1 = \frac{2 \cdot 16\,(12 + 6)}{10 + 16} = \frac{2 \cdot 16 \cdot 18}{26} = 22{,}154 \text{ feet},$$

and the gain in velocity of the other: $v_2 - c_2 = \frac{2 \cdot 10 \cdot 18}{26} = 13{,}846$ feet. From this the first body after impact will recoil with the velocity $v_1 = 12 - 22{,}154 = -10{,}154$ feet; and the other with that of $-6 + 13{,}846 = 7{,}846$ feet. Moreover, the measure of *vis viva* of the two bodies after impact $= M_1 v_1^2 + M_2 v_2^2 = 10 \cdot 10{,}154^2 + 16 \cdot 7{,}846^2 = 1031 + 985 = 2016$, as likewise of that before impact, namely: $M_1 c_1^2 + M_2 c_2^2 = 10 \cdot 12^2 + 16 \cdot 6^2 = 1440 + 576 = 2016$. Were these bodies inelastic, the first would only lose in velocity $\frac{c_1 - v_1}{2} = 11{,}077$ feet, and the other gain $\frac{v_2 - c_2}{2} = 6{,}923$ feet; the first would still retain, after impact, the velocity $12 - 11{,}077 = 0{,}923$ feet, and the second take up the velocity $-6 + 6{,}923 = 0{,}923$, and the loss of mechanical effect would be $(2016 - (10 + 16)\,0{,}923^2) : 2g = (2016 - 22{,}2) \cdot 0{,}0155 = 29{,}35$ ft. lbs.

§ 257. *Particular cases.*—The formula developed in the foregoing paragraphs, for the final velocities of impact, hold good also in the case where the one body is at rest, or where both bodies move opposed to each other, or where the mass of the one is indefinitely great compared with the other. If the mass $M_2$ be at rest, we then have $c_2 = 0$, hence for the inelastic body

$$v = \frac{M_1 c_1}{M_1 + M_2}, \text{ and for the elastic:}$$

$$v_1 = c_1 - \frac{2\, M_2 c_1}{M_1 + M_2} = \frac{M_1 - M_2}{M_1 + M_2}\, c_1, \text{ and}$$

$$v_2 = 0 + \frac{2\, M_1 c_1}{M_1 + M_2} = \frac{2\, M_1}{M_2 + M_2}\, c_1.$$

If the bodies meet, $c_2$ is therefore negative, and for an inelastic body it will follow that $v = \frac{M_1\, c_1 - M_2\, c_2}{M_1 + M_2}$, and for an elastic one:

$$v_1 = c_1 - \frac{2\, M_2\,(c_1 + c_2)}{M_1 + M_2}, \text{ and } v_2 = -c_2 + \frac{2\, M_1\,(c_1 + c_2)}{M_1 + M_2}.$$

If in this case the momenta are equal, $M_1 c_1 = M_2 c_2$, for the inelastic body then $v = 0$, *i. e.* the bodies bring each other to rest, but for elastic bodies:

$$v_1 = c_1 - \frac{2\,(M_2 c_1 + M_1 c_1}{M_1 + M_2} = c_1 - 2\, c_1 = -c_1, \text{ and}$$

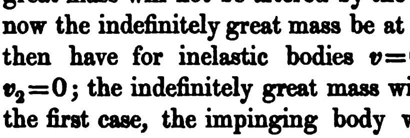
$$v_2 = -c_2 + \frac{2\,(M_2c_2 + M_1c_2)}{M_1 + M_2} = -c_2 + 2\,c_2 = +c_2\,;$$

then the bodies rebound after impact with opposite velocities. If on the other hand the masses are equal, we have then for inelastic bodies $v = \frac{c_1 - c_2}{2}$, and for elastic $v_1 = -c_2$, and $v_2 = c_1$, *i. e.* the masses rebound with their velocities interchanged.

If the masses again meet in the same direction, and if the preceding mass $M_2$ be indefinitely great, we shall then have for inelastic bodies $v = \frac{M_2c_2}{M_2} = c_2$, and for elastic $v_1 = c_1 - 2\,(c_1 - c_2) = 2\,c_2 - c_1$, $v_2 = c_2 + 0 = c_2$; the velocity therefore of the indefinitely great mass will not be altered by the collision of the finite mass. If now the indefinitely great mass be at rest, therefore, $c_2 = 0$, we shall then have for inelastic bodies $v = 0$, and for elastic $v_1 = -c_1$, $v_2 = 0$; the indefinitely great mass will then remain at rest, but in the first case, the impinging body will entirely lose its velocity, and in the second case this will be converted into an opposite one.

*Examples.*—1. With what velocity must a body of 8 lbs. impinge against another at rest of 25 lbs., in order that the last may have a velocity of 2 feet? Were the bodies inelastic, we should then have to put: $v = \frac{M_1 c_1}{M_1 + M_2}$, *i. e.* $2 = \frac{8 \cdot c_1}{8 + 25}$, hence $c_1 = \frac{33}{4} = 8\frac{1}{4}$ feet, the required velocity; but were they elastic, we should have $v_2 = \frac{2M_1 c_1}{M_1 + M_2}$; hence, $c_1 = \frac{33}{8} = 4\frac{1}{8}$ feet.—2. If a sphere $M_1$, Fig. 318, strike against a mass at rest $M_2 = nM_1$ with the velocity $c_1$, the second, a third mass $M_3 = n\,M_2 = n^2\,M_1$ with the velocity communicated by the impact, this again another mass $M_4 = n\,M_3 = n^3\,M_1$, &c, we shall have from the perfect elasticity of these masses, the velocity

FIG. 318.




$$v_2 = \frac{2M_1}{M_1 + nM_1}\,c_1 = \frac{2}{1+n}\cdot c_1\,,\; v_3 = \frac{2M_2}{M_2 + nM_2}\,v_2 = \frac{2}{1+n}\cdot v_2 = \left(\frac{2}{1+n}\right)^2 c_1\,,$$

$v_4 = \left(\frac{2}{1+n}\right)^3 c_1$, &c. If, for example, the weight of each mass be half as great as that of the succeeding one, and we have therefore the exponents of the geometrical series formed by the masses : $n = \frac{1}{2}$, it will follow that

$$v_2 = \frac{4}{3}c_1,\; v_3 = \left(\frac{4}{3}\right)^2 c_1\,,\; v_4 = \left(\frac{4}{3}\right)^3 c_1 \ldots,\; v_{10} = \left(\frac{4}{3}\right)^9 c_1 = 13{,}32\,.\,c_1.$$

§ 258. *Loss of mechanical effect.*—In the collision of inelastic

masses, a loss of *vis viva* constantly ensues, whence the masses after impact have not the power of producing so much mechanical effect, as before impact. Before impact the masses $M_1$ and $M_2$ proceeding with the velocities $c_1$ and $c_2$, contain the *vis viva*, $M_1c_1^2+M_2c_2^2$, but after impact the masses proceeding with the velocity $v=\frac{M_1c_1 + M_2c_2}{M_1 + M_2}$ have the *vis viva* $M_1v^2+M_2v^2$; hence the subtraction of these forces will give the loss in *vis viva* by the collision: $K = M_1 (c_1^2 - v^2)+M_2 (c_2^2 - v^2)=M_1 (c_1 + v) (c_1 - v) - M_2 (c_2 + v) (v - c_2)$, but $M_1 (c_1 - v) = M_2 (v - c_2) = \frac{M_1M_2 (c_1-c_2)}{M_1 + M_2}$, hence $K=(c_1+v-c_2-v)$

$$\frac{M_1M_2 (c_1-c_2)}{M_1 + M_2}=\frac{(c_1-c_2)^2 M_1M_2}{M_1 + M_2}=\frac{(c_1 - c_2)^2}{\frac{1}{M_1}+\frac{1}{M_2}}.$$

If the weight of the masses are $G_1$ and $G_2$, $M_1$ is, therefore, $=\frac{G_1}{g}$, and $M_2=\frac{G_2}{g}$, we shall from this have the loss in mechanical effect: $L=\frac{(c_1 - c_2)^2}{2g} \cdot \frac{G_1 G_2}{G_1+G_2}$. We call $\frac{G_1 G_2}{G_1+G_2}$ *the harmonic mean* of $G_1$ and $G_2$, and we may from this assert that *the loss in mechanical effect* which is produced by the impact of two inelastic masses, and which is expended upon the disfigurement of these, *is equivalent to the product of the harmonic mean of both masses, and of the height of fall which is due to the difference of the velocities of these masses.*

If one of the masses, for example $M_2$, be at rest, we shall have the loss in mechanical effect $L=\frac{c_1^2}{2g} \cdot \frac{G_1 G_2}{G_1+G_2}$, and if the mass moved $M_1$ be very great in comparison with the one at rest, $G_2$ will vanish as compared with $G_1$, and there will remain $L=\frac{c_1^2}{2g} \cdot G_2$.

For the rest we may put

$$\begin{aligned}K&=M_1 (c_1^2-v^2)+M_2 (c_2^2-v^2)=M_1 (c_1^2-2c_1v+v^2+2 c_1v-2 v^2)\\&\quad+M_2 (c_2^2-2 c_2v+v^2+2 c_2v-2 v^2)\\&=M_1 (c_1-v)^2+2 M_1v (c_1-v)+M_2 (c_2-v)^2+2 M_2v (c_2-v)\\&=M_1 (c_1-v)^2+M_2 (c_2-v)^2,\text{ because } M_1 (c_1-v)=M_2 (v-c_2).\end{aligned}$$

From this, therefore, *the vis viva lost by inelastic impacts is equivalent to the sum of the products of the masses and the squares of their loss or gain in velocity.*

*Examples.*—1. If in a machine, 16 blows per minute take place between two inelastic bodies $M_1 = \frac{1000}{g}$ lbs. and $M_2 = \frac{1200}{g}$ lbs., with the velocities $c_1 = 5$ feet, and $c_2 = 2$ feet, then the loss in mechanical effect from these blows will be:

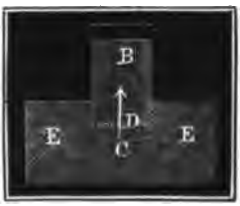
$$L = \frac{16}{60} \cdot \frac{(5-2)^2}{2g} \cdot \frac{1000 \cdot 1200}{2200} = \frac{4}{15} \cdot 9 \cdot 0{,}016 \cdot \frac{6000}{11} = 0{,}576 \cdot \frac{400}{11} = 20{,}94$$

ft. lbs. per second.—2. If two trains upon a railroad of 120000 lbs. and 160000 lbs. weight, come into collision with the velocities $c_1 = 20$, and $c_2 = 15$ feet, there will ensue a loss of mechanical effect expended upon the destruction of the locomotives and carriages, which in the case of perfect inelasticity of the impinging parts, will amount to

$$= \frac{(20+15)^2}{2g} \cdot \frac{120000 \cdot 160000}{280000} = 35^2 \cdot 0{,}016 \cdot \frac{1920000}{28} = 1344000 \text{ ft. lbs.}$$

§ 259. *Pile driving.*—The effects of impact are very often applied to ram or drive one body $B$, Fig. 319, into another $E$, a soft mass, for instance. If the resistance which the latter mass opposes to the penetration of the former be constant and $= P$, and the depth of penetration by one blow $= s$, a mechanical effect $Ps$ will be then expended. If, on the other hand, this resistance at the commencement be $= 0$, and ~~if it increase simultaneously with the depth of penetration,~~ so that at the end, after the body has penetrated the second mass a depth $s$, it be $= P$, the mechanical effect expended will be then only $\frac{(0+P)}{2} \cdot s = \frac{1}{2} Ps$. If, lastly, the initial resistance be $= P_1$, and increase simultaneously with the space, so that after describing a space $s$, it becomes $P_2$, we shall then have to put the mechanical effect $= \frac{(P_1 + P_2)}{2} s$

FIG. 319.




If the body $B$, whose mass may be $M$, begins with the velocity $v$ to penetrate a mass, ~~and if this velocity of penetration increase~~, it will, in virtue of its *vis viva*, have produced the mechanical effect $\frac{Mv^2}{2} = \frac{v^2}{2g} G$, if $G = Mg$ represent its weight.

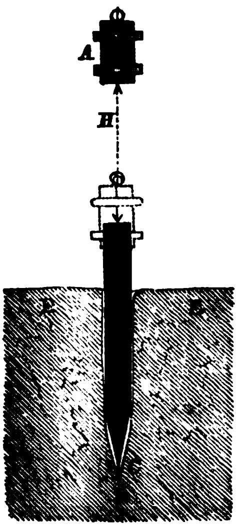
When the resistance is constant we must put : $Ps = \frac{v^2}{2g} G$; on the other hand, when the resistance beginning from nought gradually increases: $Ps = \frac{v^2}{2g} \cdot 2\, G$. ; and when it increases gradually from $P_1$ to $P_2$: $(P_1 + P_2)\, s = \frac{v^2}{2g} \cdot 2\, G$.

The initial velocity $v$ is generated if a third mass $A$, whose magnitude may be $= M_1$ and weight $= G_1$, be allowed to impinge upon the second mass $B$, with a certain velocity $c$. If, now, these masses are inelastic, we then have the velocity with which the two proceed after impact, and begin to penetrate the mass $E$:

$$v = \frac{M_1 c}{M + M_1} = \frac{G_1 c}{G + G_1}.$$

In the driving of a *pile* or *post*, Fig. 320, $B$ consists of a pile shod with iron, and $A$ of a heavy body falling from a certain height, which is called a *ram*, or block of iron. If the height of fall $= H$, we shall have:

FIG. 320.



$$\frac{v^2}{2g} = \left(\frac{G_1}{G + G_1}\right)^2 \cdot \frac{c^2}{2g} = \left(\frac{G_1}{G + G}\right)^2 \cdot H,$$

hence the mechanical effect of the pile due to the velocity $v$

$$= \left(\frac{G_1}{G + G_1}\right)^2 GH,$$

and that of the pile and ram together

$$= \left(\frac{G_1}{G + G_1}\right)^2 (G + G_1)\, H = \frac{G_1^2 H}{G + G_1}.$$

But if the resistance of the bed of earth be constant, the mechanical effect expended in the penetration of the pile will be $= Ps$, hence we shall have to put:

$$1.\ Ps = \left(\frac{G_1}{G + G_1}\right)^2 GH;\ \text{or, } 2.\ Ps = \frac{G_1^2 H}{G + G_1},$$

the first if the ram does not, during penetration, remain upon the pile, and the second if both go down together.

The weight $G + G_1$ produces, in penetrating, the mechanical effect $(G + G_1)\, s$, we may then more correctly put: $(P - G - G_1)\, s$

$= \frac{G_1^2 H}{G + G_1}$; but $G + G_1$ is small compared with $P$, and may generally be neglected. would be twice as great

Hence, were the impact perfectly elastic, we should have to put: $Ps = \left(\frac{2\,G_1}{G + G_1}\right)^2 . GH.$ Were $G$ small, compared with $G_1$, as for instance, in the driving of a nail, we should have either $Ps = G_1 H$, or $Ps = 4\,GH$. elastic

*Example.* A pile of 400 lbs. weight is driven by the last round of 20 blows of a 500 lbs. heavy ram, falling from a height of 5 feet, 6 inches deeper, what resistance will the ground offer, or what load will the pile sustain without penetrating deeper? Here $G = 400$, $G_1 = 700$ lbs., $H = 5$, and $s = \frac{0,5}{20} = 0,025$ feet, whereby it is supposed that the pile penetrates equally far for each blow. From the first formula $P = \left(\frac{700}{700 + 400}\right)^2 \frac{400 . 5}{0,025} = \left(\frac{7}{11}\right)^2 . 80000 = 32400$ lbs.; and from the second: $P = \frac{700^2 . 5}{1100 . 0,025} = \frac{4900}{11} . 200 = 89100$ lbs.

For duration, with security, such piles are only loaded from $\frac{1}{100}$ to $\frac{1}{10}$ of their strength.

FIG. 321.


§ 260. The formulæ found above are applicable to the breaking of bodies by descending weights or balls. Let $BB$, Fig. 321, be a prismatic body of the mass $M$, or weight $G = Mg$, supported at its extremities, which is bent a depth $CD = s$, by a weight $G_1$ falling from a height $AD = H$ upon its middle, and in this manner broken; the conditions under which this is possible are to be determined.

From § 190, the deflexion, or the height of the arc, is given $s = \frac{Pl^3}{48\,WE}$, from the pressure $P$ in the middle of the beam, and from its length $BB = l$, and if further its moment of flexure $WE$ is known; therefore, inversely, the pressure corresponding to a certain deflexion $s$ is: $P = \frac{48\,WEs}{l^3}$. This pressure however is not constant, but increases simultaneously with $s$, hence the mechanical effect expended in the deflexion by a depth $s$,

if supported at both ends

not $= Ps$, but only $\frac{1}{2}$ $Ps$, *i. e.* $\frac{1}{2} \cdot \frac{P^2 l^3}{48\, WE} = \frac{P^2 l^3}{96\, WE}$. This mechanical effect may now be equated to that which the falling body communicates to the beam. Since the beam rests on its extremities, we must (from § 219) consider only the third part of its mass as inert, and hence put for this mechanical effect:

$$\left(\frac{G_1}{\frac{1}{3} G + G_1}\right)^2 \cdot \tfrac{1}{3}\, GH, \text{ or } \frac{G_1^2 H}{\frac{1}{3} G + G_1}.$$

The first, if the weight $G_1$ flies back after its descent, and the second if it remains on the beam during the fracture.

If we suppose a rectangular beam of the depth $h$, and breadth $b$, we shall then have to put: $W = \frac{1}{12}\, bh^3$, and $P = \frac{2}{3}\, \frac{bh^2}{l}\, K$, hence $\frac{P^2}{W} = \frac{16}{3} \cdot \frac{bhK^2}{l^2}$; accordingly, the mechanical effect for the rupture of the beam will be $= \frac{1}{96} \cdot \frac{P^2 l^3}{WE} = \frac{bhlK^2}{18\, E}$, and we may now put:

$$1.\ GH\left(\frac{G_1}{\frac{1}{3} G + G_1}\right)^2 = \frac{bhlK^2}{6\, E}, \text{ or } 2.\ \frac{G_1^2 H}{\frac{1}{3} G + G_1} = \frac{bhlK^2}{18\, E}.$$

*Example.*—From what height must an iron weight $G_1$ of 100 lbs. be allowed to fall to break a cast-iron plate, 36 inches long, 12 inches broad, and 3 inches thick, in its middle? The modulus of elasticity of cast-iron $E = 17000000$, and the modulus of strength $K = 19000$, hence it follows that:

$$\frac{bhlK^2}{6\,E} = \frac{12 \cdot 3 \cdot 36 \cdot 19000^2}{6 \cdot 17000000} = \frac{216 \cdot 19^2}{17} = \frac{216 \cdot 361}{17} = 4587.$$

If now a cubic inch of cast iron weighs 0,275 lbs., the weight of a plate $G$ will then be $= 12 \cdot 3 \cdot 36 \cdot 0{,}275 = 1296 \cdot 0{,}275 = 356{,}4$ lbs.; hence:

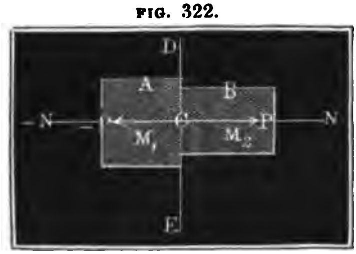
$$G\left(\frac{G_1}{\frac{1}{3} G + G_1}\right)^2 = 356{,}4 \cdot \left(\frac{100}{218{,}8}\right)^2 = 74{,}44; \text{ on the other hand,}$$

$$\frac{G_1^2}{\frac{1}{3} G + G_1} = \frac{10000}{218{,}8} = 45{,}70. \text{ Hence the height of fall required is:}$$

$$H = \frac{4587}{74{,}44} = 61{,}6 \text{ inches, or } H = \frac{4587}{3 \cdot 45{,}7} = 33{,}5 \text{ inches.}$$

§ 261. *Hardness.*—When the modulus of elasticity of the impinging bodies is known, we may then find the force of compression and its amount. Let the transverse sections of the bodies $A$ and $B$, Fig. 322, be $F_1$ and $F_2$, the lengths $l_1$ and $l_2$, and the moduli of elasticity $E_1$ and $E_2$. If both impinge against

FIG. 322.



each other with a force $P$, the compressions effected will be from § 183:

$$\lambda_1 = \frac{Pl_1}{F_1E_1}, \text{ and } \lambda_2 = \frac{Pl_2}{F_2E_2},$$

and their ratio:

$$\frac{\lambda_1}{\lambda_2} = \frac{F_2E_2}{F_1E_1} \cdot \frac{l_1}{l_2}.$$

If for simplicity we represent $\frac{FE}{l}$ by $H$, we obtain $\lambda_1 = \frac{P}{H_1}$, and $\lambda_2 = \frac{P}{H_2}$, as well as $\frac{\lambda_1}{\lambda_2} = \frac{H_2}{H_1}$. If, after the example of Whewell,* we call the quantity $\frac{FE}{l}$ the hardness of a body, it follows that *the depth of compression is inversely proportional to the hardness.*

If a mass $M = \frac{G}{g}$ impinges with the velocity $c$ upon an immoveable or indefinitely great mass, it then expends its whole *vis viva* upon the compression, hence $\frac{1}{2} Ps = \frac{Mc^2}{2} = \frac{c^2}{2g} G$. But now the space $s$ is equal to the aggregate of the compressions $\lambda_1$ and $\lambda_2$, and $\lambda_1 = \frac{P}{H_1}$, and $\lambda_2 = \frac{P}{H_2}$; hence it follows that

$$s = \lambda_1 + \lambda_2 = P\left(\frac{1}{H_1} + \frac{1}{H_2}\right) = \frac{H_1 + H_2}{H_1 H_2} \cdot P,$$

as, inversely, $P = \frac{H_1 H_2}{H_1 + H_2} s$, and the equation of condition $\frac{1}{2} \cdot \frac{H_1 H_2}{H_1 + H_2} \cdot s^2 = \frac{c^2}{2g} G$, therefore,

$$s = c\sqrt{\frac{H_1 + H_2}{H_1 H_2} \cdot \frac{G}{g}}.$$

from which $P$, $\lambda_1$ and $\lambda_2$ may be calculated.

*Example.* If a wrought-iron hammer, of 4 square inches base and 6 inches high, strikes with a velocity of 50 feet upon a plate of lead, of 2 square inches base and 1 inch thick, the following relations present themselves. The modulus of elasticity of wrought iron is $E_1 = 29000000$, and that of lead $E_2 = 700000$; hence, the hardness of these bodies is: $H_1 = \frac{F_1E_1}{l_1} = \frac{4 \cdot 29000000}{6} = 19333333$, and $H_2 = \frac{F_2E_2}{l_2} = \frac{2 \cdot 700000}{1} = 1400000$. If we put these values into the formula

* The Mechanics of Engineering, §. 207.

$s = c\sqrt{\frac{H_1+H_2}{H_1 H_2} \cdot \frac{G}{g}}$, and substitute for the weight of the hammer $= 4 \,.\, 6 \,.\, 0{,}29$ $= 7$ lbs.; therefore, $\frac{G}{g} = 7 \,.\, 0{,}031 = 0{,}217$, we shall then obtain the space of the hammer in the compression:

$s = 50\sqrt{\frac{20733333 \,.\, 0{,}224}{19333333 \,.\, 1400000}} = 50\sqrt{\frac{0{,}46443}{2706666}} = 0{,}0207$ inches $= 0{,}249$ lines. From this the force of impact or pressure follows:

$P = \frac{H_1 H_2}{H_1+H_2} \cdot s = \frac{19333333 \,.\, 1400000}{20733333} \,.\, 0{,}0207 = 27037$ lbs.; further, the compression of the hammer is: $\lambda_1 = \frac{P}{H_1} = \frac{27037}{19333333} = 0{,}0014$ inches $= 0{,}016$ lines, and that of the leaden plate: $\lambda_2 = \frac{P}{H_2} = \frac{27037}{1400000} = 0{,}0193$ inches $= 0{,}233$ lines.

§ 262. *Elastic and inelastic impact.*—If two masses $M_1$ and $M_2$ move with the velocities $c_1$ and $c_2$, the common velocity of the two at the moment of maximum compression will then be from § 256 $v = \frac{M_1 c_1 + M_2 c_2}{M_1 + M_2}$, and the mechanical effect expended on the compression from § 259:

$$L = \frac{(c_1 - c_2)^2}{2} \cdot \frac{M_1 M_2}{M_1 + M_2} = \frac{(c_1 - c_2)^2}{2g} \cdot \frac{G_1 G_2}{G_1 + G_2}.$$

This mechanical effect may be also put:

$$= \tfrac{1}{2}\, Ps = \tfrac{1}{2}\, P(\lambda_1 + \lambda_2) = \tfrac{1}{2} \cdot \frac{H_1 H_2}{H_1 + H_2}\, s^2,$$

consequently the sum of the compressions of both masses will be:

$$s = (c_1 - c_1)\sqrt{\frac{G_1 G_2}{g(G_1 + G_2)} \cdot \frac{H_1 + H_2}{H_1 H_2}},$$

from which the compressing force $P$, and the compressions of the separate masses $\lambda_1$ and $\lambda_2$, may be found.

If the masses are inelastic, these compressions will remain after impact, but if only one of the two bodies be inelastic, the other will again recover its form in the second period, and produce a mechanical effect which will generate a new change of velocity. If, for example, $M_1 = \frac{G_1}{g}$ be elastic, the mechanical effect in this second period of impact $\tfrac{1}{2}\, P\lambda_1 = \tfrac{1}{2} \cdot \frac{P^2}{H_1} = \frac{1}{2H_1}\left(\frac{H_1 H_2}{H_1 + H_2}\right)^2 s^2$ $= \frac{(c_1 - c_2)^2}{2g} \cdot \frac{G_1 G_2}{G_1 + G_2} \cdot \frac{H_2}{H_1 + H_2}$ will be given out; hence we shall

have in this case for the velocities $v_1$ and $v_2$ after impact the formula:

$$M_1 v_1 + M_2 v_2 = M_1 c_1 + M_2 c_2, \text{ and}$$

$$M_1 v_1^2 + M_2 v_2^2 = M_1 v^2 + M_2 v^2 + (c_1 - c_2)^2 \cdot \frac{M_1 M_2}{M_1 + M_2} \cdot \frac{H_2}{H_1 + H_2}$$

$$= M_1 c_1^2 + M_2 c_2^2 - (c_1 - c_2)^2 \cdot \frac{M_1 M_2}{M_1 + M_2} + (c_1 - c_2)^2 \cdot \frac{M_1 M_2}{M_1 + M_2} \cdot \frac{H_2}{H_1 + H_2},$$

i.e. $M_1 v_1^2 + M_2 v_2^2 = M_1 c_1^2 + M_2 c_2^2 - (c_1 - c_2)^2 \cdot \dfrac{M_1 M_2}{M_1 + M_2} \cdot \dfrac{H_1}{H_1 + H_2}.$

If the loss of velocity $c_1 - v_1$ be put $= x$, we shall then have the gain of velocity $v_2 - c_2 = \dfrac{M_1 x}{M_2}$, and the last equation will assume the form:

$$x\ (2 c_1 - x) - x \left(2 c_2 + \frac{M_1 x}{M_2}\right) - (c_1 - c_2)^2 \frac{M_2}{M_1 + M_2} \cdot \frac{H_1}{H_1 + H_2} = 0, \text{ or,}$$

$$\frac{M_1 + M_2}{M_2}\, x^2 - 2\ (c_1 - c_2)\ x + (c_1 - c_2)^2 \cdot \frac{M_2}{M_1 + M_2} \cdot \frac{H_1}{H_2 + H_2} = 0.$$

If this be multiplied by $\dfrac{M_2}{M_1 + M_2}$ and $\dfrac{H_1}{H_1 + H_2}$ be put

$= 1 - \dfrac{H_2}{H_1 + H_2}$, we shall then obtain the quadratic equation:

$$x^2 - 2\ (c_1 - c_2)\ \frac{M_2}{M_1 + M_2}\ x + (c_1 - c_2)^2 \left(\frac{M_2}{M_1 + M_2}\right)^2$$

$$= (c_1 - c_2)^2 \left(\frac{M_2}{M_1 + M_2}\right)^2 \cdot \frac{H_2}{H_1 + H_2}, \text{ or,}$$

$$\left(x - (c_1 - c_2)\ \frac{M_2}{M_1 + M_2}\right)^2 = (c_1 - c_2)^2 \left(\frac{M_2}{M_1 + M_2}\right)^2 \cdot \frac{H_2}{H_1 + H_2},$$

whose solution will give $x$, or the loss in velocity of the first body:

$$c_1 - v_1 = (c_1 - c_2)\ \frac{M_2}{M_1 + M_2} \left(1 + \sqrt{\frac{H_2}{H_1 + H_2}}\right),$$

and the gain in velocity of the second:

$$(v_2 - c_2 = (c_1 - c_2)\ \frac{M_1}{M_1 + M_2} \left(1 + \sqrt{\frac{H_2}{H_1 + H_2}}\right).$$

*Example.* If we assume the iron hammer in the example of the preceding paragraph to be perfectly elastic and the plate of lead inelastic, we shall then obtain the loss in velocity of the 7 lbs. heavy hammer, descending with a 50 feet velocity, since $c_2 = 0$ and $M_2 = \infty$.

$$c_1 - v_1 = c_1 \left(1 + \sqrt{\frac{H_2}{H_1 + H_2}}\right) = 50 \left(1 + \sqrt{\frac{1400000}{20733333}}\right)$$

$= 50\,(1 + 0{,}26) = 63$ feet;
hence, the velocity of the hammer after the blow is: $v_1 = c_1 - 63 = 50 - 63 = -13$ feet. The velocity of the supported plate of lead $= 0$.

§ 263. *Imperfectly elastic impact.*—If the bodies impinging [redacted] they only partially

hence the loss of the mechanical effect in question is known:

$$L = \frac{(c_1 - c_2)^2}{2} \cdot \frac{M_1 M_2}{M_1 + M_2} \cdot \frac{H_1 H_2}{H_1 + H_2} \left(\frac{1-\mu_1}{H_1} + \frac{1-\mu_2}{H_2}\right).$$

Now, in order to find the velocities $v_1$ and $v_2$ after impact, we have to combine them with each other and to solve the equations:

$$M_1 v_1 + M_2 v_2 = M_1 c_1 + M_2 c_2, \text{ and}$$

$$M_1 v_1^2 + M_2 v_2^2 = M_1 c_1^2 + M_2 c_2^2$$

$$- (c_1 - c_2)^2 \cdot \frac{M_1 M_2}{M_1 + M_2} \cdot \frac{(1-\mu_1) H_2 + (1-\mu_2) H_1}{H_1 + H_2}.$$

In the same manner as in the former §, the *loss of velocity of the first body is given:*

$$c_1 - v_1 = (c_1 - c_2) \frac{M_2}{M_1 + M_2} \left(1 + \sqrt{\frac{\mu_2 H_1 + \mu_1 H_2}{H_1 + H_2}}\right),$$

and the *gain of velocity of the body preceding:*

$$v_2 - c_2 = (c_1 - c_2) \frac{M_1}{M_1 + M_2} \left(1 + \sqrt{\frac{\mu_2 H_1 + \mu_1 H_2}{H_1 + H_2}}\right).$$

These two general formula also embrace the laws of perfectly elastic and perfectly inelastic impact. If in them we put $\mu_1 = \mu_2 = 1$, we then obtain the formula already found above for the impact of perfectly elastic bodies, but if we assume $\mu_1 = \mu_2 = 0$, we then obtain the formula for inelastic impact, &c. If both bodies have the same degree of elasticity, therefore, $\mu_1 = \mu_2$, we have more simply:

$$c_1 - v_1 = (c_1 - c_2)\, \frac{M_2}{M_1 + M_2}\, (1 + \sqrt{\mu}), \text{ and}$$

$$v_2 - c_2 = (c_1 - c_2)\, \frac{M_1}{M_1 + M_2}\, (1 + \sqrt{\mu}).$$

If, further, the mass $M_2$ is at rest, and infinitely great, it then follows that:

$$c_1 - v_1 = c_1\ (1 + \sqrt{\mu}), \text{ i. e. } v_1 = -c_1 \sqrt{\mu},$$

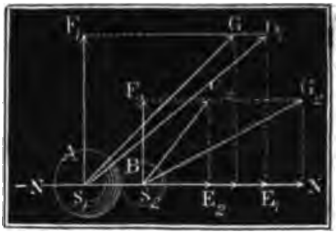
as inversely, $\mu = \left(\frac{v_1}{c_1}\right)^2$. If now $M_1$ be allowed to fall from a height $h$ upon a similar mass $M_2$, and if it reascend to a height $h_1$, we may then find from the two the co-efficient of imperfect elasticity by the formula $\mu = \frac{h_1}{h}$. Newton has already found in this manner for ivory $\mu = \left(\frac{8}{9}\right)^2 = \frac{64}{81} = 0{,}79$; for glass $\mu = \left(\frac{15}{16}\right)^2 = 0{,}9375^2 = 0{,}879$; for cork, steel, and wool, $\mu = \left(\frac{5}{9}\right)^2 = 0{,}555^2 = 0{,}309$. It must be here supposed that the impinging, or striking body, is spherical, and the body impinged upon, or the support, flat.

*Example.* What velocities will two steel plates acquire after impact, if they possessed before impact the velocities $c_1 = 10$ and $c_2 = -6$ feet, the one weighs 30 the other 40 lbs.? Here

$c_1 - v_1 = (10 + 6)\, \frac{40}{70} \left(1 + \frac{5}{9}\right) = 16 \cdot \frac{4}{7} \cdot \frac{14}{9} = \frac{16 \cdot 8}{9} = 14{,}22$ feet; hence, the velocities sought are $v_1 = c_1 - 14{,}22 = 10 - 14{,}22 = -4{,}22$ feet, and $v_2 = c_2 + 10{,}66 = -6 + 10{,}66 = 4{,}66$ feet.

§ 264. *Oblique impact.*—If the directions of motion $\overline{S_1 C_1}$ and $\overline{S_2 C_2}$ of two bodies $A$ and $B$, Fig. 323, deviate from the normal $\overline{NN}$ to the plane of contact, the impact is then *oblique*. We may reduce the theory of this to that of direct impact if we resolve

FIG. 323.




the velocities $S_1 C_1 = c_1$, and $S_2 C_2 = c_2$, in a normal and tangential direction; the lateral velocities in the direction of the normal $N\overline{N}$ communicate a certain impact, and hence are altered to the same amount as for centric impact; the velocities, on the other hand, parallel to the plane of contact communicate no impact, and hence remain unaltered. If we join the normal velocity of a body changed in accordance with the laws of centric impact to the remaining unchanged tangential velocity, we shall obtain the resultant velocities of these bodies after impact. If we represent the angles which the directions of motion make with the normal by $a_1$ and $a_2$, then $C_1S_1N = a_1$ and $C_2S_2N = a_2$, we shall obtain for the normal velocities $S_1E_1$ and $S_2E_2$ the values $c_1 \cos. a_1$ and $c_2 \cos. a_2$, for the tangential velocities on the other hand $S_1F_1$ and $S_2F_2$: $c_1 \sin. a_1$ and $c_2 \sin. a_2$. The first velocities suffer alteration from the effect of the impact, and the one passes into

$$v_1 = c_1 \cos. a_1 - (c_1 \cos. a_1 - c_2 \cos. a_2) \frac{M_2}{M_1 + M_2} (1 + \sqrt{\mu})$$

and the second into:

$$v_2 = c_2 \cos. a_2 + (c_1 \cos. a_1 - c_2 \cos. a_2) \frac{M_1}{M_1 + M_2} (1 + \sqrt{\mu}),$$

$M_1$ and $M_2$ representing the masses of the bodies.

The resultant velocity $S_1 G_1$ of the first body is given by $v_1$ and $c_1 \sin. a_1, w_1 = \sqrt{v_1^2 + c_1^2 \sin. a_1^2}$, and the velocity $S_2 G_2$ of the second body by $v_2$ and $c_2 \sin. a_2$; $w_2 = \sqrt{v_2^2 + c_2^2 \sin. a_2^2}$; the deviations from the normal are also given by the formula:

$$\tang. \phi_1 = \frac{c_1 \sin. a_1}{v_1}, \text{ and } \tang. \phi_2 = \frac{c_2 \sin. a_2}{v_2},$$

$\phi_1$ representing the angle $G_1 S_1 N$ and $\phi_2$ the angle $G_2 S_2 N$.

*Example.* Two spheres of 30 and 50 lbs. impinge against each other with the velocities $c_1 = 20$ and $c_2 = 25$ feet, which deviate from the normal by the angle $a_1 = 21° 35'$ and $a_2 = 65° 20'$, in what directions and with what velocities will the two bodies proceed after impact? The uniform component velocities are: $c_1 \sin. a_1 = 20 . \sin. 21° 35' = 7{,}357$ feet, and $c_2 \sin. a_2 = 25 . \sin. 65° 20' = 22{,}719$ feet; the variable, on the other hand, $c_1 \cos. a_1 = 20 . \cos. 21° 35'$

$= 18,598$ feet, and $c_2 \, cos. \, \alpha_2 = 25 \, . \, cos. \, 65^\circ \, 20' = 10,433$ feet. If the bodies are inelastic, then $\mu = 0$; hence, the altered normal velocities are:

$v_1 = 18,598 - (18,598 - 10,433) \frac{50}{80} = 18,598 - 5,103 = 13,495$ feet, and

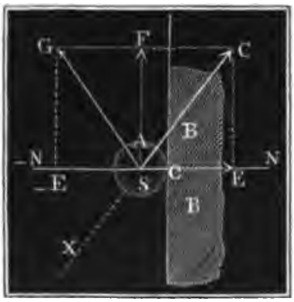
$v_2 = 10,433 + 8,165 \, . \, \frac{3}{8} = 10,433 + 3,062 = 13,495$ feet. The resultant velocities are now:

$$w_1 = \sqrt{13,495^2 + 7,357^2} = \sqrt{236,24} = 15,37 \text{ feet, and}$$
$$w_2 = \sqrt{13,495^2 + 22,719^2} = \sqrt{698,27} = 26,42 \text{ feet; and}$$

we have for their directions the $tang. \, \phi_1 = \frac{7,357}{13,495}$, $log. \, tang. \, \phi_1 = 0,73653 - 1$,

$\phi_1 = 28^\circ \, 36'$ and $tang. \, \phi_2 = \frac{22,719}{13,495}$, $log. \, tang. \, \phi_2 = 0,22622$, $\phi_2 = 59^\circ \, 17'$.

FIG. 324.




§ 266. If a mass $A$, Fig. 324, strikes against another mass, indefinitely great, or against an immoveable resistance $BB$, we have $c_2 = 0$ and $M_2 = \infty$, it then follows that
$v_1 = c_1 \, cos. \, \alpha_1 - c_1 \, cos. \, \alpha_1 (1 + \sqrt{\mu})$
$= - c_1 \, cos. \, \alpha_1 \, \sqrt{\mu}$ and
$v_2 = 0 + c_1 \, cos. \, \alpha_1 \, . \, \frac{M_1 (1 + \sqrt{\mu})}{\infty}$
$= 0 + 0 = 0$;
if now, further, $\mu = 0$, $v_1$ will also $= 0$; but if $\mu = 1$, $v_1$ will $= - c_1 \, cos. \, \alpha_1$; *i. e. in inelastic impact, the normal velocity is entirely lost; in elastic, on the other hand, it is changed in the opposite direction.* For the angle by which the direction of motion after impact deviates from the normal $tang. \, \phi_1$ is $= \frac{c_1 \, sin. \, \alpha_1}{v_1} = - \frac{c_1 \, sin. \, \alpha_1}{c_1 \, cos. \alpha_1 \sqrt{\mu}} = - tang. \, \alpha_1 \, \frac{\sqrt{1}}{\mu}$;
for inelastic bodies the $tang. \, \phi_1$ is therefore $= - \frac{tang. \, \alpha_1}{0} = \infty$,
*i. e.* $\phi_1 = 90^0$, and for elastic $tang. \, \phi_1 = - tang. \, \alpha_1$, *i. e.* $\phi_1 = - \alpha_1$. After the impact of an inelastic body against an inelastic resistance, the first proceeds with a tangential velocity $c_1 \, sin. \, \alpha_1$ in the direction $SF$ of the plane of contact; after the impact of an elastic body against an elastic resistance, the body proceeds with a uniform velocity in the direction $SG$, which lies in a plane with the normal $N\overline{N}$ and the initial direction $XS$, and makes with the normal the same angle $GS\overline{N}$ as does the direction of motion with it before impact, but on the opposite side. The angle $XS\overline{N}$,

$$m_1 c_1 + m_2 c_2 = m_1 v_1 + m_2 v_2$$

$$(v_1 - v_2) = (c_2 - c_1)\varepsilon \quad [c_2 = 0] \;\therefore\; v_1 = v_2 - c_1\varepsilon$$

$$m_1 c_1 = m_1(v_2 - c_1\varepsilon) + m_2 v_2$$

$$\therefore\; m_1 c_1 = (m_1 + m_2) v_2 - m_1 c_1 \varepsilon$$

$$\therefore\; v_2 = \frac{m_1 c_1 (1+\varepsilon)}{m_1 + m_2}$$

$$\therefore\; v_1 = \ldots c_1\varepsilon$$

$$m_1 c_1 + m_2 c_2 = m_1 v_1 + m_2 v_2$$

$$v_1 - v_2 = (c_2 - c_1)\varepsilon \;\therefore\; v_2 = v_1 + \ldots$$

$$\therefore\; m_1 c_1 + m_2 c_2 = (m_1 + m_2) v_1 - m_2 (c_2 - c_1)\varepsilon$$

$$\therefore\; v_1 = \frac{m_1 c_1 + m_2 c_2 + m_2 (c_2 - c_1)\varepsilon}{m_1 + m_2}$$

$$= \frac{m_1 c_1 + m_2 c_2 + m_2 c_2 \varepsilon - m_2 c_1 \varepsilon}{m_1 + m_2}$$

[illegible]

$$v_1 = c_1 - (c_1 - c_2)\frac{m_2}{m_1 + m_2}(1+\varepsilon)$$

$$c_1 - v_1 = (c_1 - c_2)\frac{m_2}{m_1 + m_2}(1+\varepsilon)$$

[illegible] $c_1 \cos\alpha$, $c_2 \cos\alpha$.

[illegible] § 264

$u_1^2 = \ldots$ [illegible] $\qquad u_2^2 = \ldots c_2^2 \ldots$ [illegible]

[illegible]

$v_1 - v_1' = c_2 - c_1$

$m_1 c_1 + m_2 c_2 = m_1 v_1 + m_2 v_2 \quad (1)$

$m_1 c_1^2 + m_2 c_2^2 = m_1 v_1^2 + m_2 v_2^2 \quad (2)$

or $m_1(c_1 - v_1) = m_2(v_2 - c_2) \quad (3)$

$m_1(c_1^2 - v_1^2) = m_2(v_2^2 - c_2^2) \quad (4)$

$c_1 + v_1 = c_2 + v_2$ transposing

$v_1 - v_2 = c_2 - c_1$

$F = fP = fMg$

$v_n = c_n - v_n'$ but $v_n' = ft = fgt$

§ 267

$\therefore v_2 - c_2 = \dfrac{m_1 (c_1 - c_2)(1+\varepsilon)}{m_1 + m_2}$

$c_1 = a_1 \varepsilon_1 \qquad v_1 = a_1 \omega_1$

$\therefore \omega_1 = \varepsilon_1 - (a_1 \varepsilon_1 - a_2 \varepsilon_2) \cdot \dfrac{M_2 \ldots}{M_1 \ldots a_2^2 + M_2 \ldots a_1^2} (1+\varepsilon)$

$\omega_2$ [illegible]

If $\varepsilon = 0$ [illegible]

$(a_1 \varepsilon_1 - a_2 \varepsilon_2)^2 \cdot \dfrac{M_1 \ldots M_2 \ldots}{M_1 \ldots a_2^2 + M_2 \ldots a_1^2}$

[illegible] one free.

$\therefore$ [illegible] $= 0$

before

$v_1 = c_1 - c_1$ [illegible]

$u_2$ [illegible] $= c_1$ [illegible]

$$\frac{dw_2}{du_2} = M_1 a_2^2 - M_2 y_2^2 - 2M_1 a_2^2 = 0$$

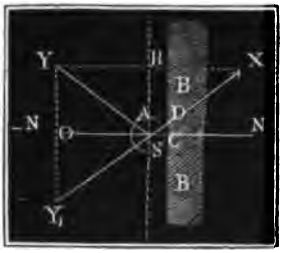
$$\therefore M_1 a_2^2 = M_2 y_2^2 \quad \text{or} \quad a_2 = y_2 \sqrt{\frac{M_2}{M_1}}$$

$$\therefore \omega_2 = c_1(1+\varepsilon) \frac{M_1 y_2 \sqrt{\frac{M_2}{M_1}}}{M_1 y_2^2 \frac{M_2}{M_1} + M_2 y_2^2}$$

$$\therefore \omega_2 = c_1(1+\varepsilon) \frac{1}{2y_2}\sqrt{\frac{M_1}{M_2}}$$

which the direction of motion before impact makes with the normal or the vertical, is called the *angle of incidence,* and the angle $GSN$, which the direction of motion after impact makes with the same, the *angle of reflexion;* and it may therefore be enunciated, *that in perfectly elastic impact, the angles of reflexion and incidence lie in the same plane with the vertical, and are equal to each other.*

FIG. 325.




In imperfectly elastic impact, the ratio $\sqrt{\mu}$ of the tangents of these angles, is equal to the ratio of the velocity given back by the expansion to ~~that of~~ the velocity ~~lost by th~~e compression. By the aid of this law, the direction may be easily found in which the body $A$, Fig. 325, must impinge against the immoveable resistance $BB$, that after impact it may pursue a certain direction $SY$. If the impact be elastic, we must let fall from a point $Y$ of the given direction the perpendicular $YO$ on the incident perpendicular $N\overline{N}$, prolong the same until the prolongation $OY_1$ is equal to the perpendicular: $SY_1$ is then the direction of the impact in question, for from this construction, the angle $NSY_1 = NSY$. If the impact be inelastic, we may make $OY_1 = \sqrt{\mu} \,.\, OY$, then $Y_1S$ will be likewise the initial direction sought, since $\frac{tang.\, a_1}{tang.\, \phi_1} = \frac{OY_1}{OY}$, and also $= \sqrt{\mu}$.

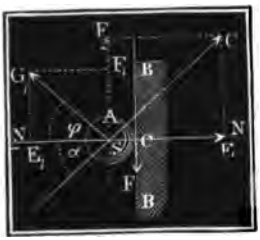
If a perpendicular $YR$ be let fall upon the line $SR$ parallel to the plane of contact, and its prolongation $RX$ be made $= \sqrt{\frac{1}{\mu}}\, RY$, we shall then also obtain, for evident reasons, in $SX$ the direction of the incidence sought.

*Remark.* The theory of oblique impact has especial application to the game of billiards. (*See* " Théorie mathématique des effets du jeu de billard," par Coriolis.) According to Coriolis, in the striking of a billiard-ball against the cushion, the ratio of the reflected velocity to that of the incident $= 0{,}5$ to $0{,}6$; therefore $\mu = 0{,}5^2 = 0{,}25$ to $0{,}6^2 = 0{,}36$. With the assistance of this value, the direction may now be known in which a ball $A$ must strike against the cushion $BB$ that it may be reflected from this towards a given point $Y$. We may let fall from the given point $Y$ the perpendicular $YR$ to the line of gravity of the ball parallel to the cushion, prolong it by $RX = \sqrt{\frac{1}{\mu}} = \frac{10}{6}$ to $\frac{10}{5}$ of its value, and draw the straight line

$Y_1X$: the point of intersection $D$ is the place where the ball $A$ must be struck that it may rebound to $Y$. By the motion of rotation of the ball, this ratio will be somewhat altered.

§ 267. In oblique impact a friction takes place between the impinging bodies, which changes the lateral velocities in the direction of the plane of contact. The friction of impact is determined like that of the friction of pressure, $P$ representing the pressure, and $f$ the co-efficient of friction, it is $F = fP$. It is distinguishable from the friction of pressure in this: that like impact, it acts only during a very short time. The changes of velocity produced by it are not immeasurably small, for the pressure $P$, and consequently the part of it $fP$, is generally very great. If we represent the impinging mass by $M$, and the normal acceleration generated by the pressure $P$, by $p$, we shall then have $P = Mp$, and hence $F = fMp$, as well as the retardation or the negative acceleration due to friction during the impact $\frac{F}{M} = fp$; *i. e.* $f$ times as great as the normal ~~pressure~~. But the two pressures have equal durations; hence, therefore, *the change of velocity effected by friction is f times as great as the change of the normal velocity effected by impact.*

FIG. 326.




In the case where a body impinges upon an immoveable mass $BB$ at the angle of incidence $a$, Fig. 326, the change in the normal velocity from the former paragraph is $w = c\ cos.\ a\ (1 + \sqrt{\mu})$; hence, the change in the tangential velocity effected by friction

$$= fw = fc\,(1 + \sqrt{\mu})\ cos.\ a.$$

The lateral velocity, therefore, after impact $c\ sin.\ a$, passes into

$$c\ sin.\ a - fc\,(1 + \sqrt{\mu})\ cos.\ a = [sin.\ a - f\ cos.\ a\,(1 + \sqrt{\mu})]\ c,$$

and in the case of perfectly elastic bodies $= (sin.\ a - 2f\ cos.\ a)\ c$, and in that of inelastic $= (sin.\ a - f\ cos.\ a)\ c$.

Bodies very often have a rotation about their centre of gravity from the effect of the friction of impact, or the motion of rotation, if present before impact, becomes consequently changed. If the moment of inertia of the round body $A$ about its centre of gravity $S = My^2$, and the radius of gyration $SC = a$, we shall have

the mass of the body reduced to the point of contact $C = \frac{My^2}{a^2}$; hence, the acceleration of rotation generated by the friction $F$ is:

$$p_1 = \frac{F}{My^2 : a^2} = \frac{fMp}{My^2 : a^2} = fp \cdot \frac{a^2}{y^2},$$

and the correspondent change of velocity:

$$w_1 = f\frac{a^2}{y^2} \cdot w = f\frac{a^2}{y^2}(1 + \sqrt{\mu})\ c\ cos.\ a.$$

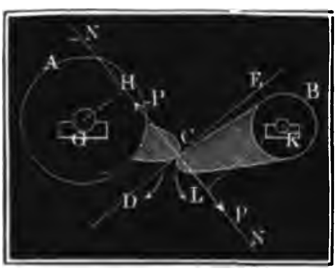
In a cylinder $\frac{a_2}{y^2} = 2$, and in a sphere $= \frac{5}{2}$, hence the change in the velocity of rotation generated by the impact against the plane is:

$$w_1 = 2f(1 + \sqrt{\mu})\ c\ cos.\ a \text{ and } = \frac{5}{2}f(1 + \sqrt{\mu})\ c\ cos.\ a.$$

*Example.* If a billiard-ball, with a 15 feet velocity and at an angle of incidence $a = 45°$, strike against the cushion, what motion after impact will it take up? If we put for the $\sqrt{\mu}$ the mean value 0,55, we shall have the ~~lateral~~ normal velocity after impact $= - \sqrt{\mu}\ c\ cos.\ a = - 0{,}55 \,.\, 15 \,.\, cos.\ 45° = - 8{,}25 \,.\, \sqrt{\tfrac{1}{2}} = - 5{,}833$ ft. and if with Coriolis we take $f = 0{,}20$, we shall then obtain the lateral velocity parallel to the cushion $= c\ sin.\ a - f(1 + \sqrt{\mu})\ c\ cos.\ a = (1 - 0{,}20 \,.\, 1{,}55)\ 10{,}607$ $= 0{,}69 \,.\, 10{,}607 = 7{,}319$ feet, and for the angle of reflexion $\phi$:

$tang.\ \phi = \frac{7{,}319}{5{,}833} = 1{,}2548$; therefore, $\phi = 51°\ 27'$, and the velocity after impact will remain $= \frac{5{,}833}{cos.\ 51°\ 27'} = 9{,}360$ feet. Besides, the ball will have further a velocity of rotation $\frac{5}{2} f \,.\, 1{,}55 \,.\, 10{,}607 = 8{,}220$ feet about its vertical line of gravity. Since the ball moves with a rolling and not a sliding motion, we must assume that besides its progressive velocity $c = 15$ feet, it possesses an equal amount of velocity of rotation, and this may likewise be resolved into the components $c\ cos.\ a = 10{,}607$ and $c\ sin.\ a = 10{,}606$. The first component answers to a rotation about an axis parallel to the cushion, and passes into $c\ cos.\ a - \frac{5}{2} f(1 + \sqrt{\mu})\ c\ cos.\ a$ $= 10{,}607 - 8{,}220 = 2{,}387$ feet, the other component $c\ sin.\ a = 10{,}607$ feet answers to a rotation about an axis normal to the cushion, and remains uniform.

FIG. 327.




§ 267. *Rotatory bodies.*—If two bodies, $A$ and $B$, capable of rotating about two fixed axes $G$ and $K$, Fig. 327, strike against each other, changes of velocity ensue, which may be determined from the moments of inertia $M_1 y_1^2$ and $M_2 y_2^2$ of the masses of these bodies about the fixed axes, with the assistance of the formulæ already found. If the

perpendiculars $GH$ and $KL$, let fall from the axis of rotation upon the line of impact, are $a_1$ and $a_2$, we then have the inert masses reduced to the points $H$ and $L$ where the perpendiculars meet the line of impact $= \frac{M_1 y_1^2}{a_1^2}$ and $\frac{M_2 y_2^2}{a_2^2}$, and if these values be substituted for $M_1$ and $M_2$ in the formula for free centric impact, we obtain the changes of the velocity of the points $H$ and $L$

$$(\S\ 264) = (c_1 - c_2) \frac{M_2 y_2^2 : a_2^2}{M_1 y_1^2 : a_1^2 + M_2 y_2^2 : a_2^2} (1 + \sqrt{\mu})$$

$$= (c_1 - c_2) \frac{M_2 y_2^2 a_1^2}{M_1 y_1^2 a_2^2 + M_2 y_2^2 a_1^2} (1 + \sqrt{\mu}) \text{ and}$$

$$(c_1 - c_2) \frac{M_1 y_1^2 : a_1^2}{M_1 y_1^2 : a_1^2 + M_2 y_2^2 : a_2^2} (1 + \sqrt{\mu})$$

$$= (c_1 - c_2) \frac{M_1 y_1^2 a_2^2}{M_1 y_1^2 a_2^2 + M_2 y_2^2 a_1^2} (1 + \sqrt{\mu}),$$

$c_1$ and $c_2$ representing the velocity of these points before impact.

But if we introduce the angular velocities, and represent these before impact by $\varepsilon_1$ and $\varepsilon_2$, and after impact by $\omega_1$ and $\omega_2$, we shall have to put $c_1 = a_1 \varepsilon_1$, $c_2 = a_2 \varepsilon_2$, and shall obtain for the loss in angular velocity of the impinging body:

$$= a_1 (a_1 \varepsilon_1 - a_2 \varepsilon_2) \frac{M_2 y_2^2}{M_1 y_1^2 a_2^2 + M_2 y_2^2 a_1^2} (1 + \sqrt{\mu}),$$

and for the body impinged upon, the gain of the same:

$$= a_2 (a_1 \varepsilon_1 - a_2 \varepsilon_2) \frac{M_1 y_1^2}{M_1 y_1^2 a_2^2 + M_2 y_2^2 a_1^2} (1 + \sqrt{\mu}),$$

consequently the angular velocities themselves, after impact, will be:

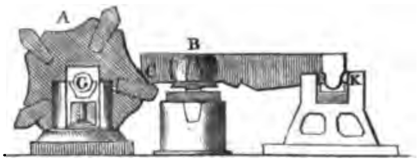
$$\omega_1 = \varepsilon_1 - a_1 (a_1 \varepsilon_1 - a_2 \varepsilon_2) (1 + \sqrt{\mu}) \frac{M_2 y_2^2}{M_1 y_1^2 a_2^2 + M_2 y_2^2 a_1^2} \text{ and}$$

$$\omega_2 = \varepsilon_2 + a_2 (a_1 \varepsilon_1 - a_2 \varepsilon_2) (1 + \sqrt{\mu}) \frac{M_1 y_1^2}{M_1 y_1^2 a_2^2 + M_2 y_2^2 a_1^2}.$$

If both bodies are perfectly elastic, we shall have $\mu = 1$, therefore $1 + \sqrt{\mu} = 2$, and if inelastic, $\mu = 0$, therefore $1 + \sqrt{\mu} = 1$. In the latter case, the loss of *vis viva* produced by impact

$$= (a_1 \varepsilon_1 - a_2 \varepsilon_2)^2 \cdot \frac{M_1 y_1^2 \cdot M_2 y_2^2}{M_1 y_1^2 a_2^2 + M_2 y_2^2 a_1^2}.$$

FIG. 328.



*Example.* The armed axle $AG$, Fig. 328, has the moment of inertia about its axis of rotation $G$, $= M_1 y_1^2 = 40000 : g$, and the tilt hammer $BK$ one about its axis $K$, $= 150000 : g$; the arm $GC$ of the axle is 2 feet, and the arm $KC$ of the hammer 6 feet, and the angular velocity of the axle at the moment of impact on the hammer = 1,05 feet; what is this velocity after impact, and what effect is lost at each blow, if there is an entire absence of elasticity? The angular velocity of the axle sought is:

4600

$$\omega_1 = 1,05 - \frac{\cancel{4 . 1,05} . 150000}{40000 \quad 36 + 150000 . 4} = 1,105 \left(1 - \frac{60}{204}\right) = 1,05 . 0,706$$

$= 0,741$ feet, and that of the hammer $= \frac{2 . 6 . 1,05 . 4}{204} = 0,247$ feet, *i. e.*, three times less than that of the axle. The loss of mechanical effect by each blow is:

$$L = \frac{(2 . 1,05)^2}{2g} . \frac{4 . 1,05 . 150000}{40000 . 36 + 150000 . 4} = 0,155 . (2,1)^2 . \frac{600000}{144 + 60}$$

$$= 0,0155 . 4,41 . \frac{150000}{51} = 201 \text{ ft. lbs.}$$

FIG. 329.


§ 268. A body $A$, in a state of free and progressive motion, Fig. 329, impinges against a body $BCK$, capable of rotating about a fixed axis $K$, the velocities after impact may be found, if, in place of $a_1 \varepsilon_1$ and $a_1 \omega_1$ in the formula of the preceding paragraph, we put the progressive velocities $c_1$ and $v_1$, and instead of $\frac{M_1 y_1^2}{a_1^2}$ the inert mass $M_1$ of the first body, the other denominations remaining the same. Hence, the velocity of the first mass after impact is:

$$v_1 = c_1 - (c_1 - a_2 \varepsilon_2)(1 + \sqrt{\mu}) . \frac{M_2 y_2^2}{M_1 a_2^2 + M_2 y_2^2},$$

and the angular velocity of the second:

$$\omega_2 = \varepsilon_2 + a_2 (c_1 - a_2 \varepsilon_2)(1 + \sqrt{\mu} . \frac{M_1}{M_1 a_2^2 + M_2 y_2^2}.$$

If the mass $M_2$ be at rest, therefore $\varepsilon_2 = 0$, we have:

$$v_1 = c_1 - c_1 (1 + \sqrt{\mu}) . \frac{M_2 y_2^2}{M_1 a_2^2 + M_2 y_2^2} \text{ and}$$

$$\omega_2 = a_2 c_1 (1 + \sqrt{\mu}) . \frac{M_1}{M_1 a_2^2 + M_2 y_2^2}.$$

If, on the other hand, $M_1$ is at rest, that is, the oscillating body the impinging one, we shall have $c_1 = 0$, and hence

$$v_1 = a_2 \varepsilon_2 (1 + \sqrt{\mu}) \cdot \frac{M_2 y_2^2}{M_1 a_2^2 + M_2 y_2^2} \text{ and}$$

$$\omega_2 = \varepsilon_2 \left(1 - (1 + \sqrt{\mu} \frac{M_1 a_2^2}{M_1 a_2^2 + M_2 y_2^2}\right).$$

The velocity communicated by impact to another at rest, depends not only on the velocity of impact and of the masses of the bodies, but also on the distance $KL = a_2$ at which the direction of impact $N\overline{N}$ is distant from the axis $K$ of the rotatory body. If the free mass be the impinging one, the rotatory mass will assume the angular velocity

$$\omega_2 = c_1 (1 + \sqrt{\mu}) \frac{M_1 a_2}{M_1 a_2^2 + M_2 y_2^2},$$

and if the oscillating mass strike against the free, this will acquire the velocity

$$v_1 = \varepsilon_2 (1 + \sqrt{\mu}) \frac{M_2 y_2^2 \cdot a_2}{M_1 a_2^2 + M_2 y_2^2},$$

but both velocities will be so much the greater, the greater

$\dfrac{a_2}{M_1 a_2^2 + M_2 y_2^2}$ or $\dfrac{1}{M_1 a_2 + \dfrac{M_2 y_2^2}{a_2}}$, therefore the less

$M_1 a_2 + M_2 \dfrac{y_2^2}{a_2}$.

If for $a_2$ we put $a \pm x$, when $x$ is very small, we shall obtain the value of the last impression:

$$M_1 (a \pm x) + \frac{M_2 y_2^2}{a \pm x} = M_1 a \pm M_1 x + \frac{M_2 y_2^2}{a}(1 \mp \frac{x}{a} + \frac{x^2}{a^2} \pm \ldots)$$

or in consequence of the smallness of the powers of $x$,

$$= M_1 a + \frac{M_2 y_2^2}{a} \pm \left(M_1 - \frac{M_2 y_2^2}{a^2}\right) x + \ldots$$

If now $a$ correspond to the least of all the values of $M_1 a_2 + \dfrac{M_2 y_2^2}{a_2^2}$, the member $\pm \left(M_1 - \dfrac{M_2 y_2^2}{a^2}\right) x$ will disappear, because the addition of the quantity ($x$) will give to it a different sign to that of a diminutive ($-x$). Therefore:

$\left(M_1 - \dfrac{M_2 y_2^2}{a^2}\right) x$ must be $= 0$, *i. e.* $\dfrac{M_2 y_2^2}{a^2} = M_1$, consequently:

$$a = \sqrt{\frac{M_2 y_2^2}{M_1}} = y_2 \sqrt{\frac{M_2}{M_1}}.$$

If at this distance one body impinge against the other, then will the latter take the greatest velocity, and be in fact

$$\omega = c_1 \, (1 + \sqrt{\mu}) \, \frac{1}{2 \, y_2} \sqrt{\frac{M_1}{M_2}},$$

in the case where the rotatory body is impinged upon; and

$$v = \tfrac{1}{2} \, y_2 \, \epsilon_2 \, (1 + \sqrt{\mu} \sqrt{\frac{M_2}{M_1}},$$

when the free body receives the blow.

The point $D$ in the line of impact of the distance corresponding to the greatest velocity, or of the arm $a$, is sometimes improperly called the centre of impact, but more properly the *point of impact.*

*Example.* What position has the point of impact if the free mass consist of an iron sphere of 16 lbs. weight, and the rotatory mass have a moment of inertia of $1000 \div g$? The distance of this point from the fixed axis of the last body: $a = \sqrt{\frac{1000}{16}} = \sqrt{62,5} = 7,906$ feet. If the impact be inelastic, and the block strike against the sphere with the velocity $\epsilon = 3$ feet, the latter will take up the velocity $v = \frac{3}{2} \, . \, 7,906 = 11,86$ feet.

§ 269. *Ballistic pendulum.*—An application of the laws laid down, is found in the theory of the *ballistic pendulum,* or the *pendulum of Robins.* It consists of a mass $MH$, turning about a horizontal axis $C$, Fig. 330, which is set into oscillating motion by a ball projected against it, which serves for the measurement of its velocity. That as inelastic a blow as possible may ensue, there is an opening made on the further side, which from time to time is filled by fresh wood or clay, &c. The ball remains after each projection sticking in this mass, and oscillating in common with the whole body. For the measurement of the velocity of the ball, it is requisite to know the

FIG. 330.

angle of elongation of this pendulum, on which account there is further a graduated arc $BD$ applied, and an index $E$ fixed to the centre of gravity of the pendulum, which slides along with the former.

From the foregoing paragraph, the angular velocity of the ballistic pendulum after the impact of the ball is: $\omega = \frac{M_1 a_2 c_1}{M_1 a_2^2 + M_2 y_2^2}$, if $M_1$ is the mass of the ball, $M_2 y_2^2$ the moment of inertia of the pendulum, $c_1$ the velocity of the ball, and $a_2$ the arm $CG$ of the impact, or the distance of the line of impact $N\overline{N}$ from the axis of revolution of the pendulum. If the distance $CM$ of the centre of oscillation $M$ of the entire mass, together with the ball from the centre of suspension $C$, *i. e.* the length of the simple pendulum, which oscillates in equal times with the ballistic, $= l$, and the angle of elongation $BCD = a$, we have the height of the isochronously oscillating simple pendulum:

$$h = CM - CH = l - l\ cos.\ a = l\ (1 - cos.\ a) = 2l\left(sin.\ \frac{a}{2}\right)^2;$$

and hence the velocity at the lowest point of its path:

$$v = \sqrt{2\,gh} = 2\ \sqrt{gl}\ sin.\ \frac{a}{2},$$

or the corresponding angular velocity:

$$\omega = \frac{v}{l} = 2\sqrt{\frac{g}{l}}\ sin\ \frac{a}{2}.$$

By equating these two values it will follow that the ~~angular~~ velocity:

$$c_1 = \frac{M_1 a_2^2 + M_2 y_2^2}{M_1 a_2} \cdot 2\ \sqrt{\frac{g}{l}} \cdot sin.\ \frac{a}{2}.$$

But now, according to the theory of the simple pendulum,

$$l = \frac{\text{moment of inertia}}{\text{statical moment}} = \frac{M_1\, a_2^2 + M_2\, y_2^2}{(M_1 + M_2)\ s},$$

$s$ being the distance of the centre of gravity $S$ from the axis of revolution, hence

$$M_1\, a_2^2 + M_2\, y_2^2 = (M_1 + M_2)\ sl, \text{ and}$$

$$c_1 = 2\left(\frac{M_1 + M_2}{M_1}\cdot\right)\frac{s}{a_2}\sqrt{gl}\,.\ sin.\ \frac{a}{2}.$$

If the pendulum makes $n$ oscillations per minute, the time of oscillation

$$\pi\sqrt{\frac{l}{g}}=\frac{60''}{n}, \text{ hence } \sqrt{gl}=\frac{60''\,.\,g}{n\pi},$$

and the required velocity of the ball

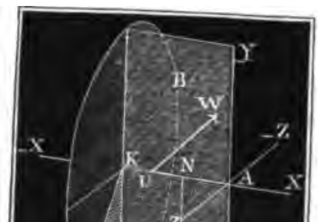
$$c_1=\frac{M_1+M_2}{M_1}\cdot\frac{120\,gs}{n\,\pi\,a_2}\cdot sin.\ \frac{a}{2}.$$

*Example.* If a ballistic pendulum, of 3000 lbs. weight, whose angle of elongation amounts to 15°, is set into oscillation by the projection of a 6 lbs. ball, if, further, the distance $s$ of the centre of gravity from the axis = 5 feet, and that of the line of projection from this axis = $5\frac{1}{2}$ feet, and lastly, the number of oscillations per minute $n$ = 40, from the above formula the velocity of the ball at the moment of the impact will be:

$$c=\frac{3006}{6}\cdot\frac{120\,.\,31{,}25\,.\,5}{40\,.\,3{,}1416\,.\,5{,}5}\,sin.\ 7\tfrac{1}{2}^{\circ}=\frac{501\,.\,3750\,.\,sin.\ 7^{\circ}\,30'}{44\,.\,3{,}1416}=1774 \text{ feet.}$$

§ 270. *Centre of percussion.*—If a body turning about a fixed axis $C$ is impinged upon by another, a reaction from the blow will generally take place upon the axis of the body, which is dependant principally upon the distance between the direction of the impact and that of the axis. Let us determine this reaction or this axial pressure in the simple case, when the direction of the blow is perpendicular to the plane passing through the axis of revolution and the centre of gravity of the body.

FIG. 331.




Let $BD$ be a plane of gravity passing through the axis of revolution $X\overline{X}$ of the body in Fig 331, $Y\overline{Y}$ a second perpendicular axis in this plane, and $Z\overline{Z}$ a third axis, at right angles to this plane of gravity. An element $M_1$ of the body is determined by the co-ordinates $AK=x_1$, $KL=y_1$, and $LM=z_1$, in this system of axes intersecting in the point $A$, and another element by the co-ordinates $x_2$, $y_2$, $z_2$, &c. If $\kappa$ be the angular acceleration, we shall have the force of inertia of the element $M_1$; $Q_1=M_1\,.\,\kappa\,.\,\overline{KM}$, and if these be resolved into the component forces $R$ parallel, and $S$ at right angles to the imaginary plane of gravity, the similarity of the triangles $KML$ and $QMR$ or $MQS$, gives $R=M_1\,.\,\kappa z_1=\kappa\,M_1 z_1$, and $S=\kappa M_1 y_1$.

From this the aggregate of the component forces parallel to the plane $= \kappa\,(M_1 z_1 + M_2 z_2 + \,..)$ and that of the component forces at right angles to this plane:

$$\kappa\,(M_1\,y_1 \;+\; M_2\,y_2 \;+\,..).$$

Since the plane $BD$ passes through the centre of gravity, the sum of the moments $M_1 z_1 + M_2 z_2 + \,.. = 0$, hence there remains only the sum of the forces $\kappa\,(M_1 y_1 + M_2 y_2 + \,..)$. If now $P$ be blow, and $W$ the resistance, or the reaction of the axis, we shall, in the first place, have to put:

$$P = W + \kappa\,(M_1\,y_1 + M_2\,y_2 \ldots).$$

The statical moment of the force:

$$Q_1 = M_1\,\kappa\,.\,\overline{KM} = M_1\,\kappa\,.\,\overline{KM}\,.\,\overline{KM} = \kappa\,.\,M_1\,\overline{KM}^2,$$

or the distance $KM$ represented by $r_1 = \kappa\,M_1 r_1^2$, the moment of the force of another particle of the mass $= \kappa\,M_2\,r_2^2$, &c., hence the statical moment of the entire inertia $= \kappa\,(M_1 r_1^2 + M_2 r_2^2 + \ldots)$. If now we put the distance $NO$ of the direction of the impact from the direction of the axis $= b$, we shall have the moment of percussion $P$ about $X\overline{X} = Pb$, whilst that of $W = 0$; we may hence also put:

$$Pb = \kappa\,(M_1 r_1^2 + M_2 r_2^2 + \,..),$$

and obtain by elimination of $\kappa$ from both equations:

$$P = W + \frac{Pb\,(M_1 y_1 + M_2 y_2 + \,..)}{M_1 r_1^2 + M_2 r_2^2 + \,..},\ \textit{i. e. the reaction sought:}$$

$$W = P\left(1 - \frac{b\,(M_1 y_1 + M_2 y_2 + ..)}{M_1 r_1^2 + M_2 r_2^2 + ..}\right).$$

If, lastly, we represent the distance $AN$ of the direction of the impact from the axis $Y\overline{Y}$ by $a$, and the distance $AU$ of the point of application $U$ of the reaction $W$ from the initial point by $u$, we shall have further the

moment $Pa$ = mom. $Wu$ + mom. $\kappa\,(M_1 x_1 y_1 \;+\; M_2 x_2 y_2 \;+\,..)$,
and the distance of the point of application sought will be:

$$u = \frac{Pa - \kappa\,(M_1 x_1 y_1 + M_2 x_2 y_2 + \,..)}{W},\ \textit{i. e.}$$

$$u = \frac{a\,(M_1 r_1^2 \,+\, M_2 r_2^2 \,+\,..) - b\,(M_1 x_1 y_1 + M_2 x_2 y_2 + \,..)}{M_1 r_1^2 + M_2 r_2^2 + \,.. - b\,(M_1 y_1 + M_2 y_2 + \,..)}.$$

The reaction $W = 0$, if $b\ (M_1 y_1 + M_2 y_2 + ..) = M_1 r_1^2 + M_2 r_2^2 + \,..$,

$$\textit{i. e. } 1.\ b = \frac{M_1\,r_1^2 \,+\, M_2\,r_2^2 \,+\,..}{M_1 y_1 \,+\, M_2 y_2 \,+\,..} = \frac{\text{moment of inertia}}{\text{statical moment}},$$

and also its moment $= 0$, if

$$Pa = \kappa\,(M_1 x_1 y_1 + M_2 x_2 y_2 + \,..),\ \textit{i. e.}\ u = 0$$

a b

$$2.\ a = \frac{M_1x_1y_1 + M_2x_2y_2 + \ldots}{M_1y_1 + M_2y_2 + \ldots}.$$

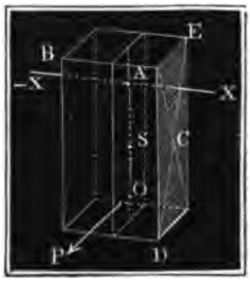
The point $O$ determined by these co-ordinates $a$ and $b$, in the plane of gravity containing the fixed axis is called the *centre of percussion.* Every blow passing through this point, and at right angles to the plane of gravity, is completely taken up by the mass, without leaving any residuary effect upon the axis, or producing any pressure. The formula (1) shews that the centre of percussion is at the same distance from the axis of revolution (compare § 251) as the centre of ~~suspension~~. Oscillation

That a hammer may not jar by its blow the hand which holds it, or react upon the wrist about which it turns, it is requisite that the blow pass through the centre of percussion.

*Examples.*—1. In a prismatic bar $CA$, Fig. 332, which turns about one of its extreme points, the centre of percussion lies about $CO = b = \frac{\frac{1}{3}l^2}{\frac{1}{2}l} = \frac{2}{3}l = \frac{2}{3}\,CA$ from the axis. If, therefore, the bar be fixed at one extremity and be struck at a

FIG. 332.


FIG. 333.

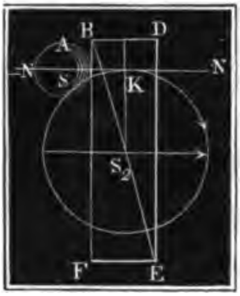



point $O$ at the distance $CO = \frac{2}{3}\,CA$, then no jar will be felt.—2. In a parallelipiped $BDE$, Fig. 333, which turns about an axis $X\overline{X}$ running parallel to its four sides and distant about $SA = s$ from the centre of gravity, the distance $AO$ of the centre of percussion $O$ from the axis $b = \frac{s^2 + \frac{1}{3}d^2}{s}$, where $d$ is the semi-diagonal of the lateral surfaces through which the axis $X\overline{X}$ passes (§ 220). If the force of the blow $P$ were to pass through the centre of gravity, the reaction would be:

$$W = P\left(1 - s\cdot\frac{s}{s^2 + \frac{1}{3}d^2}\right) = P\left(1 - \frac{s^2}{s^2 + \frac{1}{3}d^2}\right) = \frac{1}{3}\cdot\frac{Pd^2}{s^2 + \frac{1}{3}d^2} = \frac{Pd^2}{3s^2 + d^2}.$$

§ 271. *Excentric impact.*—Lastly, let us further investigate a simple case of excentric impact, when both masses are perfectly free. When two bodies $A$ and $BE$, Fig. 334, impinge upon each other so that the direction of impact $N\overline{N}$ passes through the centre of gravity $S_1$ of the one body, and beyond the centre of

FIG. 334.




gravity $S_2$ of the other body, the impact with respect to the first body is centric, and with respect to the other excentric. The effects of this excentric impact may be found from the proposition, § 214, if we assume first, that the second body is free, and that the direction of impact passes through the centre of gravity $S_2$, and secondly, that this body is fixed at its centre of gravity, and that the percutical force acts as a force of revolution. If now $c_1$ be the initial velocity of $A$, $c_2$ that of the centre of gravity of $BE$, and if the two velocities pass into $v_1$ and $v_2$ from the effect of the blow, there will remain as in § 256,

$$M_1 v_1 + M_2 v_2 = M_1 c_1 + M_2 c_2.$$

If further $\varepsilon$ be the initial angular velocity of the body $BE$, and in its revolution about the axis passing through the centre of gravity, and perpendicular to the plane $N\overline{N}S_2$, and if this velocity pass from the effect of the blow into $\omega$, and $M_2 y_2^2$ represent the moment of inertia of this body about $S_2$, or $s$ the distance $S_2K$ of the centre of gravity $S_2$ from the direction of impact, we shall then also have

$$M_1 v_1 + \frac{M_2 y_2^2}{s^2} \,.\, s\omega = M_1 c_1 + \frac{M_2 y_2^2}{s^2} s\varepsilon.$$

If both bodies are inelastic, then the points of contact of the two will have at the end equal velocities, therefore, further $v_1$ will $= v_2 + s\omega$. If we determine from the above equations $v_2$ and $\omega$ by $v_1$, and put the values obtained into the last equation, we shall then obtain:

$$v_1 = \frac{M_1 (c_1 - v_1)}{M_2} + c_2 + \frac{M_1 s^2 (c_1 - v_1)}{M_2 y_2^2} + s\varepsilon,$$

and from this may be determined the loss in velocity of the first body:

$$c_1 - v_1 = \frac{M_2 y_2^2 \, (c_1 - c_2 - s\varepsilon)}{(M_1 + M_2)\, y_2^2 + M_1 s^2},$$

the gain in progressive velocity of the second:

$$v_2 - c_2 = \frac{M_1 \, y^2 \, (c_1 - c_2 - s\varepsilon)}{(M_1 + M_2) y_2^2 + M_1 s^2},$$

and the gain in angular velocity:

$$\omega - \varepsilon = \frac{M_1 \, s \, (c_1 - c_2 - s\varepsilon)}{(M_1 + M_2) y_2^2 + M_1 s^2}.$$

For the case of perfectly elastic impact, these values are double; and for that of imperfectly elastic impact, they are $(1 + \sqrt{\mu})$ times as great.

*Example.* An iron ball $A$, of 65 lbs. weight, strikes a parallelipiped $BE$ of fir, originally at rest, with a 36 feet velocity; the length of this body is 5 feet, its breadth 3 feet, and thickness 2 feet, and the direction of the impact $N\bar{N}$ deviates by $S_2 K = s = 1\frac{3}{4}$ feet from the centre of gravity $S_2$, then the following velocities after impact are given. The specific gravity of fir may be taken $= 0,45$, the weight of the parallelipiped is therefore $= 5 \times 3 \times 2 \times 62,5 \times 0,45 = 843,75$ lbs. The square of the semi-diagonal of the lateral surface parallel to the direction of impact is:

$$y^2 = \left(\frac{5}{2}\right)^2 + \left(\frac{2}{2}\right)^2 = 7,25,$$

hence the velocity of the ball after impact is:

$$v_1 = c_1 - \frac{M_2 y^2 c_1}{(M_1 + M_2)\, y^2 + M_1 s^2} = 36 \left(1 - \frac{843,75 \,.\, 7,25}{956 \,.\, 7,25 + 65 \,.\, 1,75^2}\right)$$

$$= 36 \left(1 - \frac{843,75 \,.\, 7,25}{7130,06}\right) = 36\,(1 - 0,958) = 1,512 \text{ feet};$$

further, the velocity of the centre of gravity of the parallelipiped:

$$v_2 = \frac{M_1 y^2 c_1}{(M_1 + M_2)\, y^2 + M_1 s^2} = \frac{65 \,.\, 7,25 \,.\, 36}{7130,06} = 2,379 \text{ feet};$$

lastly, the angular velocity of this body is:

$$\omega = \frac{M_1 s c}{(M_1 + M_2)\, y^2 + M_1 s^2} = \frac{65 \,.\, 1,75 \,.\, 36}{7130,06} = 0,574 \text{ feet}.$$

# SECTION V.

## STATICS OF FLUID BODIES.

### CHAPTER I.

#### ON THE EQUILIBRIUM AND PRESSURE OF WATER IN VESSELS.

§ 272. *Fluidity.*—We regard *fluid bodies* as systems of material points, whose cohesion is so feeble, that the smallest forces are sufficient to effect a separation, and to move them amongst each other (§ 59). Many bodies in nature, such as air, water, &c., possess this property of fluidity in a high degree; other bodies, on the contrary, such as oil, fat, soft earth, &c., are fluid in a low degree. The one are called *perfectly fluid,* the other *imperfectly fluid bodies.* Certain bodies, as for instance, paste, are intermediate between solid and fluid bodies.

Perfectly fluid bodies, of which only we shall subsequently speak, are at the same time perfectly elastic, *i. e.* they may be compressed by external forces, and will perfectly resume their former volume after the withdrawal of these forces. The amount of the change of volume corresponding to a certain pressure is different for different fluids; in *liquid bodies* this is scarcely perceptible, while in *aëriform bodies,* which, on this account, are also called elastic fluids, it is very great. This slight degree of compressibility of liquid bodies is the reason why in most investigations in hydrostatics (§ 63) they are considered and treated as incompressible or inelastic. As water, of all liquids, is the one most generally diffused and the most useful for the purposes of life, it is taken as the representant of all these fluids, and in the investigations

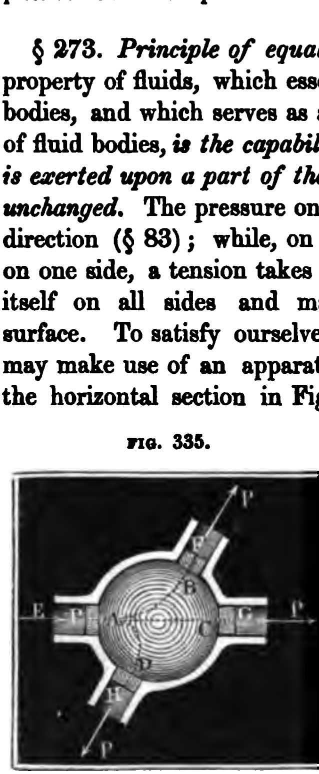
of the mechanics of fluids, water only is spoken of, whilst it is tacitly understood that the mechanical properties of other liquids are the same as those of water.

From a similar reason in the mechanics of the elastic fluid bodies ordinary atmospheric air is only spoken of.

*Remark.*—A column of water of one square inch transverse section is compressed by a weight of 15 lbs. which corresponds to the atmospheric pressure, by about 0,00005 or 50 millionths of its volume, while the same column of air under this pressure would be compressed to one half of its original volume.

§ 273. *Principle of equality of pressures.*—The characteristic property of fluids, which essentially distinguishes them from solid bodies, and which serves as a basis of the laws of the equilibrium of fluid bodies, ***is the capability of transmitting the pressure which is exerted upon a part of the surface of the fluid in all directions unchanged.*** The pressure on solids is transmitted only in its proper direction (§ 83); while, on the other hand, when water is pressed on one side, a tension takes place in the entire mass, which exerts itself on all sides and may be observed at all parts of the surface. To satisfy ourselves of the correctness of this law, we may make use of an apparatus filled with water, as is shown in the horizontal section in Fig. 335. The tubes $AE$ and $BF$, &c., equally distant and at an equal height above the horizontal base, are closed by perfectly moveable and accurately fitting pistons, the water presses, therefore, by its weight as strongly against the one piston as against the other. Let us do away with this pressure, and regard the water as devoid of weight. Let us press the one piston with a certain pressure $P$ against the water, this pressure will then be transmitted by the water to the other pistons $B$, $C$, $D$, and for the restoration of equilibrium, or to prevent the pushing back of these pistons, it is requisite that an equal and opposite pressure $P$ act against each of these pistons. We are, therefore, justified in assuming, that the pressure $P$, acting upon a point $A$ of the surface of the mass of water, produces in it a tension, and not only transmits this in the

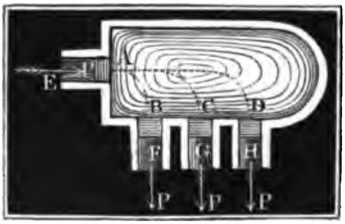
FIG. 335.



straight line $AC$, but also in every other direction $BF$, $DH$, &c., to every equal area of the surface $C$, $B$, $D$.

If the axes of the tubes $BF$, $CG$, &c., Fig. 336, are parallel to each other, the pressures which act upon their pistons may be united by addition into a single pressure; if $n$ be the number of the pistons, then the aggregate pressure upon these amounts to $P_1 = nP$, and in the case represented in the figure $P_1 = 3P$. But now the areas $F_1$ of the pressed surfaces $B$, $C$, $D$, are equal to $n$ times the pressed surface $F$ of the one piston, hence $n$ may not only be put $= \frac{P_1}{P}$, but also $= \frac{F_1}{F}$, therefore $\frac{P_1}{P} = \frac{F_1}{F}$.

FIG. 336.




If the tubes $B$, $C$, $D$, form a single one, as in Fig. 337, and if we close it by a single piston, $F_1$ then becomes a single surface, and $P_1$ is the pressure acting upon it, hence there follows this general law, ***the pressure which a fluid body exerts upon different parts of the sides of a vessel, is proportional to the area of these parts.***

FIG. 337.

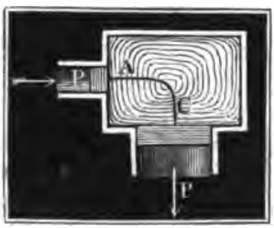
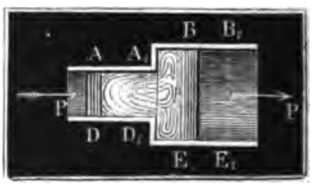



FIG. 338.




This law corresponds also to the principle of virtual velocities. If the piston $AD = F$, Fig. 338, moves inwards through a space $AA_1 = s$, it then presses the column of water $Fs$ from its tube, and if the piston $BE = F_1$ it passes outwards through the space $BB_1 = s_1$, it then leaves a space $F_1 s_1$ behind. But since we have supposed that mass of water neither allows of expansion nor compression,

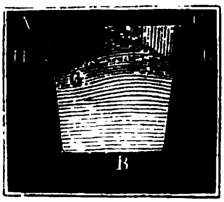
its volume then by this motion of the piston must remain unaltered, that is, the increase $Fs$ must be equal to the decrease $F_1 s_1$. But the equation $F_1 s_1 = Fs$ gives $\frac{F_1}{F} = \frac{s}{s_1}$, and by combining this proportion with the proportion $\frac{P_1}{P} = \frac{F_1}{F}$, it follows that $\frac{P_1}{P} = \frac{s}{s_1}$, hence, therefore, the mechanical effect $P_1 s_1$ = mechanical effect $Ps$. (§ 80).

*Example.* If the piston $AD$ has a diameter of $1\frac{1}{2}$ inches, and the piston $BE$ one of 10 inches, and each is pressed by a force $P$ of 36 lbs. upon the water, this piston exerts a pressure $P_1 = \frac{F_1}{F} P = \frac{10^2}{1,5^2} \cdot 36 = 1600$ lbs. If the first piston is pushed forwards 6 inches, the second will only go back by $s_1 = \frac{F}{F_1} s = \frac{9 \cdot 6}{400} = \frac{27}{200} = 0{,}135$in.

*Remark.* Numerous applications of this law will come before us in the hydraulic press, or water column machines, in pumps, &c.

§ 274. *The fluid surface.* — The gravity inherent in water causes all its particles to tend downwards, and they would actually so move unless this motion were prevented. In order to obtain a coherent mass of water, it is necessary to enclose it in vessels. The water in the vessel $ABC$, Fig. 339, is then only in equilibrium if its free surface $HR$ is perpendicular to the direction of gravity, and therefore horizontal, for so long as this surface is curved or inclined to the horizon; then there are elementary portions $E$, $F$, &c., lying higher, which from their extreme mobility in virtue of their gravity slide down on those below them, as if it were on an inclined plane $GK$.

FIG. 339.




Since the directions of gravity for great distances can no longer be regarded as parallel, we must, therefore, consider the free surface, or the level of water in a large vessel, as for example, in a great lake, no longer as a plane, but as part of a spherical surface.

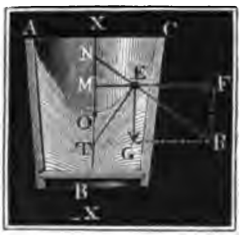
If any other force than that of gravity act upon the particles of water, the fluid surface in the state of equilibrium, will be perpendicular to the direction of the resultant arising from gravity and the concurrent force.

If a vessel $ABC$, Fig. 340, is moved forward horizontally by a

FIG. 340.


uniform accelerating force $p$, the free surface of the water in it will form an inclined plane $DF$, for in this case every element $E$ of this surface will be impelled downwards by its weight $G$, and horizontally by its inertia $P = \frac{p}{g} G$, there will then be resultant $R$, which will make with the direction of gravity a uniform angle $REG = a$. This angle is at the same time the angle $DFH$ which the surface of the water makes with the horizon. It is determined by $tang.\ a = \frac{P}{G} = \frac{p}{g}$.

FIG. 341.




If, on the other hand, a vessel $ABC$, Fig. 341, rotates uniformly about its vertical axis $X\overline{X}$, the surface of the water then forms a hollow surface $AOC$, whose sections through the axis are parabolic. If $\omega$ be the angular velocity of the vessel and the water in it, $G$ the weight of an element of water $E$, and $y$ its distance $ME$ from the vertical axis, we shall then have for the centrifugal force of this element $F = \omega^2 \frac{Gy}{g}$ (§ 231), and hence for the angle $REG = TEM = \varphi$, which the resultant $R$ makes with the vertical or the tangent to the water profile with the horizon:

$$tang.\ \phi = \frac{F}{G} = \frac{\omega^2 y}{g}. \qquad \overline{D_x y} = \frac{y}{p}$$

From this, therefore, the tangent of the angle which the line of contact makes with this ordinate is proportional to the ordinate. As this property belongs to the common parabola (§ 144), the vertical section $AOC$ of the surface of water is also a parabola whose axis coincides with the axis of revolution $X\overline{X}$.

If a vessel $ABH$ be moved in a vertical circle, Fig. 342, uniformly about a horizontal ~~parallel~~ axis $C$, the surface of the water will form in it a cylindrical surface with circular sections $DEH$. If we prolong the direction of the resultant $R$ of the gravity $G$, and the centrifugal force $F$ of an element $E$ to the intersection $O$ with the vertical $CK$ passing through the centre of revolution, we shall

FIG. 342.


then obtain the similar triangles $ECO$ and $EFR$, for which

$$\frac{OC}{EC}=\frac{FR}{EF}=\frac{G}{F},$$

but now, if we put the radius of gyration $EC=y$, and retain the last notation, $F=\frac{\omega^2 Gy}{g}$, it follows that the line

$\omega = \frac{2h}{T}$

$T = \frac{60}{u}$

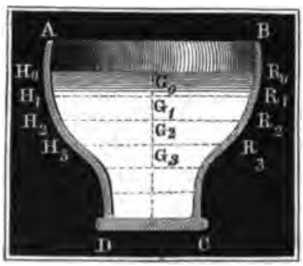
$$CO=\frac{g}{\omega^2}=g=\frac{32,2}{(3,1416)^2}\left(\frac{30}{u}\right)^2=\frac{2936}{u^2}$$

$\therefore \omega = \frac{hu}{30}$

if $u$ represents the number of revolutions per minute. As this value of $CO$ is one and the same for all the particles of water, it follows that the components of all the particles forming the section $DEH$ are directed towards $O$, and hence the section perpendicular to the directions of these forces is a circle described from $O$ as a centre. According to this, the surfaces of water in the buckets of an overshot wheel form perfect cylindrical surfaces, corresponding to one and the same horizontal axis.

§ 275. *Pressure on the bottom.*—The pressure of water in a vessel $ABCD$, Fig. 343, immediately under the water level is the least, but becomes greater and greater in proportion to the depth, and is greatest immediately above the bottom. To prove this generally, let us assume that the level of the water $H_0R_0$, whose area may be $F_0$, is uniformly pressed by a force $P_0$, for example, by the superincumbent atmosphere, or by a piston, and let us suppose the whole mass of water divided by horizontal planes, as $H_1R_1$, $H_2R_2$ into equally thick strata of water. If now $\lambda$ be the thickness or the height of such a stratum, and $\gamma$ the density of water, we shall then have the weight of the first stratum $G_1=F_0\lambda\gamma$, and hence the entire pressure on the subjacent water: $P_1=P_0+F_0\lambda\gamma$. If we divide this pressure by the area $F_1$ of the following horizontal section $H_1R_1$, we shall obtain the pressure for each unit of this surface: $p_1=\frac{P_0}{F_1}+\frac{F_0}{F_1}\lambda\gamma$, or, since $F_1$, on account of the infinitely small dis-

FIG. 343.

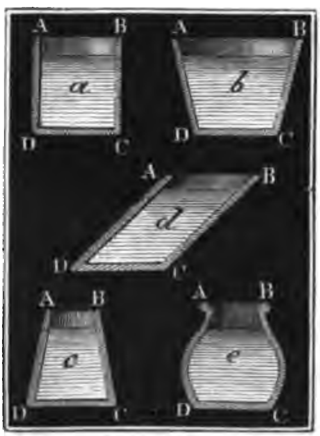


tance between $H_0R_0$ and $H_1R_1$, is infinitely little different from $F_0$, and may be substituted for this: $p_1 = p_0 + \lambda\gamma$, where $p_0$ represents the external pressure on the unit of surface. The pressure of the succeeding horizontal section $H_2 R_2$ may be determined exactly as the pressure of the stratum $H_1R_1$, if we take into consideration that the initial pressure upon the unit is now $p_1 = p_0 + \lambda\gamma$, whilst it was then only $p_0$. The pressure in the horizontal stratum $H_2R_2$ then follows: $p_2 = p_1+\lambda\gamma=p_0+\lambda\gamma+\lambda\gamma =p_0+2\lambda\gamma$; likewise the pressure in the third stratum $H_3R_3 =p_0+3\lambda\gamma$, in the fourth $= p_0+4\lambda\gamma$, and in the $n$th $= p_0+n\lambda\gamma$. But now $n\lambda$ is the depth $G_0G_n= h$ of the $n$th stratum below the level of the water, hence the pressure upon each unit of surface in the $n$th horizontal stratum may be put: $p = p_0 + h\gamma$.

The depth $h$ of an element of surface below the water level, is called the *head of water*, and the pressure of water upon any unit of surface may from this be found, if the externally acting pressure be increased by the weight of a column of water whose base is this unit, and whose height is the head of water.

The head of water $h$ on a horizontal surface, for instance, on the bottom $CD$ is at all places one and the same; hence the area of this surface $= F$, and the pressure of water against it is: $P=(p_0+h\gamma) F=Fp_0+Fh\gamma=P_0+Fh\gamma$, or if we abstract the outer pressure: $P=Fh\gamma$. *The pressure of water against a horizontal surface is therefore equivalent to the weight of the superincumbent column of water Fh.*

FIG. 344.




This pressure of water against a horizontal surface—against a horizontal bottom, for instance—or against a horizontal part of a lateral wall, is independent of the form of the vessel; whether, therefore, the vessel $AC$, Fig. 344, be prismatic as $a$, or wider above than below as $b$, or wider below than above, as $c$, or inclined as $d$, or bulging out as $e$, &c., the pressure on the bottom will be always equal to the weight of a column of water whose base is the bottom and whose height is the depth of the bottom below the level of the water. As the pressure of water transmits itself on all

## Pressure on the bottom.

Take axis Z vertical

pressure on element $dp$

$\therefore dp = \rho\, dz \therefore p = z\rho + p_1$

... $\{p = 0, \ldots\}$

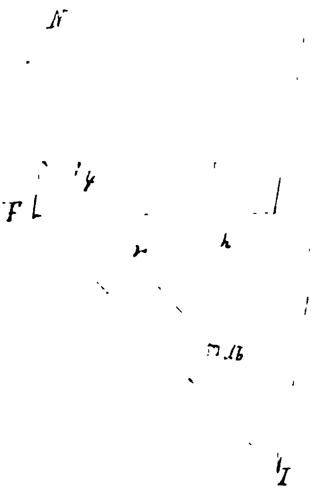
... $B$ ...

... pressure. Let $p$ = pressure on element

$\therefore \ldots\ \rho z\, ds \qquad P = \int p\, ds = \int \rho z\, ds$

$= \rho \bar{z}$ ...

## Centre of Pressure



$\rho h\, db = \rho \sin\phi\, r\, db$ = pressure on element

whole surface $\int \rho \sin\phi\, r\, db = \rho \sin\phi \int r\, db$

$= \rho \sin\phi\, \overline{BF}$ ...

$l_1$ = dist. of centre of pressure from $F$ ...

$\therefore l_1 \rho \sin\phi \int db\, r = \rho \sin\phi \int db\, r^2$ Mt. of Statics

$\therefore l_1 = \frac{\int db\, r^2}{\int db\, r}$

$l_2$ dist. from $FI$; $l_2 \rho \sin\phi \int db\, r = \rho \sin\phi \int db\, yr$

$\therefore l_2 = \frac{\int db\, yr}{\int db\, r}$ Centr. Stat. mt.

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sides, this law is therefore applicable when the surface, as $BC$, Fig. 345, is pressed upon from below upwards. Every unit of surface in the stratum lying in $BC$ is pressed by a column of water of the height $HB = RK = h$; consequently, the pressure against $CB = Fh\gamma$, $F$ being the area of the surface.

FIG. 345.

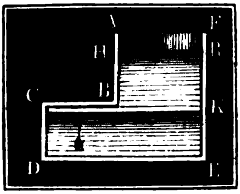
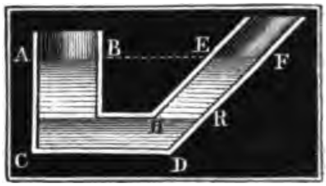


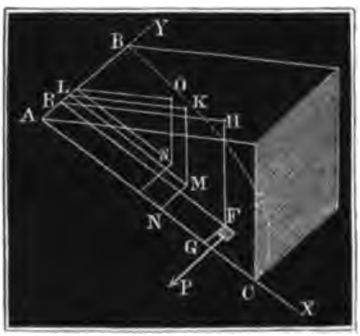
FIG. 346.




It further follows from this, that the water in tubes communicating with each other $ABC$ and $DEF$, Fig. 346, when equilibrium subsists, stands equally high, or that the two levels $AB$ and $EF$ are in one and the same horizontal plane. For the subsistence of equilibrium, it is requisite that the stratum of water $HR$ be as forcibly pressed downwards by the superincumbent column of water $ER$, as pressed upwards by the mass of water lying below it. But as in both cases the surface pressed is one and the same, so must the head of water in both cases be one and the same, therefore the level $AB$ must stand as high above $HR$ as the level $EF$.

§ 276. *Lateral pressure.*—The laws found above for the pressure of water against a horizontal surface, are not directly applicable to a plane surface inclined to the horizon; for in this case the heads of water at different places are different. The pressure $p = h\gamma$ on each unit of surface within the horizontal stratum of water, which lies a depth $h$ below the level, acts in all directions (§ 273), and consequently also perpendicular to the fixed lateral walls of the vessel, which (from § 128) perfectly counteract it. If now $F_1$ be the area of an element of a lateral surface $ABC$, Fig. 347, and $h_1$ its head of water $FH$, we shall then have the normal pressure of the water against it: $P_1 = F_1 \cdot h_1\gamma$; if $F_2$ be a second element of the surface, and $h_2$ its head of water,

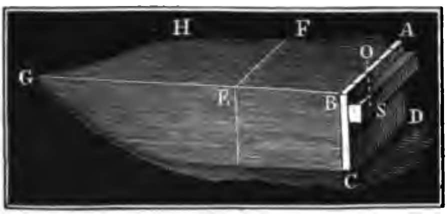
FIG. 347.



we shall then have the normal pressure on it: $P_2 = F_2 h_2 \gamma$; and for a third element $P_3 = F_3 h_3 \gamma$, &c. These normal pressures form a system of parallel forces, whose resultant $P$ is the sum of these pressures; therefore $P = (F_1 h_1 + F_2 h_2 + \ldots) \gamma$. But now, further, $F_1 h_1 + F_2 h_2 + \ldots$ is the sum of the statical moments of $F_1$, $F_2$, &c., with respect to the surface $OHR$ of the water, and $= Fh$, $F$ representing the area of the whole surface, and $h$ the depth $SO$ of its centre of gravity below the level; hence, the aggregate normal pressure against the plane surface is $P = Fh\gamma$. We mean here, by the head of water of a surface, the depth $SO$ of its centre of gravity below the level of the water; the general rule, therefore, is true that: ***the pressure of water against a plane surface is equivalent to the weight of a column of water whose base is the surface and whose height is the head of water of the surface.***

It must further be stated, that this pressure of the water is not dependent on the quantity of water which is before or below the pressed surface, that therefore, for example, a flood-gate, $AC$, Fig. 348, under otherwise similar circumstances, has to sustain the same pressure, whether the water to be dammed up be that of a small sluice $ACEF$, or that of a larger dam $ACGH$, or that of a great reservoir. From the breadth $AB = CD = b$ and height $AD = BC = a$ of a rectangular flood-gate, $F = ab$, and the head of water $SO = \frac{a}{2}$; hence, the pressure of water

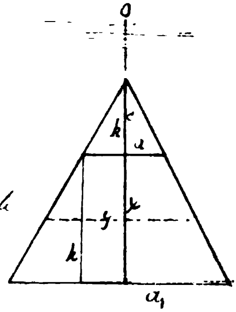
FIG. 348.




$$P = ab \cdot \frac{a}{2} \gamma = \tfrac{1}{2} a^2 b \gamma.$$

Therefore the pressure increases as the breadth, or as the square of the height of the pressed surface.

*Example.* If the water stand $3\frac{1}{2}$ feet high before a board of oak 4 feet broad, 5 feet high, and $2\frac{1}{2}$ inches thick, what will be the force required to draw it up? The volume of the board is $4 \cdot 5 \cdot \frac{5}{24} = \frac{25}{6}$ cubic feet. If now we take the density of oak saturated with water from § 58 at $62{,}5 \times 1{,}11 = 67{,}3$ lbs., the weight of this board will be: $G = \frac{25}{6} \cdot 67{,}3 = 280{,}5$ lbs. The pressure of the water against the board, and also the pressure of this last against the guides will be:



## Centre of Pressure.

Let $k$ be the depth of the upper side, $a$, $a_1$ the lower edge; $x$ = variable distance between them, $y$ any breadth. For an element of submerged surface we have $y\,dx$ and the pressure of the same $y\,dx\,(x+k)$ ∴ the moment with respect to the upper edge is $y\,dx\,(x^2+kx)$

∴ we have the distance of the centre of pressure

$$x_1 = \frac{\int_0^h y\,dx\,(x^2+kx)}{\int_0^h y\,dx\,(x+k)} = \frac{\text{moment of pressure}}{\text{pressure}}$$

To find $y$ in terms of $x$: $\quad \frac{c+x}{c} = \frac{y}{a}, \quad \frac{c+h}{c} = \frac{a_1}{a}$

$$\therefore \begin{cases} \frac{x}{c} = \frac{y}{a} - 1 \\ \frac{h}{c} = \frac{a_1}{a} - 1 \end{cases} \qquad \frac{x}{h} = \frac{\frac{y}{a}-1}{\frac{a_1}{a}-1} \qquad \therefore y = \frac{x(a_1-a)+ah}{h}$$

substituting

$$x_1 = \frac{\int_0^h (x^2+kx)[ah + x(a_1-a)]\,dx}{\int_0^h (x+k)[ah + x(a_1-a)]\,dx} = \frac{ah\left(\frac{h^3}{3}+\frac{kh^2}{2}\right)+(a_1-a)\left(\frac{h^4}{4}+\frac{kh^3}{3}\right)}{ah\left(\frac{h^2}{2}+kh\right)+(a_1-a)\left(\frac{h^3}{3}+\frac{kh^2}{2}\right)}$$

$$= \frac{a(4h^2+6kh)+(a_1-a)(3h^2+4kh)}{a(6h+12k)+(a_1-a)(4h+6k)} = \frac{ah^2+3a_1h^2+2akh+4a_1kh}{2ah+4a_1h+6ak+6a_1k}$$

$$= \frac{h}{2}\cdot\frac{h(3a_1+a)+2k(2a_1+a)}{h(2a_1+a)+3k(a_1+a)}$$

When $k = 0$ $\quad x_1 = \frac{h}{2}\cdot\frac{3a_1+a}{2a_1+a}$ the upper edge at the surface.

∴ $a = 0$ $\quad x_1 = \frac{3h+4k}{4h+6k}h$ for a triangle vertex up

∴ $a = 0$ & $k = 0$ $\quad x_1 = \frac{3}{4}h$ " " " at surface

If $a_1 = 0$ we have $x_1 = \frac{h}{2}\cdot\frac{h+2k}{h+3k}$ for a triangle vertex downwards; if $a_1 = a$ & $k = 0$ gives $x_1 = \frac{h}{2}$ a triangle whose base is at the surface.

If $a_1 = a$ $x_1 = \frac{h}{2}\cdot\frac{4h+6k}{3h+6k} = \frac{h}{3}\cdot\frac{2h+3k}{h+2k}$ for a parallelogram wholly submerged.

If $a_1 = a$ & $k = 0$ $x_1 = \frac{2}{3}h$ when one edge of the parallelogram is in the surface

If $k = \infty$ $x_1 = \frac{h}{2}\cdot\frac{4a_1+2a}{3a_1+3a} = \frac{h}{3}\cdot\frac{2a_1+a}{a_1+a}$

which is the centre of gravity, which is as it should be, when infinitely submerged the pressure would be uniform

This case is perfectly general; since the surface submerged is inclined the head is $k \sin \alpha$, $\alpha$ being the inclination

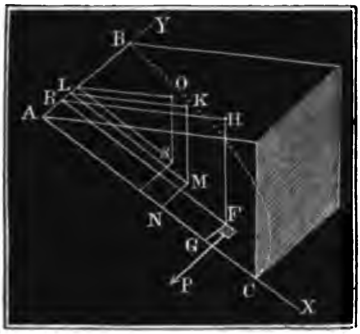
In this discussion we call the density unit

$P = \frac{1}{2} \cdot \left(\frac{7}{2}\right)^2 . 4 . 62,5 = 49 . 31,25 = 1531,25$ lbs.; if now we take the co-efficient of friction for wet wood from § 161, $f = 0,68$, the friction of this board against its guides will be $F = f P = 0,68 . 1531,25 = 1041,25$ lbs. If to this be added the weight of the board, we shall obtain the force required to pull it up $= 1041,25 + 67,3 = 1108,55$ lbs.

61.4 1102.65

§ 277. *Centre of pressure.*—The resultant $P = Fh\gamma$ of the collective elementary pressures $F_1h_1\gamma$, $F_2h_2\gamma$, &c., has, like every other system of parallel forces, a definite point of application, which is called the *centre of pressure.* Equilibrium will subsist for the whole pressure of the surface, if this point be supported. The statical moments of the elementary pressures $F_1h_1\gamma$, $F_2h_2\gamma$, &c., with respect to the plane of the level $OHR$, Fig. 349, are:

FIG. 349.




$F_1h_1\gamma . h_1 = F_1h_1^2\gamma$, $F_2h_2^2\gamma$, &c.; therefore, the statical moment of the whole pressure with respect to this plane is: $(F_1h_1^2 + F_2h_2^2 + \ldots)\gamma$. If we put the distance $KM$ of the centre $M$ of this pressure from the level of the water $= z$, we shall then have the moment of pressure $= Pz = (F_1h_1 + F_2h_2 + \ldots)z\gamma$, and by equating both moments, the depth in question of the centre $M$ below the surface;

$$1.\quad z = \frac{F_1h_1^2 + F_2h_2^2 + \ldots}{F_1h_1 + F_2h_2 + \ldots}, \text{ or} = \frac{F_1h_1^2 + F_2h_2^2 + \ldots}{Fh},$$

if, as before, $F$ represent the area of the whole surface, and $h$ the depth of its centre of gravity below the surface. To determine this pressure completely we must know further its distance from another plane or line. If we put the distances $F_1G_1$, $F_2G_2$, &c., of the elements of the surface $F_1$, $F_2$, &c., from the line $AC$ which determines the angle of inclination of the plane $= y_1$, $y_2$, &c., we shall then have the moments of the elementary pressures with respect to this line $= F_1h_1y_1\gamma$, $F_2h_2y_2\gamma$, &c., therefore, the moment of the whole surface $= (F_1h_1y_1 + F_2h_2y_2 + \ldots)\gamma$; and if we represent the distance $MN$ of the centre $M$ from this line by $v$, we shall then have the moment also $= (F_1h_1 + F_2h_2 + \ldots)v\gamma$; if, lastly, we make both moments equal we shall obtain the second ordinate:

2. $v = \frac{F_1 h_1 y_1 + F_2 h_2 y_2 + \ldots}{F_1 h_1 + F_2 h_2 + \ldots}$, or $= \frac{F_1 h_1 y_1 + F_2 h_2 y_2 + \ldots}{Fh}$.

If $a$ be the angle of inclination of the plane $ABC$ to the horizon, and $x_1$, $x_2$, &c., the distances $F_1 R_1$, $F_2 R_2$, &c., of the elements $F_1$, $F_2$, &c., as likewise $u$ the distance of the centre of pressure $M$ from the line of intersection $AB$ of the plane with the level of the water, we shall then have :

$h_1 = x_1 \sin. a$, $h_2 = x_2 \sin. a$, &c., as well as $z = u \sin. a$; and if these values be put into the expressions for $z$ and $v$, we shall then obtain :

$$u = \frac{F_1 x_1^2 + F_2 x_2^2 + \ldots}{F_1 x_1 + F_2 x_2 + \ldots} = \frac{\text{moment of inertia}}{\text{statical moment}}, \text{ and}$$

$$v = \frac{F_1 x_1 y_1 + F_2 x_2 y_2 + \ldots}{F_1 x_1 + F_2 x_2 + \ldots} = \frac{\text{centrifugal moment}}{\text{statical moment}}.$$

We may, therefore, find the distances $u$ and $v$ of the centre of pressure from the horizontal axis $AY$, and from the axis $AX$ formed by the line of fall, if we divide the statical moment of the surface with respect to the first axis, once by its moment of inertia with respect to the same axis, and a second time by its centrifugal moment with respect to both axes. The first distance is at once the distance of the centre of suspension from the line of intersection with the line of the water (§ 251). It is easy to see that the centre of pressure coincides perfectly with the centre of percussion, determined in § 270, if the line of intersection $AY$ of the surface with the level be regarded as the axis of revolution.

If the pressed surface is a rectangle $AC$, Fig. 350, with horizontal base $CD$, the centre of pressure $M$ will be found in the line $LK$ let fall upon $CD$ bisecting the basis, and will be distant $\frac{2}{3}$ of this line from the side $AB$ in the surface of water. If this rectangle does not reach the surface as in Fig. 351, if further the

FIG. 350.

FIG. 351.

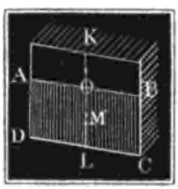
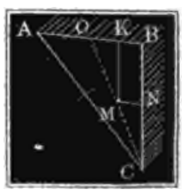


FIG. 352.



distance $KL$ of the lower base $CD$ from the surface be $l_1$, and

that of the upper base $AB = l_2$, we then have the distance $KM$ of the centre of pressure from the fluid surface:

$$u = \frac{2}{3} \cdot \frac{l_1^3 - l_2^3}{l_1^2 - l_2^2}.$$

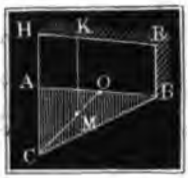
For the case of a right-angled triangle $ABC$, Fig. 352, whose cathetus $AB$ lies in the fluid surface, the distance $KM$ of the centre of pressure $M$ from $AB$ (§ 223), $u = \frac{\frac{1}{6} F . l^2}{\frac{1}{3} F . l} = \frac{1}{2} l$, if $l$ represent the height $BC$ of the triangle, and the distance of the same point from the other cathetus, as this point in every case lies in the line $CO$ bisecting the triangle, which passes from the point $O$ to the middle point of the base, $NM = v = \frac{1}{4} b$, where $b$ represents the base $AB$.

If the point $C$ lies in the surface, as in Fig. 353, therefore, the cathetus $AB$ below this point, we have

$$KM = u = \frac{\frac{1}{2} Fl^2}{\frac{2}{3} Fl} = \frac{3}{4} l \text{ and } NM = v = \frac{1}{2} \cdot \frac{b}{2} = \frac{1}{4} b.$$

FIG. 353.

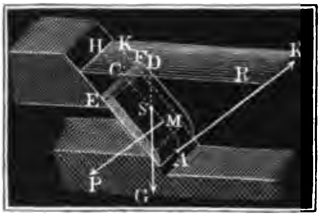

FIG. 354.




If the whole triangle $ABC$, Fig. 354, be under water, if the base

To obtain $NM = v = \frac{1}{4} b$

FIG. 355.

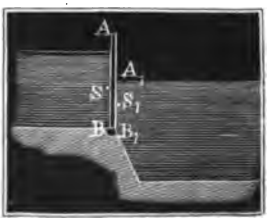



*Example.* What force $K$ must be expended to draw up a trap-door $AC$ turning about an axis $EF$, Fig. 355? Let its length $CA = 1\frac{1}{4}$ feet, its breadth $EF = 1\frac{1}{2}$ feet, its weight $= 35$ lbs.; further, the distance $CK$ of the axis of revolution $C$ from the surface $HR$, measured in the plane of the door, $= 1$ foot, and the angle of inclination of this plane to the horizon $= 68°$. The pressed surface is $F = \frac{3}{2} \cdot \frac{5}{4} = \frac{15}{8}$ square feet, and the head of water or the depth of its centre of gravity below the surface, $h = HS\, sin.\, \alpha = (HC + CS)\, sin.\, \alpha = (HC + \frac{1}{2} AC)\, sin.\, \alpha = \left(1 + \frac{1}{2} \cdot \frac{5}{4}\right) sin.\, 68°$ $= \frac{13}{8}\, sin.\, 68° = \frac{13 \,.\, 0{,}92718}{8} = 1{,}5067$ feet; hence, the pressure of water on the surface is: $P = Fh\gamma = \frac{15}{8} \,.\, 1{,}5067 \,.\, 66 = 186{,}45$ lbs. The arm of this force about the axis of revolution is the distance $CM$ of the centre of pressure $M$ from this axis; therefore $= HM - HC$

$$= \frac{2}{3} \cdot \frac{l_1^3 - l_2^3}{l_1^2 - l_2^2} - l_2 = \frac{2}{3} \cdot \frac{\left(\frac{9}{4}\right)^3 - \left(\frac{4}{4}\right)^3}{\left(\frac{9}{4}\right)^2 - \left(\frac{4}{4}\right)^2} - 1 = \frac{1}{6} \cdot \frac{729 - 64}{81 - 16} - 1$$

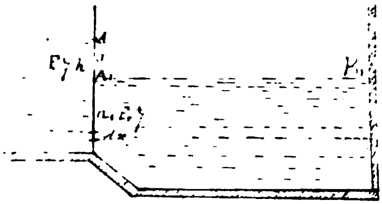
$= 0{,}705$ ft.; hence the statical moment of the pressure of water $= 186{,}45 \,.\, 0{,}705 = 131{,}46$ ft. lbs. If the centre of gravity $S$ of the trap-door lies about half the length $CS = \frac{1}{2} \cdot \frac{5}{4} = \frac{5}{8}$ feet from the axes of revolution, the arm $CD$ of the weight of the revolving door will be $= CS\, cos.\, \alpha = \frac{5}{8} \,.\, cos.\, 68° = \frac{5}{8} \,.\, 0{,}3746$ $= 0{,}2341$ ft., and hence the statical moment of this weight $= 35 \,.\, 0{,}2341 = 8{,}19$ ft. lbs. By the addition of both moments, we obtain the whole moment for drawing up the trap-door $= 131{,}46 + 8{,}19 = 139{,}65$ ft. lbs.; and if the force $K$ for this effect act at the arm $CA = 1{,}25$ feet, its amount will be $= \frac{139{,}65}{1{,}25} = 112$ lbs.

FIG. 356.




§ 279. If water presses against both sides of a plane surface $AB$, Fig. 356; there arises from the resultant forces corresponding to the two sides a new resultant, which is obtained by the subtraction of the former, because these two act oppositely to each other.

If $F$ is the area of the pressed portion on the one side of the surface $AB$, and $h$ the depth $AS$ of its centre of gravity below the level of the water; further, $F_1$ the area of the portion $A_1 B_1$ on the other side




$F = Fh\gamma \quad F_1 h_1 \gamma$

Moment about AB is $\frac{2}{3}h \cdot F'h\gamma - \frac{1}{3}$ [illegible] when [illegible] equal $F'x^2\gamma = M$

[illegible] moment [illegible] about AB $\frac{2}{3} h_1 F_1 h_1$ [illegible]

[illegible] equal $F_1\lambda_1^2$. Taken about AB

[illegible] $F_1\lambda_1^2$ [illegible] $= M$

[illegible] $\frac{M}{P}$ [illegible] $\frac{Fx^2 - F_1 x_1^2 - F_1 h_1 l}{Fh - F_1 h_1}$ [illegible] $\frac{\frac{2}{3}(Fh^2 - F_1 h_1^2) - F_1 h_1 l}{Fh - F_1 h_1}$

## [illegible]

[illegible] element pressure $= dx \cdot \gamma \cdot \frac{1}{2}h$ [illegible]

[illegible]

[illegible] $dx\gamma$ [illegible] moment [illegible] element.

$\int_0^h x\,dx - \int_0^{h_1} x\,dx$ [illegible] $\frac{\frac{h^3}{3} - \frac{h_1^3}{3}}{\frac{h^2}{2} - \frac{h_1^2}{2}}$

$\int x\,dx - \int_0^{h_1} x_1\,dx$

$= \frac{\frac{2h^3}{3} - \frac{2h_1^3}{3}}{h^2 - h_1^2}$ = moment [illegible] $\frac{F'\gamma h^2 - F_1'\gamma h_1^2 - F_1 \gamma h}{F'h - F_1'h_1}$

[illegible] $x^2$ [illegible]

[illegible] $w =$

---

Both portions equal; Resultant Pressure

$F'\gamma h$ [illegible] $F'\gamma h - F_1'\gamma h$ the Resultant pressure

[illegible] resultant [illegible] element

of the surface, and $h_1$ the depth $A_1 S_1$ of its centre of gravity below the corresponding level of the water, we then have for the resultant sought, $P = Fh\gamma - F_1 h_1 \gamma = (Fh - F_1 h_1)\,\gamma$.

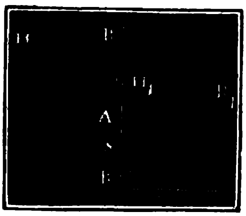
If the moment of inertia of the first portion of the fluid surface with respect to the line in which the plane of the surface intersects that of the water, $= Fx^2$, the statical moment of the pressure of water of the one side is, therefore, $= Fx^2 . \gamma$; if further, the moment of inertia of the second portion with respect to the line of intersection with the second surface of water $= F_1 x_1^2$, the statical moment of the pressure of water of the other side about the axis lying on the second surface is then $= F_1 x_1^2 \gamma$. Further, if the distance $AA_1$ of the axes $= a$, we then obtain the augmentation of the last moment in its transit from the axis $A_1$ to the axis $A$, $= F_1 h_1\, a\, \gamma$, and hence the statical moment of the pressure of water with respect to the axis in the first surface

$$= F_1 x_1^2 \gamma + F_1 h_1 . a . \gamma = (F_1 x_1^2 + F_1 a h_1)\,\gamma.$$

From this, then, it follows that the statical moment of the difference of both mean pressures $= (Fx^2 - F_1 x_1^2 - a\, F_1 h_1)\,\gamma$, and the arm of this latter force, or the distance of the centre of pressure from the axis in the first surface of water is:

$$u = \frac{Fx^2 - F_1 x_1^2 - a\,F_1 h_1}{Fh - F_1 h_1}.$$

FIG. 357.



If the portions of surface pressed are equal to one another, which takes place when, as Fig. 357 represents, the entire surface $AB$ is below the water, we have then more simply $P = F\,(h - h_1)\,\gamma$ and $u = h$; the last, because $h - h_1 = a$, and $x_1^2 = x^2 - 2\,a\,h + a^2$ (§ 217). In the last case, therefore, the pressure is equivalent to the weight of a column of water, whose base is the surface pressed, and whose height is the difference of altitude $R\,H_1$ of both surfaces of water, and the centre of pressure coincides with the centre of gravity $S$ of the surface. This law is also further correct if both surfaces of water are besides further pressed by equal forces, for example, by a piston or by the atmosphere. For this pressure upon each unit of surface $= p$, and therefore the corresponding height of a column of water $x = \frac{p}{\gamma}$ (§ 275), we have then to substitute for $h$, $h + x$, and for $h_1$, $h_1 + x$; and by subtraction, we have the residuary force

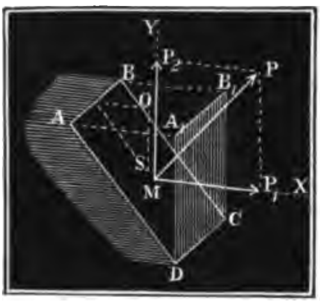
$P = (h + x - [h_1 + x])\,F\gamma = (h - h_1)\,F\gamma$. For this reason, the pressure of the atmosphere in hydrostatic investigations is generally left out of consideration.

*Example.* The height $AB$ of the upper surface of water in a canal, Fig. 356, amounts to 7 feet, the water in the lock stands 4 feet high at the sluice-gate, and the breadth of the canal and of the lock measure 7,5 feet, what mean pressure has the sluice-gate to sustain? It is $F = 7 \,.\, 7{,}5 = 52{,}5$, and $F_1 = 4 \,.\, 7{,}5 = 30$ square feet. Further, $h = \frac{1}{2} \,.\, 7 = \frac{7}{2}$ and $h_1 = \frac{4}{2} = 2$ feet, $a = 7 - 4 = 3$ feet, $x^2 = \frac{1}{3} \,.\, 7^2 = \frac{49}{3}$ and $x_1^2 = \frac{1}{3} \,.\, 4^2 = \frac{16}{3}$; hence it follows, that the mean pressure sought is: $P = (Fh - F_1 h_1)\,\gamma = (52{,}5 \,.\, \frac{7}{2} - 30 \,.\, 2) \,.\, 62{,}5 = 123{,}75 \,.\, 62{,}5 = 7734{,}375$ lbs.; and the depth of its point of application below the surface of the water is:

$$u = \frac{52{,}5 \,.\, \frac{49}{3} - 30 \,.\, \frac{16}{3} - 3 \,.\, 60}{52{,}5 \,.\, \frac{7}{2} - 60} = \frac{517{,}5}{123{,}75} = 4{,}182 \text{ feet.}$$

§ 280. *Pressure in a definite direction.*—In many cases it is of importance to know only one part of the pressure acting in a definite direction upon a surface. In order to find this component, we resolve the normal pressure $MP = P$ of the surface $AC = F$, Fig. 358, in the given direction $MX$, and in the direction $MY$ perpendicular to it into two component pressures $MP_1 = P_1$, and $MP_2 = P_2$. Let $a$ be the angle $PMX$, which the ~~normal~~ in the given direction $MX$ makes with the ~~component~~, we shall then obtain for the components; $P_1 = P\,cos.\,a$ and $P_2 = P\,sin.\,a$. Let a projection $A_1 B_1 CD$ of the surface $AB$ be made on a plane at right angles to the given direction $MX$, we shall then have for its area $F_1$, the formula $F_1 = F \,.\, cos.\, ADA_1$, or since the angle of inclination $ADA_1$ of the surface from its projection is equal to the angle $PMX = a$ between the normal pressure $P$ and its component $P_1$, we then have $F_1 = F\,cos.\,a$, or inversely: $cos.\,a = \frac{F_1}{F}$, and hence $P_1 = P \,.\, \frac{F_1}{F}$. But as the normal pressure $P = Fh\gamma$, it follows finally that $P_1 = F_1 h\gamma$, *i. e. the pressure with which water presses against a surface in a given direction, is equal*

FIG. 358.

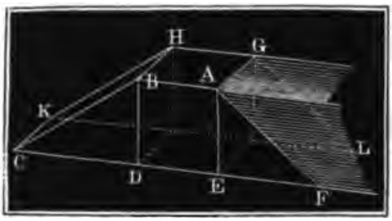


*to the weight of a column of water which has for base the projection of the surface perpendicular to the given direction, and for height, the depth of the centre of gravity of the surface below that of the water.*

It is important in most cases of application to know only the vertical or the horizontal component of the pressure of water against a surface. Since the projection at right angles to the vertical direction is the horizontal, and the projection at right angles to the horizontal direction, a vertical projection, the vertical pressure of water against a surface may be found, if the horizontal projection or its trace be considered as the surface pressed, and on the other hand the horizontal pressure of the water in any direction may be also found, if the vertical projection or the elevation of the surface at right angles to the given direction be considered as the surface pressed, but in both cases the depth of the centre of gravity of the surface below that of the water taken as the head of water.

For a prismatic pond-dam $ACH$, Fig. 359, the longitudinal profile $EG$ for the horizontal pressure of the water and the horizontal projection $EL$ of the surface of water for the vertical pressure must be regarded as the surfaces pressed. Hence if the length $AG$ of the dam $= l$, the height $AE = h$, and the front slope $EF = a$, we have then the horizontal pressure

FIG. 359.




of the water $= lh \cdot \frac{h}{2} \gamma = \frac{1}{2} h^2 l\gamma$, and its vertical pressure $= al \cdot \frac{h}{2} \gamma$

$= \frac{1}{2} alh\gamma$. If now, further, the upper breadth of the top of the dam $= b$, the slope at the back $CD = a_1$, and the density of the mass of the dam $= \gamma_1$, we then have the weight of the dam $= \left(b + \frac{a+a_1}{2}\right) hl\gamma_1$, and the whole vertical pressure of this against the horizontal bottom

$$= \frac{1}{2} alh\gamma + \left(b + \frac{a+a_1}{2}\right) hl\gamma_1 = \left[\frac{1}{2} a\gamma + \left(b + \frac{a+a_1}{2}\right) \gamma_1\right] hl.$$

If we put the co-efficient of friction $= f$, then the friction or force to push the dam forward is:

$$F = \left[\frac{1}{2}a\gamma + \left(b + \frac{a+a_1}{2}\right)\gamma_1\right] fhl.$$

In the case where the horizontal pressure of the water is to effect this displacement, we have:

$$\frac{1}{2}h^2 l\gamma = \left[\frac{1}{2}a\gamma + \left(b + \frac{a+a_1}{2}\right)\gamma_1\right] fhl, \text{ or more simply:}$$

$$h = f\left[a + \left(2b + a + a_1\right)\frac{\gamma_1}{\gamma}\right].$$

Therefore, in order that the dam may not be pushed away by the water, we must have:

$$h < f\left[a + \left(2b + a + a_1\right)\frac{\gamma_1}{\gamma}\right], \text{ or,}$$

$$b > \frac{1}{2}\left[\left(\frac{h}{f} - a\right)\frac{\gamma}{\gamma_1} - \left(a + a_1\right)\right].$$

For safety we assume that the base of the dam is quite permeable, on which account there is further a counter pressure from below upwards $= (b + a + a_1)\, lh\gamma$ to abstract, and we may put

$$h < f\left[\left(2b + a + a_1\right)\left(\frac{\gamma_1}{\gamma} - 1\right) - a_1\right].$$

*Example.* The density of the mass of a clay dam is nearly twice as great as that of water, therefore, $\frac{\gamma_1}{\gamma} = 2$ and $\frac{\gamma_1}{\gamma} - 1 = 1$; hence, for such a dam we may put simply $h < f(2b + a)$. According to experience, a dam will resist a long time if its height, slope and breadth at the top are equal to one another; if in the last formula we put $h = b = a$, then $f = \frac{1}{3}$, whence we must in other cases put $h = \frac{1}{3}\left[(2b + a + a_1)\left(\frac{\gamma_1}{\gamma} - 1\right) - a_1\right]$, and for clay dams especially, $h = \frac{1}{3}(2b + a)$, and inversely, $b = \frac{3h - a}{2}$. If the height of the dam be 20 feet and the angle of slope $\alpha = 36°$, the slope $a$ will be

$$= h \operatorname{cotg.} \alpha = 20 \,.\, \operatorname{cotg.} 36° = 20 \,.\, 1{,}3764 = 27{,}53 \text{ feet,}$$

and hence the upper breadth of the dam $b = \frac{60 - 27{,}53}{2} = 16{,}24$ feet.

§ 281. *Pressure on curved surfaces.*—The law found in the

last paragraph on the pressure of water in a definite direction is true only for plane surfaces, or for the separate elements of curved surfaces, but not for curved surfaces in general. The normal pressures on the separate elements of a curved surface may be resolved into lateral components parallel to a given direction, and into others acting in the plane normal to it; these components form a system of parallel forces, whose resultant gives the pressure in the given direction, and these components may be reduced to a resultant, but the two resultants admit of no further composition when their directions do not intersect. It is not possible in general to reduce the aggregate pressures against the elements of a curved surface to a single force, but particular cases present themselves where this composition is possible.

Let $G_1$, $G_2$, $G_3$, &c., be the projections, and $h_1$, $h_2$, $h_3$, &c., the heads of water of the elements $F_1$, $F_2$, $F_3$, &c., of a curved surface, we then have the pressure of water in the direction perpendicular to the plane of projection:

$$P_1 = (G_1 h_1 + G_2 h_2 + G_3 h_3 + \ldots)\,\gamma,$$

and its moment with respect to the plane of the surface of water:

$$P_1 u = (G_1 h_1^2 + G_2 h_2^2 + G_3 h_3^2 + \ldots)\,\gamma.$$

If the curved surface pressed upon can be decomposed into elements which have a uniform ratio to their projections, we may then put

$$\frac{F_1}{G_1} = \frac{F_2}{G_2} = \frac{F_3}{G_3}, \text{ \&c.}, = n, \text{ we then have:}$$

$$P_1 = \left(\frac{F_1 h_1}{n} + \frac{F_2 h_2}{n} + \ldots\right)\gamma = \left(\frac{F_1 h_1 + F_2 h_2 + \ldots}{n}\right)\gamma,$$

or, since the ratio of the entire curved surface $F$ to its projection

$G$, *i.e.* $\frac{F}{G}$ is $= n$,

$P_1 = \frac{Fh}{n}\gamma = Gh\gamma$; in this case we have, as for every plane surface, the pressure in any direction equivalent to the weight of a prism of water, whose basic surface is at right angles to the projection of the curved surface in the given direction, and whose height is equal to the depth of the centre of gravity of the curved surface below the surface of water.

FIG. 360.

So, for example, the vertical pressure of water against the envelope of a conical vessel $ACB$, filled with water, Fig. 360, is equal to the weight of a column of water which has the bottom for its base, and two thirds of the length of the axis $CM$ for height, because the horizontal projection of the envelope of a right cone upon its base, as likewise the envelope, may be resolved into exactly similar triangular elements, and because the centre of gravity $S$ of the surface of the cone is distant two thirds of the height of the cone from the vertex, (§ 110). If $r$ be the radius of the base, and $h$ the height of the cone, we shall then have the pressure against the bottom $= \pi r^2 h\gamma$, and the vertical pressure against the envelope $= \frac{2}{3}\pi r^2 h\gamma$; but as the bottom is rigidly connected with the sides, and both pressures act opposed to each other, the force with which the vessel is pressed downwards by the water is:

$$= \left(1 - \frac{2}{3}\right)\pi r^2 h\gamma = \frac{1}{3}\pi r^2 h\gamma =$$

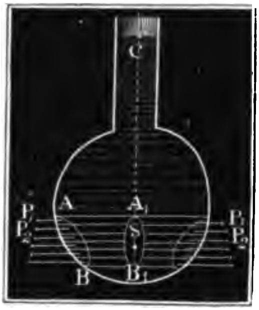
to the weight of the whole mass of water. If the bottom be separated by a fine cut from the envelope, this will then press with its full force $\pi r^2 h\gamma$ downwards, or on its support, and on the other hand it would be necessary to hold down the envelope with a force $\frac{2}{3}\pi r^2 h\gamma$ to prevent its being raised off.

FIG. 361.


*Remark.* From this the force which the steam of a steam-engine or the water of a water-column machine exerts on the piston, is independent of the form of the piston. Whether the surface of pressure be augmented by being hollowed out or rounded, the pressure with which the steam or water pushes forward the piston is equivalent to the product of the cross section or horizontal projection of the piston and the pressure on a unit of surface. The pressure on the larger surface of a funnel-shaped piston $AB$, Fig. 361, whose greater radius $CA = CB = r$ and lesser radius $GD = GE = r_1$, is $= \pi r^2 p$, and the reaction upon the envelope $= \pi(r^2 - r_1^2)p$; hence, the residuary effective pressure is $= \pi r^2 p - \pi(r^2 - r_1^2)p = \pi r_1^2 p =$ the cross section of the cylinder multiplied by the pressure on a unit of surface.

§ 282. *Horizontal and vertical pressure.*—Whatever may be the

FIG. 362.




form of a curved surface $AB$, Fig. 362, the horizontal pressure of the water against it is always equivalent to the weight of a column of water, whose base is the vertical projection $A_1B_1$ of the surface perpendicular to the given direction of pressure, and whose height of pressure is the depth $CS$ of the centre of gravity $S$ of the projection below the surface of water. The correctness of this follows directly from the formula $P_1 = (G_1h_1 + G_2h_2 + \ldots)\ \gamma$, when we consider that the height of pressure $h_1$, $h_2$, &c., of the elements of the surface are also the heights of pressure of their projections, that therefore $G_1h_1 + G_2h_2 + \ldots$ is the statical moment of the whole projection, *i. e.* the product $Gh$ of the vertical projection $G$ and the depth $h$ of its centre of gravity below the surface of water. We have here, therefore, again to put $P_1 = Gh\gamma$, and to consider $h$ as the height of pressure of the vertical projection.

The vertical section which divides a vessel containing water into two equal or unequal parts, is at once the vertical projection of the two parts, but the horizontal pressure on one part of the wall of the vessel is proportional to the product of its vertical projection and ~~to~~ the depth of its centre of gravity below the surface of the water, consequently the horizontal pressure on a part of the wall of the vessel is exactly equal in amount to the oppositely acting horizontal pressure on the part opposite, and consequently the two forces balance each other in the vessel; the whole vessel is therefore equally pressed by the enclosed water in all horizontal directions.

If an opening $O$ be made in the side of a vessel $HBR$, Fig. 363, the part of the pressure corresponding to the section of this opening disappears, and the pressure on the oppositely situated portion of the surface $F$ now comes into action. Whilst, therefore, the water flows out at the lateral aperture, an equal distribution of the horizontal pressure no longer takes place over the whole extent, and there ensues a reaction opposite to the motion of the flowing water: $P = Fh\gamma$, $F$ being the

FIG. 363.

projection of the aperture, and $h$ the height of pressure of its projection. By this reaction the vessel may be set into motion.

The vertical pressure of water is $P_1 = G_1 h_1 \gamma$ against an element of surface $F_1$, Fig. 364, of the side of the vessel, since the horizontal projection $G_1$ may be regarded as the transverse section, and the height of pressure $h_1$ as the height, and therefore $G_1 h_1$ as the volume of a prism, equivalent to the weight of a column of water $HF_1$ incumbent on the element and reaching the surface of the water. The elements of the surface which make up a finite part of the bottom, or side of the vessel, hence suffer a vertical pressure which is equivalent to the weight of all the incumbent columns of water, *i. e.* to the weight of a column of water incumbent on the whole portion. Let this volume $= V_1$, we then obtain for the vertical pressure $P = V_1 \gamma$. For another portion $A_1 B_1$, which lies vertically above the former, we have the vertical pressure opposed to it $Q = V_2 \gamma$; but if both portions are rigidly connected with each other, there results from the two forces the force acting vertically downwards $R = (V_1 - V_2)\gamma = V\gamma =$ *to the weight of the columns of water contained between the two portions of the surface.* If lastly we apply this law to the whole vessel, it follows that *the aggregate vertical pressure of the water against the vessel is equivalent to the weight of the enclosed mass of water.*

FIG. 364.


§ 283. *Thickness of pipes.*—The application of the laws of the pressure of water to pipes, boilers, &c., is of particular importance. That these vessels may adequately resist the pressure, and be prevented bursting from its effect, we must give a certain thickness to their sides, corresponding to the head of water and the internal width. The bursting of a pipe may take place in various ways, viz. transversely or longitudinally; the latter happens more frequently than the former, as will be soon understood from what follows.

The width of the pipe $BD = 2r$, Fig. 365, and the head of water $CK = h$, therefore the pressure on a unit of surface $p = h\gamma$, we then have the whole pressure in the direction of the axis of the pipe $= \pi r^2 p = \pi r^2 h \gamma$; if the thickness of the side $AB = DE = e$,

FIG. 365.


we then have the transverse section of the mass of the pipe $= \pi\ (r+e)^2 - \pi r^2$ $= 2\,\pi r e + \pi e^2 = 2\pi r e\left(1 + \frac{e}{2r}\right)$, and if lastly we put the modulus of ~~elasticity~~ rupture $= K$, we then have the pressure for rupture over the whole section of the pipe $= 2\,\pi r e\left(1 + \frac{e}{2\,r}\right)\ K$, for this reason we have now to put:

$$2\,\pi r e\left(1 + \frac{e}{2\,r}\right) K = \pi r^2 p,$$

or approximately and more simply $2\,e\,K = rp$, and hence the thickness of the pipe $e = \frac{rp}{2\,K} = \frac{rh\gamma}{2\,K}$. In order, therefore, to avoid any transverse rent in the pipe or in the boiler, the thickness of the sides must be made $e > \frac{rp}{2\,K}$. Of all longitudinal rents, *AE, LH,* &c., those running diametrically, such as *AE,* take place the most easily, because they have the smallest area, whence we must only take these into account. Let us consider a portion of a pipe of the length $l$, and let us have regard to the occurrenee of a rent of the length $l$, we then obtain a ~~transverse section of~~ the surface of separation $= le$, and hence the force for rupture in this surface $leK$. For two oppositely situated rents this force is $= 2\,leK$, whilst the pressure of water for each half of the pipe is proportional to the transverse section $2\,rl$, and hence is $= 2\,rlp$. By equating the two expressions, it follows that $2\,leK = 2\,rlp$, i. e. $e\,K = rp$, therefore the thickness $e = \frac{rp}{K}$. To provide against longitudinal rents, the sides must be made as thick again, as to provide against transverse rents.

From the formula $e = \frac{rp}{K} = \frac{rh\gamma}{K}$, it follows that *the strength of similar pipes are as the widths and as the heads of water or pressures upon a unit of surface.* A pipe three times the width of another, which has five times the pressure to sustain on each unit of surface that the other has, must have its sides fifteen times as thick.

Hollow spheres, which have to sustain a pressure $p$ from within

on each unit of surface, require a thickness $e=\frac{rp}{2K}$, because here the projection of the surface of pressure is the greatest circle $\pi r^2$, and the surface of separation the ring $2\pi re\left(1+\frac{e}{2r}\right)$, or approximately for a smaller thickness $= 2\pi re$.

The formula found give for $p = o$, also $e = o$, for this reason, therefore, pipes which have no internal pressure to sustain, may be made indefinitely thin; but as each pipe must sustain a certain pressure from its own weight, we must still give to it a certain thickness $e_1$, to obtain the strength of a tube which will resist under all circumstances. Hence, for cylindrical pipes or boilers we must put $e = e_1 + \frac{rh\gamma}{K}$, or more simply, if $d$ represents the inner width of the pipe, $n$ the pressure in atmospheres, each corresponding to a column of water 33 ft. high, and $\mu$ a number from experiment $e = e_1 + \mu nd$.

From experiments made we must take for pipes of

| | | |
|---|---|---|
| Iron plate . . . | $e=0{,}00086\ nd+0{,}12$ | inches |
| Cast iron . . . | $e=0{,}00238\ nd+0{,}33$ | " |
| Copper . . . | $e=0{,}00148\ nd+0{,}16$ | " |
| Lead . . . | $e=0{,}00242\ nd+0{,}20$ | " |
| Zinc . . . | $e=0{,}00507\ nd+0{,}16$ | " |
| Wood . . . | $e=0{,}0323\ nd+1{,}04$ | " |
| Natural stones . . | $e=0{,}0369\ nd+1{,}15$ | " |
| Artificial stones . . | $e=0{,}0538\ nd+1{,}53$ | " |

*Example.* If a perpendicular water-column machine has cast-iron pipes of 10 inches inner width, how thick must these be at 100, 200, and 300 feet depths? From the formula, for 100 feet pressure, this thickness is:

$$= 0{,}00238 \cdot \frac{100}{33} \cdot 10 + 0{,}33 = 0{,}07 + 0{,}33 = 0{,}40 \text{ inches};$$

for 200 feet, $= 0{,}14 + 0{,}33 = 0{,}47$ inches; and for 300 feet pressure, $= 0{,}22 + 0{,}33 = 0{,}55$ inches. Cast-iron conducting pipes are commonly proved at 10 atmospheres, for which reason, $e = 0{,}0238 \cdot d + 0{,}33$ inches; therefore, for pipes of 10 inches width, the thickness $e = 0{,}24 + 0{,}33 = 0.57$ inches must be given.

*Remarks.* The thickness of the sides of steam-boilers will be considered in the Second Part. Concerning the theory of the strength of pipes, a treatise by Brix, in the "Verhandlungen des Vereins zur Beförderung des Gewerbfleiszes in Preuszen," Jahrgang 1834, may be consulted. The technical relations and the proving of pipes is fully treated of in Hagen's "Handbuch der Wasserbaukunst," vol. I. and in Genieys' "Essai sur les moyens de conduire, &c., les eaux."

General discussion of the conditions of equilibrium of any fluid when acted upon by given forces

Take axis of Z vertical let x y z be the coordinates of any point of the fluid.

Divide the fluid into elements by planes parallel to the coordinate planes

An element is $dx\,dy\,dz$. Let $\rho$ be the density of the fluid at this point (some function of $x\,y\,z$)

$dm = \rho\,dx\,dy\,dz$ Let $X\,dm$ $Y\,dm$ $Z\,dm$ be the resulting action on $dm$ along the coordinate axes. The parallelopipedon $dx\,dy\,dz$ is pressed on all sides. The pressures maintain it in equilibrium. Let $p$ = pressure on a unit of surface at the pt. $(x\,y\,z)$ acting vertically down ~~ward on the upper surface~~.

$\therefore p\,dx\,dy$ = pressure on upper surface of an element. $p$ is a function of $x\,y\,z$

pressure on the lower surface $(p + \frac{dp}{dz} dz)$ [illegible]

since $z$ becomes $z + dz$ & $p$ becomes $p + \frac{dp}{dz}$ [illegible]

resultant vertical pressure = [illegible]

of the pressures on the two bases =

$\frac{dp}{dz} dz\, dx\, dy$ ∴ since there is [illegible]

$\frac{dp}{dz} dz\, dx\, dy = Z\, dm = Z\, \rho\, dx\, dy\, dz$

or $\frac{dp}{dz} = \rho Z$ [illegible] in the same manner

we obtain $\frac{dp}{dy} = \rho Y$ [illegible] $\frac{dp}{dx}$ [illegible]

multiplying [illegible] by $dx$, $dy$, $dz$ [illegible]

and adding, $\frac{dp}{dx} dx + \frac{dp}{dy} dy + \frac{dp}{dz} dz$ [illegible]

[illegible] $(X\, dx + Y\, dy + Z\, dz)$ [illegible] $= dp$ ∴ $p =$ [illegible]

---

If [illegible] a level surface [illegible]

one which is every where [illegible]

to the resultant of the [illegible]

[illegible]

[illegible] a surface [illegible]

[illegible] drawn & $ds$ its an element [illegible] ∴

$ds \cos x = dx$ ∴ $\cos x = \frac{dx}{ds}$

$ds \cos y = dy$ ∴ $\cos y = \frac{dy}{ds}$

$ds \cos z = dz$ ∴ $\cos z = \frac{dz}{ds}$

Also if R [illegible] resultant, etc.

[illegible] $\frac{X}{R}, \frac{Y}{R}, \frac{Z}{R}$ are the cosines of the angles which R makes with the coordinate axes and the [illegible] made [illegible] the curve with the resultant force

$\therefore \cos [illegible] = \frac{dx}{ds}\cdot\frac{X}{R} + \frac{dy}{ds}\cdot\frac{Y}{R} + \frac{dz}{ds}\cdot\frac{Z}{R}$ (b)

[illegible] cos = [illegible] member

[illegible] $Xdx + Ydy + Zdz = 0 \quad \therefore dp = 0$

or $p$ = const. [illegible]

II

Conversely if we [illegible]

[illegible]

[illegible] constant;

we shall then have [illegible]

[illegible] cannot [illegible]

$\therefore$ [illegible] R [illegible]

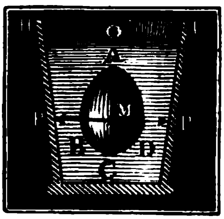
$\therefore$ [illegible] we prove that the resultant is normal to the [illegible].

# CHAPTER II.

## ON THE EQUILIBRIUM OF WATER WITH OTHER BODIES.

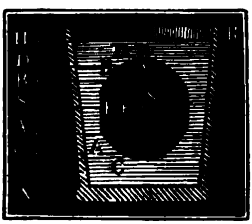
§ 284. *Buoyancy.*—A body immersed under water is pressed upon by the water on all sides, and now the question arises as to the amount, direction and point of application of the resultant of all these pressures. Let us imagine this resultant to consist of a vertical and a horizontal component, and determine these forces according to the rules of § 282. The horizontal pressure of the water against a surface is equivalent to the horizontal pressure against its vertical projection, but now every projection of a body $AC$, Fig. 366, is at the same time the projection of the fore part $ADC$ and the back part $ABC$ of its surface; hence, also, the horizontal pressure of water against the back portion of the surface of a body is equal in amount to that of the front portion, and as both pressures are exactly opposite, their resultant $=0$. As this relation takes place for every arbitrary horizontal direction, and the vertical projection corresponding to this, it follows that the resultant of all the horizontal pressures is nothing; that, therefore, the body $AC$ below the water is equally pressed in all horizontal directions, and for this reason exerts no effort to move forward in a horizontal direction.

FIG. 366.



To find the vertical pressure of the water against the body $BCS$, Fig. 367, let us suppose it made up of vertical elementary prisms, $AB$, $CD$, &c., and determine the vertical pressures on their terminating surfaces $A$ and $B$, $C$ and $D$. If the lengths of these prisms are $l_1$, $l_2$, &c., the depths of their upper extremities $B$, $D$, &c., below the surface of water $HR$: $h_1$, $h_2$, &c., and the horizontal transverse sections $F_1$, $F_2$, &c., we then have for the vertical pressures acting from above downwards against the extremities $B$, $D$, &c., $= F_1 h_1 \gamma$, $F_2 h_2 \gamma$, &c.; on the other hand, the pressures acting from below upwards and

FIG. 367.



against the extremities $A, C$, &c. $= F_1 (h_1+l_1)\gamma, F_2 (h_2+l_2)\gamma$, &c.; and it now follows, from a composition of these parallel forces, that the resultant $P$

$$= F_1 (h_1+l_1)\gamma + F_2 (h_2+l_2)\gamma + \dots - F_1 h_1 \gamma - F_2 h_2 \gamma - \dots$$
$$= (F_1 l_1 + F_2 l_2 + \dots)\gamma = V\gamma,$$

if $V$ represents the volume of the immersed body or the water displaced.

*Therefore the buoyancy or the force with which the water strives to push a body immersed from below upwards, is equivalent to the weight of water displaced, or to a quantity of water which has the same volume as the submerged body.*

Further, to find the point of application of this resultant, let us put the distances $AA_1$, $CC_1$, &c., of the elementary columns $AB$, $CD$, &c., from a vertical plane $HN$: $a_1$, $a_2$, &c., and determine the moments of the forces with respect to this plane. If $S$ is the point of application of the upward pressure, and $SS_1 = x$ its distance from that principal plane, we shall then have:

$$V\gamma \cdot x = F_1 l_1 \gamma \cdot a_1 + F_2 l_2 \gamma \cdot a_2 + \dots; \text{ and hence,}$$

$$x = \frac{F_1 l_1 a_1 + F_2 l_2 a_2 + \dots}{F_1 l_1 + F_2 l_2 + \dots} = \frac{V_1 a_1 + V_2 a_2 + \dots}{V_1 + V_2 + \dots}, \text{ if } V_1, V_2,$$

&c., represent the contents of the elementary columns. Since (from § 100) the centre of gravity is accurately determined by the same formula, it follows *that the point of application $S$ of the upward pressure coincides with the centre of gravity of the water displaced.*

§ 285. The weight $G$ of the body acting in an opposite direction associates itself with the buoyaney of the body immersed or under water, and from the two there arises a resultant $R = G - V\gamma$ or $= (\varepsilon - 1)\, V\gamma$, if $\varepsilon$ be the specific gravity of the body.

FIG. 368.

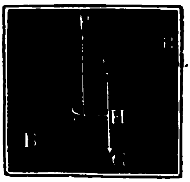

If the mass of the body be homogeneous, the centre of gravity of the displaced water will coincide with that of the body, and hence this point will be the point of application of the resultant $R$; but if there be not homogenity, then these centres of gravity do not coincide, and the point of application of the resultant $R$ deviates from both centres of gravity. Let us put the horizontal distance $SH$, Fig. 368, of both centres of gravity from each other, $= b$, and the horizontal distance $SA$ of the point of application $A$

sought from the centre of gravity $S$ of the displaced water $= a$, we shall have the equation $Gb = Ra$, from which is given:

$$a = \frac{Gb}{R} = \frac{G\,b}{P - P}.$$

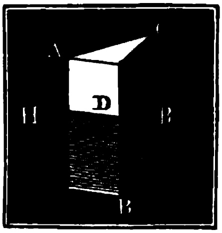
If the immersed body be left to its own gravity, the three following cases may present themselves. Either the specific gravity of the body is equal to that of the water, or it is greater, or it is less than the specific gravity of the water. In the first case the buoyancy is equal, in the second it is less, and in the third it is greater than the weight of the ~~water~~ body. Whilst, in the first case, equilibrium subsists between the weight and the buoyancy, the body must in the second case sink with the force $G - V\gamma = (\epsilon - 1)\,V\gamma$, and, in the third case, rise with the force $V\gamma - G = (1 - \epsilon)\,V\gamma$. The rising goes on only as long as the mass of water $V_1$, cut off from the plane of the surface and displaced by the body, has the same weight as the entire body. The weight $G = V\epsilon\,\gamma$ of the body $BB_1$, Fig. 369, and the buoyancy $P = V_1\,\gamma$ now constitute a couple, by which the body is made to revolve until the directions of both coincide, or until the centre of gravity of the body lies in one and the same vertical line with the centre of gravity of the displaced water.

FIG. 369.



The line passing through the centre of gravity of the floating body and through that of the displaced water, is called *the axis of floatation*; and on the other hand, the section of the body formed by the plane of the surface of the water, *the plane of floatation*. Every plane which divides a body, so that one part is to the whole as the specific gravity of the body to that of the fluid, and that the centres of gravity of the two parts lie in a line normal to this plane, is a plane of floatation of the body.

FIG. 370.




§ 286. *Depth of floatation.*—If the figure and weight of a floating body be known, the depth of immersion may be calculated beforehand, with the help of the previous rule. If $G$ be the weight of the body, we may then put the volume of the displaced water $V = \frac{G}{\gamma}$; if we combine with it the stereometrical formula for the volume $V$, we shall obtain the equation of condition. Hence, for the prism $ABC$, Fig. 370,

with vertical axis, for example, $V = Fy$, if $F$ represent the section and $y$ the depth $BD$ of immersion, $Fy = \frac{G}{\gamma}$ and $y = \frac{G}{F\gamma}$. For a pyramid $ABC$, Fig. 371, whose vertex floats under the water, $V = \frac{1}{3} f y^3$, if $f$ represents the section at the distance of unity from the vertex; hence it follows, that:

$$\tfrac{1}{3} f y^3 = \frac{G}{\gamma}, \text{ and hence the depth } CE = y = \sqrt[3]{\frac{3\,G}{f\gamma}}.$$

FIG. 371.

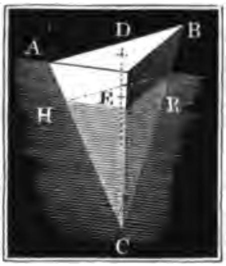



FIG. 372.


For a pyramid $ABC$, Fig. 372, floating with its base below the water, the distance is given $CE = y_1$ of the vertex from the surface, from the height $h$ of the entire pyramid, if we put:

$$V = \tfrac{1}{3} f (h^3 - y_1^3),\; y_1 = \sqrt[3]{h^3 - \frac{3\,G}{f\gamma}}.$$

For a sphere $AB$, Fig. 373, with the radius $CA = r$, $V = \pi y^2 \left(r - \frac{y}{3}\right)$, hence we shall have to solve the cubic equation $y^3 - 3 r y^2 + \frac{3\,G}{\pi\gamma} = 0$, to find the depth of immersion $DE$ of the sphere.

FIG. 373.

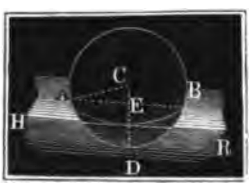
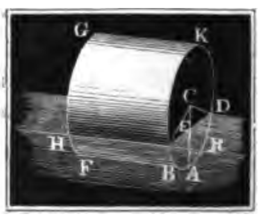



FIG. 374.




For a floating cylinder $AG$, with horizontal axis, Fig. 374, of a radius $BC = DC = r$, if $a^0$ be the angle $BCD$ subtended at the

centre by the arc immersed, the depth of immersion $AE = y = r(1 - cos. \frac{1}{2}\alpha)$, but to find the arc immersed, we must put the volume of the water displaced = to the segment $\frac{r^2\alpha}{2}$ less the triangle $\frac{r^2 sin. \alpha}{2}$, multiplied by the length $GK = l$ of the cylinder; therefore, $(\alpha - sin. \alpha)\frac{lr^2}{2} = \frac{G}{\gamma}$, and solve the equation $\alpha - sin. \alpha = \frac{2G}{lr^2\gamma}$, by approximation, with respect to $\alpha$.

*Examples.*—1. A wooden sphere, of 10 inches diameter, floats $4\frac{1}{2}$ inches deep, the volume of water displaced by it is then:

$$V = \pi\left(\frac{9}{2}\right)^2\left(5 - \frac{9}{6}\right) = \frac{\pi . 81 . 7}{8} = \frac{567 . \pi}{8} = 222{,}66 \text{ cubic inches,}$$

whilst the solid contents of the sphere are $\frac{\pi d^3}{6} = \frac{\pi . 10^3}{6} = 523{,}6$ cubic inches. From this, 523,6 cubic inches of the mass of the sphere weigh as much as 222,66 cubic inches of water, and it follows that the specific gravity of the former is: $\varepsilon = \frac{222{,}66}{523{,}6} = 0{,}425$..

2. How deep will a wooden cylinder of 10 inches diameter and specific gravity $\varepsilon = 0{,}425$, sink? $\frac{\alpha - sin. \alpha}{2} = \frac{\pi r^2 l . \varepsilon\gamma}{lr^2\gamma} = \pi\varepsilon = 0{,}425 . \pi = 1{,}3352$; now a table of segments given for ~~the area~~ $\frac{\alpha - sin. \alpha}{2} = 1{,}32766$ of a circular segment, the angle subtended at the centre by the arc $\alpha^0 = 166^\circ$, and for $\frac{\alpha - sin. \alpha}{2} = 1{,}34487$, the same angle $= 167^\circ$; hence, simply, the angle subtended at the centre corresponding to the slice 1,3352 is:

$\alpha^0 = 166^\circ + \frac{1{,}3352 - 1{,}32766}{1{,}34487 - 1{,}32766} . 1^\circ = 166^\circ + \frac{754^\circ}{1721} = 166^\circ\, 26'$; therefore the depth of immersion:

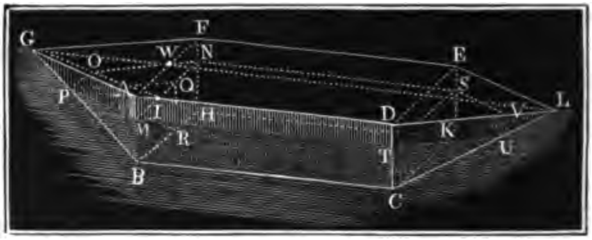
$$y = r(1 - cos. \tfrac{1}{2}\alpha) = 5(1 - cos. 83^\circ\, 13') = 5 . 0{,}8819 = 4{,}41 \text{ inches.}$$

§ 287. The determination of the depth of immersion occurs chiefly in the case of ships, boats, &c. If these have a regular form, the depth may be calculated from geometrical formulæ; but if this regularity fails, or the law of configuration is not known, or if the form is very complex, the depth of immersion must then be determined by experiment.

An example of the first case is in the punt $ACLEG$, represented in Fig. 375, bounded by plane surfaces. It consists of a parallelepiped $ACE$, and of two four-sided pyramids $BFG$ and $CEL$,

forming the head and the stern, and its plane of floatation is com-

FIG. 375.




posed of a parallelogram *MS*, and two trapeziums *MO* and *SU*, and cuts off a bulk of water, consisting of a parallelepiped *MCS*, and two triangular prisms *PNR*, and two quadrilateral pyramids *BQP*. If we put the length *AD* of the middle portion $= l$, the breadth $AF=b$, and the depth $AB=h$; further, the length *GW* of each of the two ends $= c$, and the depth of immersion, *i. e.* $BM=CT=y$. The immersed part *MCS* of the middle portion will be: $= \overline{MN} \times \overline{MT} \times \overline{MB} = lby$. The base of the quadrilateral pyramid *BQP* is $B\overline{M} \,.\, B\overline{R}$, and the height *PJ*, hence the solid contents of this pyramid $= \frac{1}{3}\, B\overline{M} \,.\, B\overline{R} \,.\, P\overline{J}$. But now:

$$BM=y,\ BR=\frac{BP}{BG} \,.\, BH=\frac{BM}{BA} \,.\, BH=\frac{y}{h} \,.\, b=\frac{b\,y}{h},$$

and likewise:

$$PJ=\frac{BM}{BA} \,.\, GW=\frac{y}{h}\,c=\frac{c\,y}{h},$$

hence the contents of both pyramids are:

$$=2 \,.\, \tfrac{1}{3} \,.\, y \,.\, \frac{b\,y}{h} \,.\, \frac{c\,y}{h}=\tfrac{2}{3}\,\frac{bcy^3}{h^2}.$$

The transverse section of the triangular prism

$$RNO \text{ is } = \tfrac{1}{2}\, RQ \,.\, \overline{PJ}=\tfrac{1}{2}\, y \,.\, \frac{c\,y}{h}=\frac{cy^2}{2\,h}, \text{ and the side}$$

$$RH=QN=b-\frac{b\,y}{h}=b\left(1-\frac{y}{h}\right),$$

hence the solid contents of both prisms are:

$$=2 \,.\, \frac{cy^2}{2\,h} \,.\, b\left(1-\frac{y}{h}\right)=\frac{bcy^2}{h}\left(1-\frac{y}{h}\right).$$

By addition of the three volumes found, the volume of the water displaced is known:

$$V = bly + \frac{2}{3}\frac{bcy^3}{h^2} + \frac{bcy^2}{h} - \frac{bcy^3}{h^2} = \left(l + \frac{c\,y}{h} - \frac{1}{3} \cdot \frac{cy^2}{h^2}\right) by.$$

Now the gross weight of the boat $= G$, we then have to put:

$$\left(l + \frac{c\,y}{h} - \frac{1}{3} \cdot \frac{cy^2}{h^2}\right) by\gamma = G, \text{ or,}$$

$$y^3 - 3\,hy^2 - \frac{3\,lh^2}{c} \cdot y + \frac{3\,h^2G}{bc\gamma} = 0.$$

The depth of immersion $y$ is determined from the loading by the solution of the last cubic equation.

*Examples.*—1. If the length of the middle portion $l = 50$ feet, the length of each end $c = 15$ feet, the breadth $b = 12$ feet, and the depth $h = 4$ feet, with a depth of immersion $y = 2$ feet, the whole weight amounts to
$G = [50 + 15 \cdot \frac{2}{4} - \frac{1}{3} \cdot 15 \cdot (\frac{2}{4})^2] \cdot 12 \cdot 2 \cdot 62{,}5 = (50 + 7{,}5 - 1{,}25) \cdot 24 \cdot 62{,}5 = 87235$ lbs.
2. If the clear weight of the former boat amount to 50000 lbs. we shall have for the depth of immersion: $y^3 - 12y^2 - 160y + 202{,}02 = 0$. By trial, it is easily found that this equation may be answered pretty accurately by $y = 1{,}17$, whence the depth of immersion sought may be taken as great.

*Remark.* To know the weight of the load of a ship, a scale is attached to both sides, which is called a water-gauge. The divisions are made from experiment, while it is observed what loads correspond to definite immersions.

§ 288. *Stability.*—The floating of bodies takes place either in an upright or an oblique position; and further, with or without stability. A body, a ship for example, floats uprightly, if one plane through the axis of ~~symmetry~~ is a plane of symmetry of the body, a body floats obliquely if it is not divided by any of the planes, which may be carried through the axis of floatation into two congruent halves. A body floats with stability, if it strives to maintain its state of equilibrium (compare § 130), if, therefore, mechanical effect is to be expended to bring it out of this position, or if it returns of itself into a position of equilibrium after having been drawn out of one. On the other hand, a body floats without stability if it passes into a new position of equilibrium after having been brought out of one by a shock or blow.

If a body *ABC*, Fig. 376, floating at first uprightly, is brought into an inclined position, the centre of gravity $S$ of the water displaced passes from the plane of symmetry *EF*, and assumes a position $S_1$ on the larger half immersed. The buoyancy applied at $S_1$ $P = V\gamma$, and the weight applied at the centre of gravity of the body $G = -P$ form a couple by which (§ 90) a constant revolu-

tion is produced. About whatever point this revolution may take place, the point $C$, yielding to the weight $G$, will always go down, and $S_1$, or another point $M$ of the vertical $S_1P$, obedient to the force $P$, will rise, therefore the plane of symmetry, or of the axis $EF$ of the ship, will be drawn downwards at $C$, and upwards at $M$, and hence it will remain upright if $M$, as in the figure, lie above $C$, or

FIG. 376.

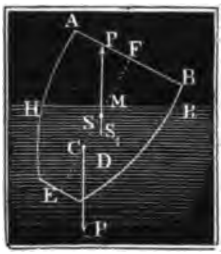
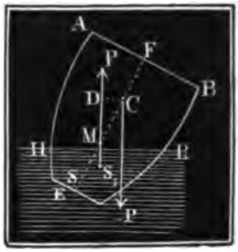



FIG. 377.

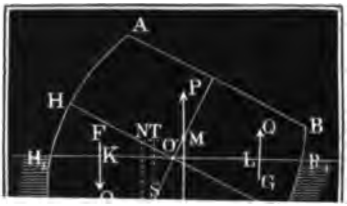



ncline itself still more as in Fig. 377, if $M$ lie below $C$. From this, then, the stability of a floating body, or ship, is dependant on the point $M$, in which the vertical through the centre of gravity $S_1$ of the displaced water intersects the plane of symmetry. This point is called the *metacentrum*. A ship, or another body, floats, therefore from this, with stability, if its metacentrum lies above the centre of gravity of the ship, and without stability if it lies below, lastly if the two points coincide, it is in a state of indifferent equilibrium.

The horizontal distance $CD$ of the metacentrum $M$ from the centre of gravity $C$ of the ship, is the arm of the force of the couple constituted of $P$ and $G = -P$, and hence the moment of the last is the measure of its stability $= P \, . \, \overline{CD}$. If we represent the distance $CM$ by $c$, and the angle of revolution $SMS_1$ of the ship, or of the plane of its axis, by $\phi^0$, we obtain for the measure of stability $S = Pc \sin. \phi$; and this is, therefore, the greater, the greater the weight, the greater the distance of the metacentrum from the centre of gravity of the ship, and the greater the angle of inclination of this last.

§ 289. In the last formula, $S = Pc \sin. \phi$, the stability of the ship depends principally on the distance of the metacentrum from the centre of gravity of the ship, it is hence of importance to obtain a formula for the determination of this distance. In the transit of the ship $ABE$, Fig. 378, from the upright into the

FIG. 378.




inclined position, the centre of gravity $S$ advances to $S_1$, the space $HOH_1$ is drawn out of the water, and that of $ROR_1$ sinks below, and the buoyancy on the one side is thereby diminished by a force $Q$ acting at the centre of gravity $F$ of

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$$S' = P c \sin \phi = P\left(\frac{F_1 a}{F} + e \sin \phi\right)$$

a-
ce

$CS$ of the centre of gravity $C$ of the ship from the centre of gravity of the displaced water by $e$, we obtain the measure of the stability

$$S = Pc \sin. \phi = P\left(\frac{F_1 a}{F} + e \sin. \phi\right).$$

If the angle of revolution be small, the transverse sections $HOH_1$ and $ROR_1$ may be regarded as equally small triangles; if we represent the breadth $HR = H_1R_1$ of the ship at the place of immersion by $b$, we may then put

$$F_1 = \tfrac{1}{2} \cdot \tfrac{1}{2} b \cdot \tfrac{1}{2} b \phi = \tfrac{1}{8} b^2 \phi, \text{ and } KL = a = 2 \cdot \tfrac{2}{3} \frac{b}{2} = \tfrac{2}{3} b,$$

as also $\sin. \phi = \phi$, from which the stability is :

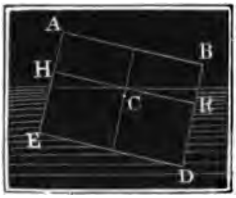
$$S=P\left(\tfrac{1}{12}\frac{b^3\phi}{F}+e\phi\right)=\left(\frac{b^3}{12\,F}+e\right)P\phi.$$

If the centre of gravity $C$ of the ship coincides with the centre of gravity $S$ of the displaced water, we then have $e=o$, hence: $S=\frac{b^3}{12\,F}\cdot P\phi$, and if the centre of gravity of the ship lies ~~below~~ that of the displaced water, we then have $e$ negative; hence: $S=\left(\frac{b^3}{12\,F}-e\right)P\phi$. It follows also that the stability of a ship is nothing, if $e$ be negative and at the same time $e=\frac{b^3}{12\,F}$.

It is seen from the results obtained that the stability comes out greater, the broader the ship is, and the lower its centre of gravity lies.

*Example.* A rectangular figure $AD$, Fig. 379, of the breadth $AB=b$, height $AE=h$, and depth of immersion $EH=y$, $F=by$, and $e=-\frac{h-y}{2}$; hence, the amount of stability is:

FIG. 379.



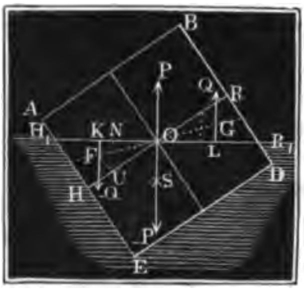

$$S=P\phi\left(\frac{b^3}{12\,b\,y}-\frac{h}{2}+\frac{y}{2}\right),$$ or if the specific gravity of the mass of the body be put $=\varepsilon$,

$$S=P\phi\left(\frac{b^2}{12\,h\,\varepsilon}-\frac{h}{2}(1-\varepsilon)\right).$$

Hence, the stability ceases if $b^2=6\,h^2\varepsilon\,(1-\varepsilon)$, i. e., if $\frac{b}{h}=\sqrt{6\,\varepsilon\,(1-\varepsilon)}$. For $\varepsilon=\frac{1}{2}\;\frac{b}{h}=\sqrt{\frac{6}{2}\cdot\frac{1}{2}}=\sqrt{\frac{3}{2}}=1{,}225$; if, therefore, the breadth is not 1,225 of the height, the body will float without any stability.

§ 290. *Oblique floatation.*—The formula $S=P\left(\frac{F_1a}{F}\pm e\sin.\varphi\right)$ for the stability of a floating body may be also applied to find the different positions of floating bodies, for if we put $S=0$ we obtain the equation for a second position of equilibrium, whose solution leads to the determination of the corresponding angle of inclination. The equation, therefore, $\frac{F_1a}{F}\pm e\sin.\phi=0$, must be solved with respect to $\phi$.

FIG. 380.




The transverse section of a parallelepiped $AD$, Fig. 380, is $F=HRDE=H_1R_1DE=by$, if $b$ be the breadth $AB=HR$, and $y$ the perpendicular depth $EH=DR$; further, the trans-

verse section $F_1 = HOH_1 = ROR_1$ as a rectangular triangle with the cathetus $OH = OR = \frac{1}{2} b$, and the cathetus:

$$HH_1 = RR_1 = \tfrac{1}{2}\, b \text{ tang. } \phi, \quad F_1 = \tfrac{1}{8}\, b^2 \text{ tang. } \phi.$$

If, further, the centre of gravity $F$ is distant from the base $FU = \frac{1}{3} HH_1 = \frac{1}{6} b \text{ tang. } \phi$, and if from $O$ about $OU = \frac{2}{3} OH = \frac{1}{3} b$, it follows that the horizontal distance of the centre of gravity $F$ from the middle $O$, $= OK = ON + NK = OU \cos. \phi + FU \sin. \phi$ $= \frac{1}{3} b \cos. \phi + \frac{1}{6} b \text{ tang. } \phi \sin. \phi$, and the arm:

$$a = KL = 2\, OK = \tfrac{2}{3}\, b \cos. \phi + \tfrac{1}{3}\, b \frac{\sin. \phi^2}{\sin. \phi}.$$ cos

According to this the equation for the oblique position of equilibrium is:

$$\frac{\frac{1}{8} b^2 \text{ tang. } \phi \left(\frac{2}{3} b \cos. \phi^2 + \frac{1}{3} b \sin. \phi^2\right)}{by \cos. \phi} - e \sin. \phi = 0,$$

or, $\frac{\sin. \phi}{\cos. \phi} = \text{tang. } \phi$ being substituted,

$$\sin. \varphi \left[\left(\tfrac{1}{12} + \tfrac{1}{24} \text{ tang. } \varphi^2\right) b^2 - ey\right] = 0;$$

which equation will be satisfied by:

$$\sin. \varphi = 0 \text{ and by tang. } \varphi = \sqrt{2} \sqrt{\frac{12\, e y}{b^2} - 1}.$$

The first equation, when $\varphi = 0$, corresponds to upright, and the second to oblique floatation. The possibility of the latter requires that $\frac{ey}{b^2} > \frac{1}{12}$. If now $h$ be the height of the parallelepiped, and $\varepsilon$ its specific gravity, we then have:

$$y = \varepsilon\, h \text{ and } e = \frac{h - y}{2} = (1 - \varepsilon)\, \frac{h}{2}, \text{ hence it follows that}$$

$$\text{tang. } \varphi = \sqrt{2} \sqrt{\frac{6\, \varepsilon\, (1 - \varepsilon)\, h^2}{b^2} - 1},$$

and the equation of condition of oblique floatation is:

$$\frac{h}{b} > \sqrt{\frac{1}{6\, \varepsilon\, (1 - \varepsilon)}}.$$

*Examples.*—1. If the floating parallelepiped is as high as it is broad, and has a specific gravity $\varepsilon = \frac{1}{2}$, then the *tang.* $\phi$ is $= \sqrt{2}\, \sqrt{3 \cdot \frac{1}{2} - 1} = \sqrt{3 - 2} = 1$; hence, $\phi = 45°$.

2. If the height $h = 0{,}9$ of the breadth $b$, and the specific gravity $\frac{1}{2}$, we have then *tang.* $\phi = \sqrt{3 \cdot 0{,}81 - 2} = \sqrt{0{,}43} = 0{,}6557$; hence, $\phi = 33° \, 15'$.

§ 291. *Specific gravity.*—The law of buoyancy of water may be applied to the determination of the density, or the specific gravity of bodies. From § 284, the upward pressure of water is equal to the weight of liquid displaced; hence if $V$ is the volume of a body and $\gamma_1$ the density of the liquid, we then have the buoyancy $P = V\gamma_1$. If now $\gamma_2$ be the density of the mass of the bodies, we then have the weight of the body $G = V\gamma_2$; hence the ratio of the densities $\frac{\gamma_2}{\gamma_1} = \frac{G}{P}$, i. e. ***the density of the body immersed is to the density of the fluid as the absolute weight of the body to the buoyancy or loss of weight by immersion.***

Therefore, $\gamma_2 = \frac{G}{P}\gamma_1$, and $\gamma_1 = \frac{P}{G}\gamma_2$; or if $\gamma$ be the density of water, $\varepsilon_1$ the specific gravity of the liquid, and $\varepsilon_2$ that of the body, then will $\gamma_1 = \varepsilon_1\gamma$, and $\gamma_2 = \varepsilon_2\gamma$, $\varepsilon_2 = \frac{G}{P}\varepsilon_1$, and $\varepsilon_1 = \frac{P}{G}\varepsilon_2$. If, therefore, the weight of a body ~~or~~ and its loss of weight by immersion is known, then the density or the specific gravity of the mass of a body may be found from the density or specific gravity of the liquid, and inversely, the density or specific gravity of the first, from the density or specific gravity of the last.

If the fluid in which the solid body is weighed is water, we then have $\varepsilon_1 = 1$, and $\gamma_1 = \gamma = 1000$ kilogrammes, or 62,5 lbs., according as we take the cubic metre or cubic foot for unit of volume, hence for this case the density of the body is:

$$\gamma_2 = \frac{G}{P}\gamma = \frac{\text{absolute weight}}{\text{loss of weight}} \text{ times the density of water,}$$

and the specific gravity:

$$\varepsilon_2 = \frac{G}{P} = \frac{\text{absolute weight}}{\text{loss of weight}}.$$

To estimate the buoyancy or loss of weight, as well as to determine the weight $G$, we make use of an ordinary balance, only that below one of the scale pans of this balance there is appended a hook, to which the body may be suspended by a fine thread or fine wire, whilst it dips into the water contained in a vessel underneath. A balance arranged for the weighing of bodies in water is commonly called a *hydrostatic balance.*

If the body whose specific gravity we wish to determine is lighter than water, we may connect it mechanically with another

heavy body, so as to make it sink. If this heavy body loses the weight $P_2$, and the system the weight $P_1$, the loss of weight of the lighter body is: $P = P_1 - P_2$, now if $G$ represents the ~~loss of~~ weight of the lighter body, we have then its specific gravity:

$$\varepsilon_2 = \frac{G}{P} = \frac{G}{P_1 - P_2}.$$

If the specific gravity of a mechanical combination, or a composition of two bodies, and the specific gravities of their constituents $\varepsilon_1$ and $\varepsilon_2$ are known, from the weight of the whole, the weights $G_1$ and $G_2$ may be estimated. In every case $G_1 + G_2 = G$, and also the volume $\frac{G_1}{\varepsilon_1 \gamma}$ + volume $\frac{G_2}{\varepsilon_2 \gamma}$ = volume $\frac{G}{\varepsilon \gamma}$, therefore: $\frac{G_1}{\varepsilon_1} + \frac{G_2}{\varepsilon_2} = \frac{G}{\varepsilon}$. By combining these equations we have:

$$G_1 = G\left(\frac{1}{\varepsilon} - \frac{1}{\varepsilon_2}\right) \div \left(\frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_2}\right), \text{ and}$$

$$G_2 = G\left(\frac{1}{\varepsilon} - \frac{1}{\varepsilon_1}\right) \div \left(\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}\right).$$

*Examples.*—1. If a piece of limestone, weighing 310 grains, becomes 121,5 grains lighter when under water, its specific gravity is $\varepsilon = \frac{310}{121,5} = 2,55$.—2. To find the specific gravity of a piece of oak, round which a piece of lead has been wrapped, and which has lost by being weighed in water 10,5 grains; if now the wood itself weighed 426,5 grains, and the system under water was 484,5 grains lighter than in the air, the specific gravity of the mass of wood would be:

$$\varepsilon = \frac{426,5}{484,5 - 10,5} = \frac{426,5}{474} = 0,9.$$

3. An iron vessel, completely filled with quicksilver and perfectly closed, has a net weight of 500 lbs., and has lost 40 lbs. in the water; if now the specific gravity of cast-iron = 7,2, and that of quicksilver is 13,6, the weight of the empty vessel is:

$$G_1 = 500\left(\frac{40}{500} - \frac{1}{13,6}\right) : \left(\frac{1}{7,2} - \frac{1}{13,6}\right)$$
$$= 500\,(0,08 - 0,07353 : (0,1388 - 0,0735)$$
$$= \frac{500 \,.\, 0,00647}{0,0653} = \frac{3235}{65,3} = 49,5 \text{ lbs.},$$

and the weight of the enclosed quicksilver:

$$G_2 = 500\,(0,08 - 0,1388) : (0,07353 - 0,1388) = \frac{500 \,.\, 0,0588}{0,0653}$$
$$= \frac{2940}{6,53} = 450,2 \text{ lbs.}$$

*Remark* 1. For the determination of the specific gravities of liquids, meal, corn,

&c., the mere weighing in open air is sufficient, because we may give to the bodies any volume at will, by filling vessels with them. If an empty bottle weighs $= G$, and the same filled with water $G_1$, and the weight $G_2$ if it contain any othe substance, we shall then have the specific gravities of masses of these: $\varepsilon = \frac{G_2 - G}{G_1 - G}$. For example, to find the specific gravity of rye (not rye grains), a bottle is filled with the grains, and after much shaking, then weighed. After deduction of the weight of the empty bottle, the weight of the rye was $= 120{,}75$ grms., and the weight of an equal quantity of water $= 155{,}65$; the ~~weight~~ of the rye is accordingly $= \frac{120{,}75}{155{,}65} = 0{,}776$; and therefore 1 cubic foot of this grain weighs

$$= 0{,}776 \,.\, 62{,}5 = 48{,}5 \text{ lbs.}$$

*Remark* 2. The problem solved by Archimedes of finding the ratio of the constituents from the specific gravity of a mixture, and from the specific gravity of its constituents, admits only of a limited application to chemical combinations, metallic alloys, &c., because a contraction or expansion of the mass generally takes place, so that the volume of the mixture is no longer equal to the sum of the volumes of the constituents.

*Remark* 3. The further extension of this subject, namely, its application to the measurement of volume, &c., belongs to physics and chemistry.

§ 292. *Areometer.*—Areometers are principally used to determine the density of liquids. These instruments are hollow bodies, formed about a symmetrical axis, whose centres of gravity lie very low, and by floating perpendicularly in liquids, give their density. They are made of glass, brass, &c., and are called, according to the various purposes for which they are intended, hydrostatic balances, salzometers, hydrometers, alcohometers, &c. There are two kinds of hydrometers, viz. the weight and the scale hydrometer. The first are often used for the determination of the weights, ~~viz.~~ as well as the specific gravities of solid bodies.

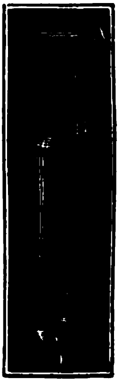
FIG. 381.


1. If $V$ be the volume of the portion of an hydrometer $ABC$, Fig. 381, floating freely, and immersed up to a certain mark $O$ in the water, $G$ the weight of the whole balance, $P$ the weight placed upon the plate while floating in the water, whose density may be $= \gamma$, and $P_1$ the weight required to ~~be put on~~ to make it float in any other liquid of the density $\gamma_1$, we shall then have

$$V\gamma = P + G \text{ and } V\gamma_1 = P_1 + G; \text{ hence,}$$

$$\frac{\gamma_1}{\gamma} = \frac{P_1 + G}{P + G}.$$

FIG. 382.

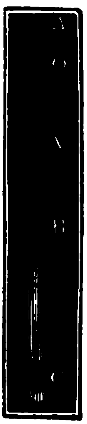


2. If $P$ be the weight which must be put upon the plate to make the hydrometer $ABC$, Fig. 382, sink up to a mark $O$, and $P_1$ the weight which must be put upon $A$, together with the body to be weighed, to obtain the same immersion, we shall then have simply the weight of this body $G_1 = P - P_1$. But if $P_1$ must be augmented by $P_2$, when the body to be weighed is put into the cup $D$ under the surface, to preserve the depth of immersion unchanged, $P_2$ will then be the buoyancy, and hence the specific gravity of the body:

$$\varepsilon = \frac{G_1}{P_2} = \frac{P - P_1}{P_2}.$$

Those hydrometers which have a cup suspended below for the determination of the specific gravities of solid bodies, minerals for instance, are called Nicholson's hydrometers.

FIG. 383.



3. Let the weight of an hydrometer $ABC$, Fig. 383, $= G$, and the volume immersed, if this balance floats in water, $= V$, then $G = V\gamma$. If the balance rise by $OX = x$, when immersed in a heavier liquid, for the transverse section $F$ of the stem, the volume immersed is $= V - Fx$, and hence $G = (V - Fx)\,\gamma_1$; the two formulæ, divided by one another, give the density of the liquid:

$$\gamma_1 = \frac{V}{V - Fx} \cdot \gamma = \gamma : (1 - \frac{F}{V}\,x) = \gamma : (1 - \mu x).$$

If the liquid in which the hydrometer is immersed be lighter than the water, it will sink in it to a depth $x$, for which reason, $G = (V + Fx)\,\gamma$, and hence we must put $\gamma_1 = \gamma : (1 + \mu\, x)$.

To find the co-efficient $\mu = \frac{F}{V}$, the balance is loaded with a weight $P$ of quicksilver, which is poured in and takes the lowest position, so that while floating in water, a considerable length $l$ of the stem to which the scale is applied, sinks lower down. If now we put $P = Fl\gamma$, we shall then obtain:

$$\mu = \frac{F}{V} = \frac{P}{Vl\gamma} = \frac{P}{Gl}.$$

*Examples.*—1. If a Nicholson's hydrometer weighs 65 grains, 13,5 grains must be

**taken off the plate, that it may sink to the same depth in alcohol as it does in water; the specific gravity of alcohol is $= \frac{65 - 13,5}{65} = 1 - 0,208 = 0,792$.—2. The normal weight of a Nicholson's balance is 1500 grains, *i. e.* 1500 grains require to be put on to make the instrument sink to 0; from this 1030 grains must be taken by the weighing of a piece of brass placed upon the upper plate, and 121,5 to be added if this body is placed on the lower plate. The absolute weight of this piece of brass is therefore = 1030 grains, and its specific gravity $= \frac{1030}{121,5} = 8,47$. — 3. A scale areometer, of 1162 grains weight, after having been lightened by 463 grains, rises 6 inches, and has therefore the co-efficient $\mu = \frac{465}{1162.6} = \frac{465}{6772} = 0,00686$. After complete filling and restoration of the weight of 1162 grains, it ascends, when floating in a saline solution, $2\frac{5}{12}$ inches; hence, the specific gravity of this is:**

$$= 1 \div \left( (1 - 0,00686 \times \frac{29}{12} \right) = 1 \div 0,983 = 1,02.$$

***Remark.* The further extension of this subject belongs to physics, chemistry, and technology.**

§ 293. *Liquids of different densities.*—If several liquids, of different densities, are in the same vessel, without their exerting any chemical action upon each other, from the ready displacement of their particles, they arrange themselves above each other, according to their specific gravities, viz. the densest below, then the less dense, and then the lightest. The limiting surfaces are also in a state of equilibrium, as likewise the free surface horizontal; for as long as the surface of limitation $EF$ between the masses $M$ and $N$, Fig. 384, is inclined, columns of fluid, of different densities, like $GK$, $G_1K_1$, rest on the horizontal stratum $HR$, and hence the pressure on this stratum will not be everywhere the same; and lastly, no equilibrium will subsist.

FIG. 384.


In communicating tubes $AB$ and $CD$, Fig. 385, the liquids

FIG. 385.

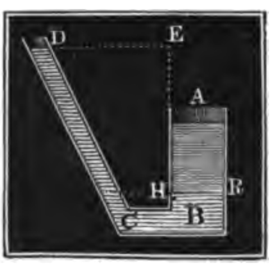



FIG. 386.

arrange themselves one above the other, according to their densities, only their surfaces $A$ and $D$ do not lie in one and the same level. If $F$ be the area $HR$ of the transverse section of a piston, Fig. 386, in the one branch $AB$ of two communicating tubes, and the height of pressure or the height $EH$ of the surface of the water in the second tube $CD$ above $HR$, $= h$, we then have the pressure against the surface of the piston $P = Fh\gamma$. On the other hand, if we replace the pressure of the piston by a column of liquid $AH$, Fig. 386, of the height $AH = h_1$ and the density $\gamma_1$, we then have $P = Fh_1\gamma_1$; and equating both expressions, we obtain the equation $h_1\gamma_1 = h\gamma$, or the proportion $\frac{h_1}{h} = \frac{\gamma}{\gamma_1}$.

***Therefore, the ~~pressure of~~ heights of pressure in communicating tubes, for the subsistence of equilibrium between two different liquids, or the heights of the columns of liquid measured from the common plane of contact, are inversely as the densities or specific gravities of these liquids.***

As mercury is about 13,6 times the density of water, a column of mercury, in communicating tubes will hold in equilibrium a column of water of 13,6 times the height.

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# CHAPTER III.

## ON THE EQUILIBRIUM AND PRESSURE OF AIR.

§ 294. *Tension of gases.*—The atmospheric air which surrounds us, as well as all kinds of air or gases, possesses, in virtue of the repulsive force of its parts or molecules, a tendency to occupy a greater and greater space; hence, we can obtain a limited mass of air only by confining it in perfectly closed vessels. The force with which gases endeavour to dilate themselves is called their ***elasticity, tension,*** or ***expansive force.*** It exhibits itself by pressure against the sides of the vessels which enclose it, and so far differs from the elasticity of solids and liquids, that it manifests its action in every condition of density, while the elasticity of the last-mentioned bodies in a certain state of expansion is nothing. The pressure or tension of air and other gases is measured by

the *barometer*, the *manometer*, and the *valve*. The barometer is chiefly used for determining the pressure of the atmosphere.

FIG. 387.

The common, or as it is called, the *cistern barometer*, Fig. 387, consists of a glass tube, closed at one end $A$ and open at the other $B$, which, when filled with mercury, is inverted, and its open end immersed in a cistern likewise containing mercury. By the inversion of this instrument, there remains in the tube a column of mercury $BS$, which (§ 393) is sustained in equilibrium by the pressure of the air on the surface of mercury $HR$. The space $AS$ above the mercurial column is deprived of air, or a vacuum; hence, there is no pressure on this column from above, for which reason, the height of the mercurial column above the surface of mercury $HR$ in the cistern, serves for a measure of the air's pressure.

To measure this height with precision and convenience, an accurately divided scale is appended, which runs lengthwise along the tube. A more particular description of the different barometers, and an explanation of their uses, &c., belongs to the department of physics.

§ 295. It has been found by the barometer, that for a certain mean state of the atmosphere and at places very little above the level of the sea, the air's pressure is held in equilibrium by a column of mercury, 76 centimetres, or about 28 Paris inches = 29 Prussian inches = 30 English inches nearly, (29,994 exactly). As the specific gravity of mercury is nearly 13,6 (13,598), it follows that the pressure of the air is equivalent to the weight of a column of water, 0,76 . 13,6 = 10,336 metres = 31,73 Paris feet = 32,84 Prussian feet = 33,824 English feet.

The tension of the air is very often measured by the pressure it exerts upon a unit of surface. Since a cubic centimetre of mercury weighs 0,0136 kilogrammes, the pressure of the atmosphere, or the weight of a column of mercury 76 centimetres high on a base of 1 centimetre square, = 0,0136 . 76 = 1,0336 kilogrammes, and since a cubic inch of mercury weighs $\frac{66 \,.\, 13,6}{1728} = 0,5194$ Prussian lbs., or $\frac{62,5 \,.\, 13,6}{1728} = 0,491$ lbs. English, the mean pressure of the atmosphere is then = 29 . 0,5194 = 15,05 Prussian

lbs. on the square inch, = 2167 lbs. on the square foot, and in English measure = 30 . 0,491 = 14,73 on the square inch, = 21~~1~~21,12 lbs. avd. on the square foot. 14,7~~6~~3 lbs. per square inch is the standard usually adopted

In mechanics, the mean pressure of the atmosphere is commonly taken as unity, and other expansive forces referred to this and assigned in atmospheric pressures, or *atmospheres*. Hence, to a pressure of $n$ atmospheres corresponds a mercurial column of 30 . $n$ inches, or a weight of 14,73 lbs. on each square inch; and inversely, to a mercurial column of $h$ inches corresponds a tension of $\frac{h}{28} = 0{,}03571\, h$ or $= \frac{h}{30} = 0{,}0333\, h$ atmospheres, and to a pressure of $p$ lbs. on the square inch, a tension of

$$\frac{h}{15{,}05} = 0{,}0644\, p \text{ or } \frac{h}{14{,}73} = 0{,}0678 \text{ atmospheres.}$$

The equation $\frac{h}{28} = \frac{p}{15{,}05}$ or $\frac{h}{30} = \frac{p}{14{,}73}$ give the formulæ of reduction $h = 1{,}8604\, p$ inches and $p = 0{,}5375\, h$ lbs., or $h = 2{,}036\, p$ inches, and $p = 0{,}491\, h$ lbs. English. For a tension $h$ inches $= p$ lbs., the pressure against a plane surface of $F$ square inches: $P = Fp = 0{,}491\, Fh$ lbs. English, or $= 0{,}5375\, Fh$ lbs. Prussian.

*Examples.*—1. If the water in a water-pressure engine stands 250 feet above the surface of the piston, the pressure against the surface will then be $= \frac{250}{334} = 7{,}48$ atmospheres.—2. If the blast of a cylindrical bellows has a tension of 1,2 atmospheres, its pressure on every square inch $= 1{,}2 \,.\, 17{,}676 = 14{,}73$ lbs., and on the surface of the piston of 50 inches diameter $= \frac{\pi \,.\, 50^2}{4} \,.\, 17{,}67 = 34695$ lbs. As the atmosphere exerts a counter-pressure $\frac{\pi \,.\, 50^2}{4} \,.\, 14{,}73 = 28922{,}3$ lbs., the pressure on the piston is $= 34695 - 20922{,}3 = 5772{,}7$ lbs.

§ 296. *Manometer.*—To find the tension of gases or vapours enclosed in vessels, instruments similar to the barometer are made use of, which are called *manometers*. These instruments are filled with mercury or water, and are either open or closed; but in the latter case, the upper part is either a vacuum or full of air. The vacuum manometer, Fig. 388, differs little from the ordinary barometer. To measure by this instrument the tension of air in a reservoir, a tube $GK$ is fitted in, one end of which $G$ passes into the reservoir, and the other $K$ projects above the surface of mercury $CE$ in the cistern of the

instrument. The space $EFHR$ above the mercury is hereby put into communication with the air-holder, and the air in it assumes the tension of the air in the holder, and forces into the tube a column of mercury $OS$, which sustains in equilibrium the pressure of the air which is to be measured.

The siphon manometer, $ABC$, Fig. 389, open above, gives the

FIG. 388.


FIG. 389.

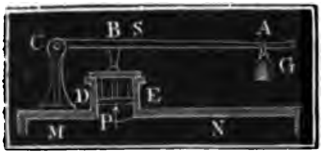

FIG. 390.


excess of tension above the pressure of the atmosphere in the vessel $MN$, because the pressure of the atmosphere on $S$, joined to that of the mercurial column $RS$, is in equilibrium with the tension. If $b$ be the height of the barometer, and $h$ that of the manometer, or the difference of heights $RS$ of the surfaces of mercury in both branches of the manometer, we shall then have the tension of the air communicating with the shorter branch measured by the height of a column of mercury: $b_1 = b + h$, or the pressure measured on a square inch $p = 0{,}491\ (b + h)$ lbs.; or if $b$ be the mean height of the barometer,

$$p = 14{,}73 + 0{,}491\ h \text{ lbs.}$$

Cistern manometers, Fig. 390, $ABCE$, are more common than siphon manometers. As the air here acts through a greater quantity of mercury or water, as it may be, upon the column of fluid, its oscillations do not so quickly affect the column of fluid, and its measurement, when thus at rest, is rendered both easier and more accurate. For the sake of convenience of measuring by, or reading off from the scale, a float is not unfrequently attached to it, which rests on the mercury, and is connected with an index hand, accompanying the scale by means of a thread passing over a small roller.

FIG. 391.




The expansive force of a gas or vapour enclosed in $MN$ may be likewise determined, but with less accuracy, by the help of a valve $DE$, Fig. 391, if the sliding weight is so placed that it is in equilibrium with the pressure of the air or vapour. If $CS = s$ be the distance of the centre of gravity of the lever from the fulcrum $C$, $CA = a$ the arm of the weight, and $Q$ the weight of the lever with its valve, we then have the statical moment with which the valve is pressed down by the weight $= Ga + Qs$; if now the pressure of the gas or vapour from below $= P$, the pressure of the atmosphere from above $= P_1$, and lastly, the arm $CB$ of the valve $= d$, we then have the statical moment with which the valve strives to lift itself up $= (P—P_1)\, d$, and by equating the moments of both :

$$Pd - P_1 d = Ga + Qs, \text{ and } P = P_1 + \frac{Ga + Qs}{d}.$$

If $r$ represent the radius $\frac{1}{2}\, DE$ of the valve, $p$ the internal and $p_1$ the external tension, measured by the pressure on a square inch, we then have :

$$P = \pi r^2 p \text{ and } P_1 = \pi r^2 p_1; \text{ hence } p = p_1 + \frac{Ga + Qs}{\pi r^2 d}.$$

*Examples.*—1. If the height of mercury of a manometer, open above, is 3,5 inches, but that of the barometer 27 inches, the corresponding expansive force is then $h = b + h_1 = 27 + 3{,}5 = 30{,}5$ inches, or $p = 0{,}820 \,.\, h = 0{,}520 \,.\, 30{,}5 = 15{,}8$ lbs.—2. If the height of a water-manometer is 21 inches, the expansive force corresponding to this, with the height of the barometer at 27 inches, is

$$h = 27 + \frac{21}{13{,}6} = 28{,}54 \text{ inches} = 15{,}34 \text{ lbs.}$$

3. If the statical moment of an unloaded safety-valve is 10 inch lbs., the statical moment of a 10 lbs. sliding weight 15 . 10 = 150 inch lbs., the arm of the valve measured, from the valve to the fulcrum, 4 inches, and the radius of the valve 1,5 inches, then the difference of the pressures on both surfaces of the valve is :

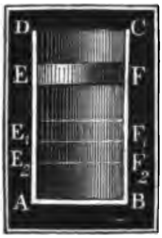
$$p - p_1 = \frac{150 + 10}{\pi\, (1{,}5)^2 \,.\, 4} = \frac{160}{9\pi} = 5{,}66 \text{ lbs.}$$

Were the pressure of the atmosphere $p_1 = 14$ lbs., the tension of the air below the valve would from this amount to $p = 19{,}66$ lbs.

§ 297. *Law of Mariotte.*—The tension of gases increases with their density; the more a certain quantity of air is compressed or condensed, the greater is its tension; and ~~the greater its~~

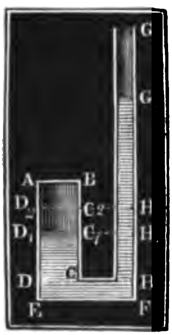
~~tension~~, the more it is allowed to expand or become rarefied, the less does its expansive force exhibit itself. The ratio in which the tension and the density, or the volume of the gases stand to each other, is expressed by the law discovered by Mariotte, and named after him. This law assumes that *the density of one and the same quantity of air or gas is proportional to its tension*; or, as the spaces which are occupied by one and the same mass are inversely proportional to the densities, *that the volume of one and the same mass of gas is inversely as its expansive force.* Accordingly, if a certain quantity of air becomes compressed to one half its original volume, its density is therefore doubled, its tension is also as great again as at first; and on the other hand, if a certain quantity of air be expanded to ~~one-third~~ of its original bulk, therefore, its density reduced one-third, its elasticity will be equal to one-third only of its original tension. Ordinary atmospheric air, for example, under the piston *EF* of a cylinder *AC*, Fig. 392, which originally presses with 15 lbs. on every square inch, will press on the piston with a force of 30 lbs., if this piston be pushed to $E_1 F_1$, and the enclosed air compressed to one half its original volume, and this force will amount to 3 . 15 = 45 lbs., if the piston come to $E_2 F_2$, and describe two-thirds of the whole height.

FIG. 392.




If the area of the piston be 1 square foot, the pressure of the atmosphere against it will amount to = 144 . 15,~~6~~ = ~~2246,4~~ 2160 lbs.; hence, to press down the piston one half the height of the cylinder it will require ~~2246,4~~ 2160 lbs., and to push it ~~up~~ two-thirds of this height ~~4492,8~~ lbs. to be exerted. 4320

FIG. 393.




The law of Mariotte may be likewise proved by pouring mercury into the tube *GH* communicating with the air of a cylinder *AC*, Fig. 393. If a column of air *AC* be originally enclosed by the quantity of mercury *DEFH*, ~~which~~ has the same tension as the external air, and afterwards be compressed to one-half or one-fourth its volume by the addition of fresh mercury, we shall then find that the distances of the surfaces of $G_1 H_1$, $G_2 H_2$, &c. of mercury are equivalent to the single and treble height of the barometer *b*, &c., that therefore if we add

to this the single height, corresponding to the external pressure of the air, the tension will be twice or four times as great as that due to its original volume.

The tensions are $h$ and $h_1$ or $p$ and $p_1$, $\gamma$ and $\gamma_1$ the corresponding densities, and $V$ and $V_1$ the volumes appertaining to one and the same quantity of air, we then have, according to the law laid down:

$$\frac{\gamma}{\gamma_1}=\frac{V_1}{V}=\frac{h}{h_1}=\frac{p}{p_1};\text{ hence}$$

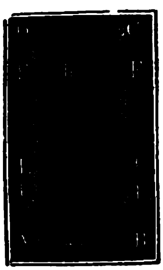
$$\gamma_1=\frac{h_1}{h}\gamma=\frac{p_1}{p}\gamma \text{ and } V_1=\frac{h}{h_1}V=\frac{p}{p_1}V.$$

From this the density and also the volume may be reduced from one tension to another.

*Examples.*—1. If in a blowing machine, the manometer stand at 3 inches, whilst the barometer is at 28 inches, the density of the wind is $=\frac{28+3}{28}=\frac{31}{28}=1{,}107$ times that of the external air.—2. If a cubic foot of atmospheric air, with the barometer at 28 inches, weighs $\frac{62.5}{770}$ lbs.; with the barometer at 34 inches it will weigh:

$$\frac{62.5}{770}\cdot\frac{34}{28}=\frac{21250}{21560}=0{,}985\text{ lbs.}$$

FIG. 394.



§ 298. The mechanical effect which must be expended to condense a certain quantity of air to a certain degree, and the effects which the air by its expansion will produce, cannot be directly assigned, because the expansive force varies at every moment of condensation or extension; we must therefore endeavour to find a special formula for the calculations of this value. Let us imagine a certain quantity of air $AF$, enclosed in a cylinder $AC$, Fig. 394, by a piston $EF$, and let us enquire what effect must be expended to push forward the piston through a certain space $EE_1=FF_1$. If the original tension $=p$, and the original height of the capacity of the cylinder $=s_0$, and the tension after describing the space $EE_1=p_1$, and the residuary volume of air $=s_1$, the proportion $p_1:p=s_0:s_1$ then holds true, and gives $p_1=\frac{s_0}{s_1}p$.

While describing a very small space $E_1E_2=x$, the tension $p_1$ may be regarded as invariable, and hence the mechanical effect to be

expended is $= Ap_1 x = \frac{Aps_0 x}{s_1}$, when $A$ represents the surface of the piston.

It follows from the properties of logarithms, that a very small magnitude $y = hyp.\ log.\ (1+y) = 2{,}3026\ Log.\ (1+y)$, if *hyp. log.* represents the hyperbolic, and *Log.* the common logarithms; we may consequently put

$$Aps_0 \frac{x}{s_1} = Aps_0\ hyp.\ log. \left(1 + \frac{x}{s_1}\right)$$

$$= 2{,}3026\ Aps_0\ log. \left(1 + \frac{x}{s_1}\right)$$

But now:

$$hyp.\ log. \left(1 + \frac{x}{s_1}\right) = hyp.\ log. \left(\frac{s_1 + x}{s_1}\right) = hyp. log. (s_1 + x) - hyp. log. s_1;$$

hence the elementary mechanical effect is:

$$= Aps_0\ [hyp.\ log.\ (s_1 + x) - hyp.\ log.\ s_1].$$

Let us imagine the whole space $EE_1$ to be made up of very small parts, such as $x$, and therefore put $EE_1 = nx$, we shall find the mechanical effects corresponding to all these parts, if in the last formula we substitute for

$s_1, s_1 + x, s_1 + 2\,x, s_1 + 3\,x, \ldots$ to $s_1 + (n-1)\,x$, and for
$s_1 + x, s_1 + 2\,x, s_1 + 3\,x$, &c., to $s_1 + nx$, or $s_0$,

and by summation the whole expenditure of mechanical effect in describing the space $s_0 - s_1$:

$$L = Aps_0 \begin{cases} hyp.\ log.\ (s_1 + x) - hyp.\ log.\ s_1 \\ hyp.\ log.\ s_1 + 2\,x) - hyp.\ log.\ (s_1 + x) \\ hyp.\ log.\ (s_1 + 3\,x) - hyp.\ log.\ (s_1 + 2\,x) \\ \quad\cdot \qquad\qquad \cdot \\ \quad\cdot \qquad\qquad \cdot \\ \quad\cdot \qquad\qquad \cdot \\ hyp.\ log.\ (s_1 + nx) - hyp.\ log.\ [s_1 + (n-1)\,x] \end{cases}$$

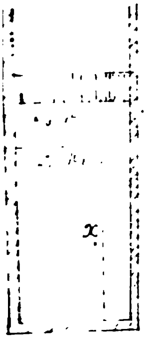
$$= Aps_0\ [hyp.\ log.\ (s_1 + nx) - hyp.\ log.\ s_1]$$

$$= Aps_0\ (hyp.\ log.\ s_0 - hyp.\ log.\ s_1) = Aps_0\ hyp.\ log. \left(\frac{s_0}{s_1}\right),$$

since one member in the one line always cancels one member in the following one.

Since $\frac{s_0}{s_1} = \frac{h_1}{h} = \frac{p_1}{p}$, this mechanical effect may be put:

§ 278




Let p = original tension $s_0$ = height of the cylinder; let the section = 1, let x be [illegible] from the bottom

let $p_1$ = tension at ht x

by Mariotte $\frac{p_1}{p} = \frac{s_0}{x}$ ∴ $p_1 = \frac{p s_0}{x}$ i.e. $\frac{p_1}{p} = \frac{s_0}{x}$

∴ [illegible] of work = $\int p_1 dx = -p s_0 \int \frac{dx}{x} = -p s_0 \log x + C$

[illegible] $s_0$ L [illegible] $\int_{s_1}^{s_0} = L = p s_0 \log \frac{s_0}{s_1} = p s_0 \log \frac{p_1}{p}$

Conversely the work of [illegible] compression [illegible]. If $p_1$ = [illegible]

[illegible] $p \frac{s_1}{x}$ [illegible] $\int p\,dx = L =$

$\int p_1 s_1 \frac{dx}{x} = p_1 s_1 \log x + C$ = 0 when $x = s_1$

$\int_{s_1}^{s_0} = p_1 s_1 \log \frac{s_0}{s_1}$ ($p_1 s_1 = p s_0$) ∴ $L = p s_0 \log \frac{p_1}{p}$

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$$L = Aps_0 \ hyp. \ log. \left(\frac{h_1}{h}\right) = Aps_0 \ hyp. \ log. \left(\frac{p_1}{p}\right).$$

If we put the space described by the piston $s_0 - s_1 = s$, we shall hence find that the mean force of the piston $p$ condensing the air is in the proportion

$$\frac{h_1}{h} = \frac{p_1}{p}, \ P = \frac{L}{s} = Ap \frac{s_0}{s} hyp. \ log. \ \left(\frac{p_1}{p}\right).$$

Let $A = 1$ (square foot) and $s_0 = 1$ (foot), we obtain the mechanical effect produced

$$L = p \ hyp. \ log. \left(\frac{p_1}{p}\right) = 2{,}3026 \, p \, log. \left(\frac{p_1}{p}\right).$$

This formula gives the mechanical effect which must be expended to convert a unit or cubic foot of air of a lower pressure or tension $p$ into a higher tension $p_1$, and to reduce it thereby to the volume $\left(\frac{p}{p_1}\right)$ cubic feet. On the other hand:

$$L = p_1 \ hyp. \ log. \left(\frac{p_1}{p}\right) = 2{,}3026 \, p_1 \ log. \ \left(\frac{p_1}{p}\right)$$

expresses the effect which a unit of volume of gas gives out or produces when it passes from a higher pressure $p_1$ to a lower $p$.

To reduce by condensation a mass of air of the volume $V$, and tension $p$ to the volume $V_1$, and the tension $p_1 = \frac{V}{V_1} p$, the mechanical effect requisite to be expended is $V p \ hyp. \ log. \left(\frac{V}{V_1}\right)$, and when, inversely, the volume $V_1$ at a tension $p_1$ is converted by expansion into the volume $V$, and into the tension $p = \frac{V_1}{V} p_1$, the effect:

$$V p \ hyp. \ log. \ \left(\frac{V}{V_1}\right) = V_1 p_1 \ hyp. \ log. \left(\frac{V}{V_1}\right)$$

will be given out.

*Examples.*—1. If a blast converts 10 cubic feet of air per second, of 28 inches tension, into air of 30 inches tension, the effect to be expended upon this for every second will be $= 17280 \, . \, 0{,}820 \, . \, 28 \ hyp. \ log. \left(\frac{30}{28}\right) = 251596$ (*hyp. log.* 15 — *hyp. log.* 14) $= 251596 \, (2{,}708050 - 2{,}639057) = 251596 \, . \, 0{,}068993 = 17358$ inch lbs. $= 1446{,}5$ ft. lbs.—2. If a mass of vapour in a steam engine below the surface of a

piston $A = \pi 8^2 = 201$ square inches, stands 15 inches high, and with a tension of three atmospheres, pushes up the piston 25 inches, the mechanical effect evolved, and which is expended on the piston, is:

$$L = 201{,}3 \; . \; 15{,}6 \; . \; 15 \; hyp.\; log. \left(\frac{15+25}{15}\right) = 1411020 \; hyp.\; log.\; \tfrac{8}{3}$$

$= 1411020 \,.\, 0{,}980383 = 138334$ inch lbs. $= 11527$ feet lbs., and the mean force of the piston, without regard to its friction and the counter pressure, is:

$$= \frac{138334}{25} = 59832 \text{ lbs.}$$

§ 299. *Strata of air.*—Air enclosed in a vessel is at different depths of different density and tension, for the upper strata press together the lower on which they rest, so that there is only one and the same density and tension in one and the same horizontal stratum, and both increase with the depth. But in order to discover the law of this increase of density downwards, or the decrease upwards, we must adopt a method very similar to that of the former paragraph.

FIG. 395.


Let us imagine a vertical column of air $AE$, Fig. 395, of the transverse section $AB=1$, and of the height $AF=s$. Let the density of the lower stratum $= \gamma$, and the tension $= p$, and the density of the upper stratum $EF = \gamma_1$, and the tension $= p_1$, we shall then have $\frac{\gamma_1}{\gamma}=\frac{p_1}{p}$. If $x$ is the height $EE_1$ of the stratum $E_1F$, we have its weight, and hence also the diminution of its tension corresponding to:

$$y = 1 \; . \; x \; . \; \gamma_1 = \frac{x\,\gamma\,p_1}{p}, \text{ and inversely,}$$

$$x = \frac{p}{\gamma} \,.\, \frac{y}{p_1}, \text{ or as in the former paragraph:}$$

$$x = \frac{p}{\gamma}\; hyp.\; log. \left(1+\frac{y}{p_1}\right) = \frac{p}{\gamma}\; [hyp.\; log.\; (p_1+y) - hyp.\; log.\; p_1].$$

Let us put for $p_1$, successively

$$p_1+y,\; p_1+2\,y,\; p_1+3\,y, \;\&c., \text{ to } p_1+(n-1)\,y,$$

and add the corresponding heights of the strata or values of $x$, and we shall then obtain the height of the entire column of air, as in the former §.

$$s=\frac{p}{\gamma}\,(hyp.\ log.\ p - hyp.\ log.\ p_1)=\frac{p}{\gamma}\ hyp.\ log.\left(\frac{p}{p_1}\right),\ \text{also}$$

$$s=\frac{p}{\gamma}\ hyp.\ log.\left(\frac{b}{b_1}\right)=2{,}302\frac{p}{\gamma}\ log.\left(\frac{b}{b_1}\right)$$

if $b$ and $b_1$ are the heights of the barometer corresponding to the tensions $p$ and $p_1$.

If, inversely, the height $s$ is given, the expansive force and density of the air corresponding to it may be calculated. It is

$\gamma_1\, dx \quad \therefore \quad dx = \frac{dp_1}{\gamma_1}, \quad \frac{\gamma_1}{\gamma} = \frac{p_1}{p}$

$\frac{p}{\gamma}\frac{dp_1}{p_1} \qquad x = -\frac{p}{\gamma}\log p_1 + C$

$S = \frac{p}{\gamma}\log\frac{p}{p_1}$ limits

$\therefore \frac{p}{p_1} = \varepsilon^{\frac{s\gamma}{p}} = \frac{\gamma}{\gamma_1} \qquad \therefore \gamma_1 = \gamma\,\varepsilon^{-\frac{s\gamma}{p}}$

$s = 58604 \,.\, Log.\left(\frac{30}{25}\right) = 58604 \,.\, 0791813 = 58604 \,.\, 0{,}12 = 7032{,}48$ feet.—2. The density of the air on a mountain 10,000 feet high is: $Log.\ \frac{\gamma}{\gamma_1} = \frac{10000}{58604} = 0{,}1706$; hence, $\frac{\gamma}{\gamma_1} = 1{,}481$, and $\frac{\gamma_1}{\gamma} = \frac{1}{1{,}481} = 0{,}675$; it is therefore only $67\frac{1}{2}$ per cent. of the density of that at the foot.

§ 300. *Gay-Lussac's law.*—Heat or temperature has an essential influence on the density and expansive force of gases. The more air enclosed in a vessel becomes heated, the greater does its expansive force exhibit itself, and the higher that the temperature of the air enclosed by a piston in a vessel is raised, the more it expands, and pushes against the piston. From the experiments of Gay-Lussac, which in later times have been repeated by Rudberg, Magnus and Regnault, it results that for equal densities the expansive force, and for equal expansive forces the volume of one and the same quantity of air increases as the temperature. We may place this law by the side of that of Mariotte, and name it, for distinction's sake, Gay-Lussac's law.

According to the latest experiments, the expansive force of a definite volume of air increases by being heated from the freezing to the boiling point, by 0,367 of its original value, or for this increase of temperature the volume of a definite quan-

tity of air increases, the tension remaining the same, by 36,7 per cent. If the temperature is given in centigrade degrees, of which there are 100 between the freezing and boiling point, it follows that the expansion for each degree is = 0,00367, and for $t$ degrees temperature = 0,00367 . $t$; if we make use of Fahrhenheit's thermometer, which contains between the freezing and boiling point $180^0$, for each degree the expansion is = .002039, and for $t$ degrees temperature = .002039 . $t$. This co-efficient is true only for atmospheric air; slightly greater values correspond to other gases, and even for atmospheric air, this co-efficient increases slightly with the temperature.

If a mass of air of the original volume $V_0$, and of the temperature $0^0$ be heated $t$ degrees without assuming a different tension, the new volume is then $V = (1 + 0{,}00367\, t)\, V_0$, and if it acquire the temperature $t_1$, it will then assume the volume: $V_1 = (1 + 0{,}00367\, t_1)\, V_0$, and by dividing the ratio of the volumes:

$$\frac{V}{V_1} = \frac{1 + 0{,}00367\, t}{1 + 0{,}00367\, t_1},$$

on the other hand, the corresponding ratio of density:

$$\frac{\gamma}{\gamma_1} = \frac{V_1}{V} = \frac{1 + 0{,}00367\, t_1}{1 + 0{,}00367\, t}.$$

If moreover a change take place in the tensions, if $p_0$ is the tension at zero, $p$ that at the temperature $t$, and $p_1$ that at $t_1$, we then have:

$$V = (1 + 0{,}00367\, t)\frac{p_0}{p}\, V_0, \text{ further } V_1 = (1 + 0{,}00367\, t_1)\frac{p_0}{p_1}\, V_0,$$

hence:

$$\frac{V}{V_1} = \frac{1 + 0{,}00367\, t}{1 + 0{,}00367\, t_1} \cdot \frac{p_1}{p}, \text{ and } \frac{\gamma}{\gamma_1} = \frac{1 + 0{,}00367 \,.\, t_1}{1 + 0{,}00367 \,.\, t} \cdot \frac{p}{p_1}, \text{ or,}$$

$$\frac{\gamma}{\gamma_1} = \frac{1 + 0{,}00367 \,.\, t_1}{1 + 0{,}00367 \,.\, t} \cdot \frac{b}{b_1}.$$

*Example.* If a mass of air, of 800 cubic feet, and of ~~10~~ 15 lbs. tension, and $10^\circ$ temperature, is raised by the blast and by the warming apparatus of a blast-furnace to a tension of 19 lbs. and to a temperature of $200^\circ$, it will at length assume the greater volume:

$$V_1 = \frac{1 + 0{,}00367 \,.\, 200}{1 + 0{,}00367 \,.\, 10} \cdot \frac{15}{19} \cdot 800 = \frac{1{,}734}{1{,}0367} \cdot \frac{12000}{19} = 1056 \text{ cubic feet.}$$

§ 301. *Density of the air.*—By aid of the formula at the end of

the former paragraph, $\gamma$ may now be calculated by the density corresponding to a given temperature and tension of the air. By accurate weighings and measurements we have the weight of a cubic metre of atmospheric air at a temperature of $0^0$, and 0,76 metre height of barometer = 1,2995 kilogrammes. Since a cubic foot (Prussian) = 0,030916 cubic metre and 1 kilogramme = 2,13809 lbs. The density of the air for the relations given is: = 0,030916 . 2,13809 . 1,2995 = 0,08590 lbs. If now the temperature is = $t^0$ cent., the density for the French measure: $\gamma = \frac{1,2995}{1 + 0,00367\, t}$ kilogrammes; and for the Prussian measure:

$\gamma = \frac{0,08590}{1 + 0,00367 \,.\, t}$ lbs. and for the English: $\gamma = \frac{0,081241}{1 + 0,00204 t}$ lbs.

If now the expansive force varies from the mean, if, therefore, the height of the barometer is not 0,76 metres, but $b$, we shall obtain:

$$\gamma = \frac{1,2995}{1 + 0,00367 \,.\, t} \cdot \frac{b}{0,76} = \frac{1,71 \,.\, b}{1 + 0,00367\, t} \text{ kilog.}$$

or if $b$, as is commonly the case, be given in Paris inches:

$$\gamma = \frac{0,003058 \,.\, b}{1 + 0,00367\, t} \text{ lbs.}$$

Very often the expansive force is expressed by the pressure $p$, on a square centimetre or square inch, for this reason the factor $\frac{p}{1,0336}$, or $\frac{p}{14,73}$, or $\frac{p}{15,05}$ must be introduced, and it then follows that:

$$\gamma = \frac{1,2995}{1 + 0,00367 \,.\, t} \cdot \frac{p}{1,0336} = \frac{1,2572\, p}{1 + 0,00367 \,.\, t} \text{ kilog. or}$$

$$\gamma = \frac{0,08565}{1 + 0,00367 \,.\, t} \cdot \frac{p}{15,05} = \frac{0,005691\, p}{1 + 0,00367 \,.\, t} \text{ lbs. Prussian.}$$

For the same temperature and tension, the density of steam is $\frac{5}{8}$ of the density of atmospheric air, for which reason we have for steam:

$$\gamma = \frac{0,8122}{1 + 0,00367\, t} \cdot \frac{p}{1,0336} = \frac{0,7857\, p}{1 + 0,00367\, t} \text{ kilog. or}$$

$$\gamma = \frac{0,05353}{1 + 0,00367 \,.\, t} \cdot \frac{p}{15,05} = \frac{0,003557\, p}{1 + 0,00367 \,.\, t} \text{ lbs. Prussian.}$$

$$= \frac{0,050775}{1 + 0,00204\, t} \cdot \frac{p}{14,73} = \frac{0,003447\, p}{1 + 0,00204\, t} \text{ lbs. English.}$$

*Examples.*—1. What weight has the wind contained in a cylindrical regulator, of 40 feet length and 6 feet width, at a temperature of $10^\circ$ and 18 lbs. pressure? The density of this wind is:

$$= \frac{0{,}005691 \,.\, 18}{1{,}0367} = \frac{0{,}10244}{1{,}0367} = 0{,}0988 \text{ lbs.};$$

the capacity of the regulating vessel is $= \pi \,.\, 3^2 \,.\, 40 = 1131$ cubic feet, hence, the quantity of wind $= 0{,}0988 \,.\, 1131 = 112$ lbs.—2. A steam engine uses per minute 500 cubic feet of vapour, of $107^\circ$ temperature and 36 inches pressure, how many pounds of water is required for the generation of this steam? The density of this steam is:

$$= \frac{0{,}05353}{1 + 0{,}00367 . 107^\circ} \,.\, \frac{36}{28} = \frac{0{,}05353 \,.\, 36}{1{,}393 \,.\, 28} = 0{,}0494 \text{ lbs.};$$

hence, the weight of 500 cubic feet, or the weight of the corresponding quantity of water, $= 500 \,.\, 0{,}0494 = 24{,}7$ lbs.

FIG. 396.


§ 302. By aid of the results obtained in the last paragraph, the theory of the air manometer may be explained. This instrument consists of a barometer tube of uniform bore *AB*, Fig. 396, filled above with air and below with mercury, and of a vessel *CE* likewise containing mercury, which is put in communication with the gas or vapour whose tension we wish to find. From the height of the columns of mercury and of air, the expansive force may be estimated as follows. The instrument is commonly so arranged, that the mercury in the tube stands at the same level as the mercury in the cistern, when the temperature of the enclosed air $t = 10^0$, and the tension in the space *EH* equal to the mean atmospheric pressure $b = 0{,}76$ metres $= 30$ inches.

But if for a height of the barometer $b$, from *EH* a column of mercury $h_1$ has ascended into the tube, and the length of the column of the residuary air is $h_2$, we have then its tension

$$= \frac{h_1 + h_2}{h_2} b, \text{ and hence } b_1 = h_1 + \frac{h_1 + h_2}{h_2} b. \quad (A)$$

If a change of temperature takes place, the temperature from observation of $h_1$ and $h_2$ is not as at first $= t$, but $t_1$, we then have the tension of the column of air

$$AS = \frac{h_1 + h_2}{h_2} \,.\, \frac{1 + 0{,}00367 \,.\, t_1}{1 + 0{,}00367 \,.\, t} \,.\, b,$$

and hence the height of the barometer in question:

$$b_1 = h_1 + \frac{h_1 + h_2}{h_2} \,.\, \frac{1 + 0{,}00367 \,.\, t_1}{1 + 0{,}00367 \,.\, t} \,.\, b\,.$$

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pressure

,588 $h_1$

, indicates

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... pressure of air in tube of the manometer ...

∴ by Mariotte's law we have

$$x = \frac{h_1 + h_2}{h_2} p \quad (1)$$

but p is measured by ... the barometer

... ∴ (1) becomes $x = \frac{h_1 + h_2}{h_2} b$ ... (2)

The column of mercury in the barometer b, is in equilibrium with the outside pressure of the air.

The column of mercury in the ... $h_1$ is in equilibrium with the outside pressure of the air minus the tension of the air in the manometer.

$$\therefore b_1 = h_1 + x$$

$$\text{or } b_1 = h_1 + \frac{h_1 + h_2}{h_2} b \quad \text{......(A)}$$

*Example*
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FIG.


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For $b = 28$ inches (Paris), and $t = 10^0$ C., it follows that

$$b_1 = h_1 + 27\ (1 + 0{,}00367\ t_1)\ \frac{h}{h_2},\ \text{whereby}\ h = h_1 + h_2,$$

represents the whole length of the tube measured to the surface of mercury *HR*.

From the height of the barometer $b_1$, it follows that the pressure on the square inch (Prussian) is

$$p = \frac{15{,}6}{28}\ h_1 + 15{,}6 \,.\, \frac{27}{28}\ (1 + 0{,}00367\ t_1 \,.\, \frac{h}{h_2} = 0{,}538\ h_1$$
$$+ 14{,}51\ (1 + 00367\ t)\ \frac{h}{h_2}\ \text{lbs.}$$

*Example.*—If an air manometer, of 25 inches length, at 21° temperature, indicates a column of air of 12 inches in length, then the corresponding height of the barometer is:

$b_1 = 25 - 12 + 27\ (1 + 0{,}00367 \,.\, 21)\ \frac{25}{12} = 13 + 9 \,.\, 1{,}07707 \,.\, \frac{25}{4}$
$= 13 + 60{,}58 = 73{,}58$ inches, and the pressure on a square inch
$= 0{,}538 \,.\, 73{,}38 = 39{,}59$ lbs.

# SECTION VI.

## DYNAMICS OF FLUID BODIES.

## CHAPTER I.

### THE GENERAL LAWS OF THE EFFLUX OF WATER FROM VESSELS.

§ 303. *Efflux.*—The doctrine of the efflux of fluids from vessels constitutes the first principal division of hydro-dynamics. We distinguish first between the efflux of *air* and the efflux of *water*, and then again between the efflux under *uniform* and that under *variable pressure*. We next treat of the efflux of water under constant pressure. The pressure of water may be assumed as constant when the same quantity of water is admitted on one side as flows out from the other, or when the quantity of water flowing out in a certain time is small compared with the size of the vessel. The chief problem, whose solution will be here treated, is that of determining the *discharge* through a given *orifice*, under a given pressure, in a definite time.

If the discharge in each second $= Q$, we then have the expenditure, after the lapse of $t$ seconds, under invariable pressure: $Q_1 = Qt$. But to obtain the efflux per second, it is necessary to know the dimensions of the orifice, and the velocity of the particles of fluid issuing from it. For the sake of simplicity of investigation, we shall for the present assume that the particles of water flow out in straight and parallel lines, and on this account form a *prismatic vein* or stream of fluid. If now $F$ be the transverse section of the

vein and $v$ the velocity of the water, or of each particle of water, the discharge per second will form a prism of the base $F$ and height $v$, and therefore $Q$ will be $= Fv$ units of volume and $G = Fv\gamma$ units of weight, $\gamma$ being the density of the water or the effluent liquid.

*Examples.*—1. If water flows through a sluice, of 1,7 square feet aperture, with a 14 feet velocity, the discharge will be $Q = 14 \cdot 1{,}7 = 23{,}8$ cubic feet, and hence the discharge per hour will be $= 23{,}8 \cdot 3600'' = 85680$ cubic feet.—2. If 264 cubic feet of water were to be discharged through an orifice of 5 square inches in

...lso by initial velocity dh ... dh,

$v_1 = \sqrt{2(g_1 dh + g_2 dh)} - \sqrt{2g(h \cdot h_1)} = \sqrt{2gh + c^2}$

$Fv \quad \therefore \quad C = \frac{F}{G} v$ put this value

Let $F \cdot G$

$v \quad \frac{G}{F}$ ; Let

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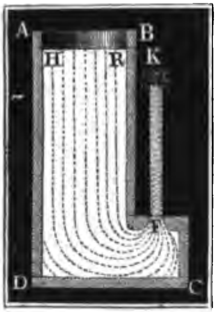
mass $Q\gamma$ accumulates in its transit from a state of rest into that of the velocity $v$, is $\frac{v^2}{2g} Q\gamma$ (§ 71). If no loss of mechanical effect take place in its passage through the orifice, both mechanical effects will be equal, and therefore $h\,Q\gamma = \frac{v^2}{2g} Q\gamma$; *i. e.*, $h = \frac{v^2}{2g}$, and, inversely, $v = \sqrt{2gh}$, or in feet, $h = 0{,}0155\,v^2$, and $v = 8{,}03\,\sqrt{h}$.

*The velocity, therefore, of water issuing through an orifice is equivalent to the final velocity of a body falling freely from the height of the water.*

The correctness of this law may be proved by the following experiments. If we apply an orifice directed upwards to the vessel $AC$, Fig. 398, the jet $FK$ will ascend vertically, and nearly

attain the level $HR$ of the water in the vessel, and we may assume that it would exactly attain this height, were all resistances, such as those of the air, friction at the sides of the vessel, disturbances from the descending water, &c., were entirely removed. But since a body ascending to a perpendicular height $h$, has the initial velocity $v = \sqrt{2gh}$ (§ 17), it accordingly follows that the velocity of efflux is $v = \sqrt{2gh}$.

FIG. 398.




For a different head of water $h_1$, the velocity is $v_1 = \sqrt{2gh_1}$, hence we have $v : v_1 = \sqrt{h} : \sqrt{h_1}$; *therefore, the velocities of efflux are to each other as the square roots of the heads of water.*

*Examples.*—1. The discharge which takes place in each second through an orifice 10 inches square, under a pressure of 5 feet, is:

$Q = Fv = 10 \,.\, 12 \sqrt{2gh} = 120 \,.\, 8{,}03 \sqrt{5} = 963{,}60 \,.\, 22{,}36 = 2093{,}4$ cubic inches.

2. That 252 cubic inches may be discharged through an orifice of 6 square inches in each second, the head of water required is:

$$h = \frac{1}{2g}\left(\frac{Q}{F}\right)^2 = 0{,}0155 \left(\frac{252}{6}\right)^2 = 0{,}0155\,(42)^2 = 26{,}81 \text{ inches.}$$

§ 305. *Velocity of influx and efflux.*—If water flows in with a certain velocity $c$, the mechanical effect $\frac{c^2}{2g}$, corresponding to the velocity due to the height $h_1 = \frac{c^2}{2g}$ must be added to the mechanical effect $h \,.\, Q\gamma$; hence we have to put:

$$(h + h_1)\, Q\gamma = \frac{v^2}{2g}\, Q\gamma, \text{ or } h + h_1 = \frac{v^2}{2g},$$

and therefore the velocity of efflux:

$$v = \sqrt{2g\,(h + h_1)} = \sqrt{2gh + c^2}.$$

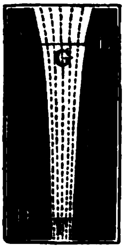
Since the quantity of water flowing into a vessel kept constantly full is as great as that $Q$ which flows out, we may put $Gc = Fv$, where $G$ represents the area of the transverse section $HR$ (Fig. 397) of the water pouring in. Accordingly if we put $c = \frac{F}{G} v$ we shall then obtain:

$$h = \frac{v^2}{2g} - \left(\frac{F}{G}\right)^2 \frac{v^2}{2g} = \left[1 - \left(\frac{F}{G}\right)^2\right]\frac{v^2}{2g};$$

and hence: $v = \dfrac{\sqrt{2gh}}{\sqrt{1 - \left(\frac{F}{G}\right)^2}}$.

According to this formula, the velocity increases, the greater the ratio of the sections $\frac{F}{G}$ becomes; the velocity is least, viz. $= \sqrt{2gh}$, if the transverse section $F$ of the orifice of efflux is small compared with the transverse section $G$ of the orifice of influx, and it approximates more and more to an infinitely great velocity, the smaller the difference is between these orifices. If $F = G$, therefore $\frac{F}{G} = 1$, then $v = \frac{\sqrt{2gh}}{0} = \infty$, and therefore also $c = \infty$. This infinite value must be understood to express that the water must flow to and from a bottomless vessel $AC$, Fig. 399, with an infinite velocity, that the stream of fluid $CF$ may entirely fill up the orifice of discharge $F$. If we put $v = \frac{Gc}{F}$, we shall then obtain:

FIG. 399.




$$h = \left[\left(\frac{G}{F}\right)^2 - 1\right]\frac{c^2}{2g}, \text{ hence } F = \frac{G}{\sqrt{1 + \frac{2gh}{c^2}}},$$

which expression indicates that the transverse section $F$ of the stream flowing out, for an infinite velocity of influx, is constantly less than the transverse section $G$ of the stream flowing in, and hence, that the discharging orifice is not quite filled when it is greater than $\dfrac{G}{\sqrt{1 + \frac{2gh}{c^2}}}$.

*Remark.* The accuracy of this formula, given by Daniel Bernoulli,

$v = \dfrac{\sqrt{2gh}}{\sqrt{1 - \left(\frac{F}{G}\right)^2}}$, has of late been brought into doubt by many philosophers. I have endeavoured to prove how unfounded are the objections made, in an article "Efflux," in the "Allgemeinen Maschinenencyclopädie."

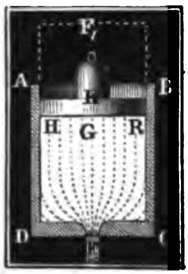
*Example.* If water runs from a circular orifice, 5 inches in width, in the bottom of

a prismatic vessel of 60 square inches transverse section, under a pressure of 6 feet, the velocity is then:

$$v = \frac{8,03\sqrt{h}}{\sqrt{1-\left(\frac{25\pi}{4.60}\right)^2}} = \frac{8,03 \ . \ 2,449}{\sqrt{1-(0,327)^2}} = \frac{19,666}{\sqrt{0,8931}} = \frac{19,666}{10945} = 20,8 \text{ feet.}$$

§ 306. *Velocity of efflux, pressure and density.*—The above formulæ are only true if the pressure of the air on the fluid surface is as great as its pressure against the orifice; but if these pressures are different from one another, we have then to extend these formulæ. If the upper surface $HR$, Fig. 400, is pressed by a piston $K$ with a force $P_1$, which case, for example, presents itself in that of the fire-engine, we may then imagine it to be replaced by the pressure of a column of water. If $h_1$ be the height of this column, and $\gamma$ the density of the liquid, we may therefore put $P_1 = Gh_1\gamma$.

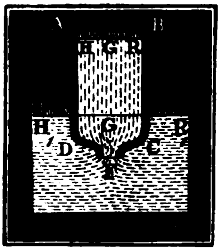
FIG. 400.




If we substitute for $h$ the head of water augmented by $h_1 = \frac{P_1}{G\gamma}$, $h + h_1 = h + \frac{P_1}{G\gamma}$, we then obtain for the velocity of efflux: $v = \sqrt{2g\left(h + \frac{P_1}{G\gamma}\right)}$, when, moreover, we suppose $\frac{F}{G}$ to be very small. If, further, we represent the pressure on each unit of surface of $G$ by $p_1$, we have more simply $\frac{P_1}{G} = p_1$, and hence $v = \sqrt{2g\left(h + \frac{p_1}{\gamma}\right)}$. Again, if we represent the pressure of water at the level of the orifice by $p$, we may then put $p = \left(h + \frac{p_1}{\gamma}\right)\gamma$; therefore, $h + \frac{p_1}{\gamma} = \frac{p}{\gamma}$, whence $v = \sqrt{2g\frac{p}{\gamma}}$.

*The velocity of efflux, therefore, increases as the square root of the pressure on each unit of surface, and inversely as the square root of the density of the fluid.* Under equal pressures, therefore, a fluid of a density represented by 4, runs out half as fast as one of density 1. Since the air is 770 times lighter than water, it would if it were an inelastic, flow out $\sqrt{770} = 27\frac{3}{4}$ faster than water. If the water does not flow freely, but under water, in consequence of a counter-pressure, a diminution of the velocity of

FIG. 401.




efflux then takes place. If the mouth of the vessel *AC*, Fig. 401, is the depth $FG = h$ below the surface of the upper water *HR*, and $FG_1 = h_1$ below the surface $H_1R_1$ of the lower water, we then have the pressure downwards $p = h\gamma$, and the counter-pressure upwards $p_1 = h_1\gamma$, hence the force of efflux is: $p - p_1 = (h - h_1)\gamma$, and the velocity of efflux $v = \sqrt{2g\left(\frac{p-p_1}{\gamma}\right)} = \sqrt{2g(h-h_1)}$.

For efflux under water the difference of level $h - h_1$ between the surfaces must be regarded as the head of water.

If the water on the side of the outer orifice be pressed by the force $p$, and on the side of the inner orifice or of the surface of water by the force $p_1$, we have then generally:

$$v = \sqrt{2g\left(h + \frac{p_1 - p}{\gamma}\right)}.$$

*Examples.*—1. If the piston of a 12 inch cylinder, or that of a fire-engine, be pressed down with a force of 3000 lbs., and there were no obstacle in the tubes or pipes, the water would then issue through the mouth-piece of the tube and be directed vertically upwards with a velocity

$$v = \sqrt{2g\frac{p_1}{\gamma}} = \sqrt{2g\frac{P_1}{G\gamma}} = 8{,}03\sqrt{\frac{3000}{\frac{\pi}{4}\cdot 62{,}5}} = 8{,}03\sqrt{\frac{600\cdot 4}{\pi\ 12{,}5}}$$

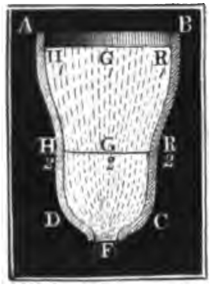
$= 65{,}28$ feet, and ascend to the height $h = 0{,}0155 \cdot v^2 = 66{,}18$ feet.—2. If water rushes into a rarefied space; for example, into the condenser of a steam-engine, whilst it is pressed from above or on its exposed surface by the atmosphere, the last formula $v = \sqrt{2g\left(h + \frac{p_1 - p}{\gamma}\right)}$ for the velocity of efflux is then to be applied. If the head of water $h = 3$ feet, and the external barometer stand at 27 inches, and the internal at 4 Paris inches, we shall now have $\frac{p_1 - p}{\gamma} = 27 - 4 = 23$ Paris inches $= \frac{23}{12}\ 1{,}035 = 1{,}9837$ Prussian inches $= 2{,}005$ English inches, or a column of water $= 13{,}5 \cdot 2{,}005 = 27{,}67$ feet, and the velocity of the water rushing into the vacuum $v = 8{,}03\sqrt{3 + 27{,}67} = 44{,}4$ feet.—3. If the water in the feed-pipe of a steam-engine boiler stands 12 feet above the surface of the water in the boiler, and the the pressure of steam be 20 lbs. and the pressure of the atmosphere only 15 lbs. on the square inch, the velocity with which the water enters into the boiler will be:

$$v = 8{,}03\sqrt{12 + \frac{(18\ .\ 20)\cdot 144}{62{,}5}} = 8{,}03\sqrt{12 - \frac{5\cdot 144}{62{,}5}} = 8{,}03\sqrt{.\ 48}$$

$= 4{,}96$ feet.

§ 307. *Hydraulic pressure.*—When water enclosed in a vessel is in motion, it then presses more feebly against the sides than when

FIG. 402.




at rest. We must, therefore, distinguish the hydrodynamic or hydraulic pressure from the hydrostatic pressure of water. If $p_1$ be the pressure on each unit of surface $H_1R_1 = G_1$, Fig. 402, $p$ the pressure without the orifice $F$, and $h$ the head of water $FG_1$, we then have for the velocity of efflux

$$v = \sqrt{2g\left(h + \frac{p_1 - p}{\gamma}\right)} \div \sqrt{1 - \left(\frac{F}{G_1}\right)^2}, \text{ or}$$

$h + \dfrac{p_1 - p}{\gamma} = \left[1 - \left(\dfrac{F}{G_1}\right)^2\right] \dfrac{v^2}{2g}$; if, further, in another transverse section $H_2 R_2 = G_2$, which lies at a height $FG_2 = h_1$ above the orifice, the pressure $= p_2$, we then have likewise:

$$h_1 + \frac{p_2 - p}{\gamma} = \left[1 - \left(\frac{F}{G_2}\right)^2\right] \frac{v^2}{2g}.$$

If we subtract one expression from the other, it then follows that:

$$h - h_1 + \frac{p_1 - p_2}{\gamma} = \left[\left(\frac{F}{G_2}\right) - \left(\frac{F}{G_1}\right)^2\right] \frac{v^2}{2g},$$

or, if the head of water $G_1G_2$ of the stratum $H_2R_2 = G_2$ be represented by $h_2$, the measure of the hydraulic ~~pressure~~ head of water at $H_2R_2$ is:

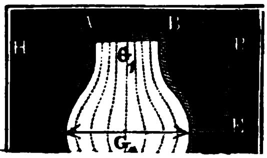
$$\frac{p_2}{\gamma} = h_2 + \frac{p_1}{\gamma} - \left[\left(\frac{F}{G_2}\right)^2 - \left(\frac{F}{G_1}\right)^2\right] \frac{v^2}{2g}.$$

But now $\dfrac{Fv}{G_1}$ is the velocity $v_1$ of the water at the surface $G_1$, and $\dfrac{Fv}{G_2}$ the velocity $v_2$ of the water at the section $G_2$; hence more simply we may put $\dfrac{p_2}{\gamma} = \dfrac{p_1}{\gamma} + h_2 - \left(\dfrac{v_2^2}{2g} - \dfrac{v_1^2}{2g}\right)$.

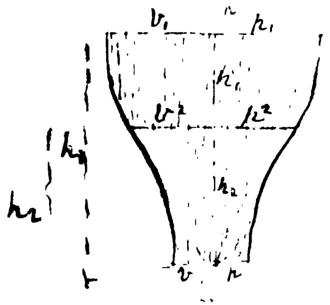
*Therefore, from this it follows that the hydraulic head of water* $\dfrac{p_2}{\gamma}$ *at any place in the vessel is equivalent to the hydrostatic head of water* $\dfrac{p_1}{\gamma} + h_2$ *diminished by the difference of the height due to the velocity at this point, and at the place of entrance.* If the upper surface of the water $G_1$ is great, we may neglect the velocity of influx, and hence may put $\dfrac{p_2}{\gamma} = \dfrac{p_1}{\gamma} + h_2 - \dfrac{v_2^2}{2g}$, and the *hy-*

*draulic head of water is less by the height due to the velocity than the hydrostatic head of water.* The faster, therefore, water flows in conduit pipes, the less it presses against the sides of the pipes. From this cause, pipes very often burst, or begin to leak, when its motion in them is checked, or when the pipes are stopped up, &c.

FIG. 403.



By means of the apparatus of efflux *ABCD*, Fig. 403, we may have ocular [illegible] of the difference between



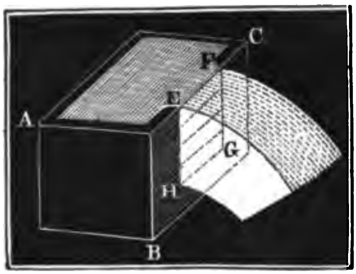
$\frac{v_2^2}{2g}$ is $< \frac{v_1^2}{2g}$. If, on the other hand, the transverse section $G_3$ be $< G_1$, and the water therefore flow through $G_3$ quicker than through $G_1$, we shall then have the height of the column of water in the small tube $E_1$ whose inner orifice is at $G_3$, $y = h_3 - \left(\frac{v_3^2}{2g} - \frac{v_1^2}{2g}\right)$ less than $h_3$, and hence it will not reach to the level *HR* of $G_1$. Again, if $G_4$ be very small, and therefore the corresponding velocity $v_4$ very great, then $\frac{v_4^2}{2g} - \frac{v_1^2}{2g}$ may be $> h_4$, and hence the corresponding hydraulic head of water $z$ may be negative, *i.e.* the air may press more from without than the water from within. A column of water will therefore ascend in the tube $E_2K$, which is inserted below, and whose outer orifice is under water, which in conjunction with the pressure of the water, will balance that of the external atmospheric. If this small tube be short, the water, which may be coloured for this purpose, will ascend from the vessel *K* underneath, through the tube, enter the reservoir of efflux, and will arrive at *F* and be discharged.

FIG. 404.


*Remark.* If the discharging vessel $ACE$, Fig. 404, consists of a wide reservoir $AC$ and of a narrow vertical tube $CE$, the hydraulic pressure at all places in this tube is then negative. If we disregard the pressure of the atmosphere $p_1$, the pressure of the water in the vicinity of the mouth $F$ may be put $= 0$, because the whole head of water here $GF = h$ will be expended in generating the velocity $v = \sqrt{2gh}$; on the other hand, at a place $D_1E_1$ at the height $G_1G = h_1$ below the surface of water, the hydraulic pressure $= h_1 - h = -(h - h_1)$ negative; if, therefore, a hole be bored in this tube, no water will run out, but air will be drawn in rather, which will arrive at $F$ and flow out. This negative pressure will be greatest directly below the water, because $h_1$ is there least.

§ 308. By means of the formula $Q = Fv = F\sqrt{2gh}$, the discharge issuing in one second can only then be calculated directly when the orifice is horizontal, because here only the velocity throughout the whole transverse section $F$ is the same; but if the transverse section of the orifice has an inclination to the horizon, for example, if it is at the side of the vessel, the particles of water at different depths will then flow out with different velocities, and the discharge $Q$ can no longer be considered as a prism, and hence, therefore, the formula $Q = Fv = F\sqrt{2gh}$ cannot be applied directly. The most simple case of this kind is presented in the efflux through a cut in the side of a vessel, or in what is called a weir, Fig. 405. This cut forms a rectangular aperture of efflux $EFGH$, whose breadth $EF = GH$ is represented by $b$, and height $EH = FG$ by $h$. If we divide this surface $bh$ by horizontal lines into a great number $n$ of equally broad laminæ, we may suppose the velocity in each of these to be the same. Since the heads of water of these laminæ from above downwards are $\frac{h}{n}, \frac{2h}{n}, \frac{3h}{n}$, &c., we then have the corresponding velocities $\sqrt{2g\frac{h}{n}}, \sqrt{2g\cdot\frac{2h}{n}}, \sqrt{2g\cdot\frac{3h}{n}}$; and since, further, the area of a lamina $= b\cdot\frac{h}{n} = \frac{bh}{n}$, we then have the discharges:

FIG. 405.



$\frac{bh}{n}\sqrt{2g\frac{h}{n}}, \frac{bh}{n}\sqrt{2g\cdot\frac{2h}{n}}, \frac{bh}{n}\sqrt{2g\cdot\frac{3h}{n}}$, &c.; consequently the discharge through the entire section:

$$Q = \frac{bh}{n}\left(\sqrt{2g\frac{h}{n}} + \sqrt{2g\cdot\frac{2h}{n}} + \sqrt{2g\cdot\frac{3h}{n}} + \ldots\right)$$

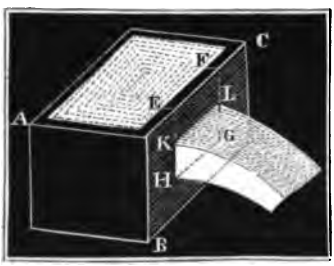
$$bh\sqrt{2gh}\,(\sqrt{1} + \sqrt{2} + \sqrt{3} + \ldots + \sqrt{n})$$

$$Q = b\sqrt{2g}\int_{h_2}^{h_1} x^{\frac{1}{2}}dx = \frac{2}{3}b\sqrt{2g}(h_1^{\frac{3}{2}} - h_2^{\frac{3}{2}})$$

$$v = \frac{2}{3}\sqrt{2g}\cdot\frac{h_1^{\frac{3}{2}} - h_2^{\frac{3}{2}}}{h_1 - h_2}$$

If by the term *mean velocity* ($v$) *be understood that velocity which must subsist at all places, that as much water, in consequence, does issue as with the variable velocities of efflux within the whole profile*; we may then put: $Q = bh \cdot v$, and, consequently, $v = \frac{2}{3}\sqrt{2gh}$, i.e. *the mean velocity of water issuing through a rectangular cut in the side of a vessel is* $\frac{2}{3}$ *of the velocity at the sill or lower edge of the cut.*

FIG. 406.




If the rectangular aperture of efflux $KG$, Fig. 406, with horizontal sill does not reach the surface of the water, we may find the discharge by regarding the aperture as the difference of the two cuts $EFGH$ and $EFLK$. Hence, if $h_1$ is the depth $HE$ of the lower, and $KE = h_2$ that of the upper edge, we then have the discharge from these apertures $\frac{2}{3}b\sqrt{2gh_1^3}$, and $\frac{2}{3}b\sqrt{2gh_2^3}$, and hence the quantity of water for the rectangular orifice $GHKL$:

$Q = \frac{2}{3}b\sqrt{2gh_1^3} - \frac{2}{3}b\sqrt{2gh_2^3} = \frac{2}{3}b\sqrt{2g}\,(h_1^{\frac{3}{2}} - h_2^{\frac{3}{2}})$, and the mean velocity of efflux:

$$v = \frac{Q}{b\,(h_1 - h_2)} = \frac{2}{3}\sqrt{2g}\cdot\frac{h_1^{\frac{3}{2}} - h_2^{\frac{3}{2}}}{h_1 - h_2}.$$

If $h$ is the mean head of water $\frac{h_1 + h_2}{2}$, or the depth of the centre of the orifice below the surface of water, and $a$ the height of the orifice $HK = h_1 - h_2$, we may then put:

$$v = \frac{2}{3} \sqrt{2g} \cdot \frac{\left(h + \frac{a}{2}\right)^{\frac{3}{2}} - \left(h - \frac{a}{2}\right)^{\frac{3}{2}}}{a}, \text{ or approximately:}$$

$$= \left[1 - \frac{1}{96}\left(\frac{a}{h}\right)^2\right] \sqrt{2gh}.$$

*Example.* If a rectangular orifice is 3 feet wide and $1\frac{1}{4}$ feet high, and the lower edge lies $2\frac{3}{4}$ feet below the surface of water, the discharge is then:

$Q = \frac{2}{3} . 8,03 . 3\, (2,75^{\frac{3}{2}} - 1,5^{\frac{3}{2}}) = 16,06\, (4,560 - 1,837) = 16,06 . 2,723 = 43,73$ cubic ft.
From the formulæ of approximation the mean velocity of efflux is:

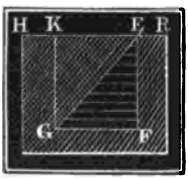
$v = \left[1 - \frac{1}{96}\left(\frac{1,25}{2,125}\right)^2\right] . 8,03 \sqrt{2,125} = (1 - 0,0036) . 11,683 = 11,683 - 0,042$
$= 11,641$ feet, and hence the discharge $Q = 3 . \frac{5}{4} . 11,641 = 43,65$ cubic feet.

*Remark.*—If the cut in the side is inclined to the horizon at an angle $\delta$, we shall then have to substitute the height of the aperture $\frac{h_1 - h_2}{\sin. \delta}$ for its vertical projection, whence we must put $Q = \frac{2}{3} \frac{b \sqrt{2g}}{\sin. \delta} (\sqrt{h_1^3} - \sqrt{h_2^3})$. If the transverse section of the reservoir parallel to the aperture be not considerably greater than the section of the aperture, we shall then have to take into account the velocity $v_1 = \frac{F}{G} v$ with which the water flows to it, and for this reason put:

$$Q = \frac{2}{3} b \sqrt{2g} \left[\left(h_1 + \frac{v_1^2}{2g}\right)^{\frac{3}{2}} - \left(h_2 + \frac{v_1^2}{2g}\right)^{\frac{3}{2}}\right].$$

§ 309. *Triangular lateral orifice.*—Besides rectangular lateral orifices, we have in practice triangular and circular. Let us first consider the efflux through a triangular orifice $EFG$, Fig. 407, with horizontal base, whose vertex $E$ lies in the surface of the water. Let the base $FG = b$ and the height $EF = h$, let us divide the last into $n$ equal parts, and carry through the points of division, lines parallel to the base, we then resolve the entire surface into small elements of the areas:

FIG. 407.




$$\frac{b}{n} \cdot \frac{h}{n}, \frac{2b}{n} \cdot \frac{h}{n}, \frac{3b}{n} \cdot \frac{h}{n}, \text{ \&c.,}$$

and the heads of water:

$$\frac{h}{n}, \frac{2h}{n}, \frac{3h}{n}, \text{ \&c.}$$

The discharge [redacted]

If the base of the orifice $EGK$ lies in the surface and the vertex lower by $h$, we then have the discharge $\frac{2}{3}\, bh \sqrt{2\, gh}$ flowing through the rectangle $EFGK$,

$$Q_1 = \tfrac{2}{3}\, bh \sqrt{2\, gh} - \tfrac{2}{5}\, bh \sqrt{2\, gh} = \tfrac{4}{15}\, bh \sqrt{2\, gh}.$$

Through the trapezium $ABCD$, Fig. 408, whose upper base $AB = b_1$, lies in the surface of the water, and whose lower base is $CD = b_2$, and height $DE = h$, we may find the discharge by regarding the orifice as composed of a rectangle and two triangles, viz:

$$Q = \tfrac{2}{3}\, b_2 h \sqrt{2\, gh} + \tfrac{4}{15}\,(b_1 - b_2)\, h \sqrt{2\, gh} = \tfrac{2}{15}\,(2\, b_1 + 3\, b_2)\, h \sqrt{2\, gh}.$$

FIG. 408.

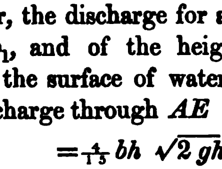
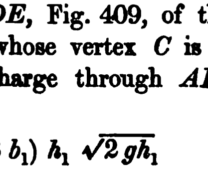



FIG. 409.




Further, the discharge for a triangle $CDE$, Fig. 409, of the base $DE = b_1$, and of the height $h_1$, and whose vertex $C$ is distant $h$ from the surface of water: $Q$ = discharge through $ABC$ less the discharge through $AE$

$$= \tfrac{4}{15}\, bh \sqrt{2\, gh} - \tfrac{2}{15}\,(2\, b + 3\, b_1)\, h_1 \sqrt{2\, gh_1}$$

$$= \tfrac{2}{15} \sqrt{2g}\, \left[2\, b\, (h^{\frac{3}{2}} - h_1^{\frac{3}{2}}) - 3\, b_1 h_1^{\frac{3}{2}}\right].$$

As the breadth $AB = b$ may be determined by the proportion $b : b_1 = h : (h - h_1)$, it follows that

$$Q=\frac{2\sqrt{2g}\,.\,b_1}{15}\left(\frac{2\,h\,(h^{\frac{3}{2}}-h_1^{\frac{3}{2}})}{h-h_1}-3\,h_1^{\frac{3}{2}}\right)$$

$$=\frac{2\,\sqrt{2\,g}b_1}{15}\left(\frac{2\,h^{\frac{5}{2}}-5\,h\,h_1^{\frac{3}{2}}+3\,h_1^{\frac{5}{2}}}{h-h_1}\right).$$

Lastly, for a triangle $ACD$, Fig. 410, whose vertex lies above the base, the quantity discharged is

$$Q=\tfrac{2}{3}\sqrt{2g}\,.\,b_1\,(h^{\frac{3}{2}}-h_1^{\frac{3}{2}})-\frac{2\,\sqrt{2g}\,.\,b_1}{15}\left(\frac{2\,h^{\frac{5}{2}}-5\,h\,h_1^{\frac{3}{2}}+3\,h_1^{\frac{5}{2}}}{h-h_1}\right)$$

$$=\frac{2\,\sqrt{2g}\,.\,b_1}{15}\left(\frac{3\,h^{\frac{5}{2}}-5\,h_1\,h^{\frac{3}{2}}+2\,h_1^{\frac{5}{2}}}{h-h_1}\right).$$

FIG. 410.

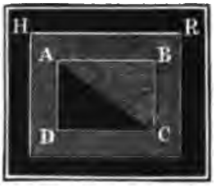



FIG. 411.


*Example.* What quantity of water flows through the square $ABCD$, Fig. 411, whose vertical diagonal $AC$ = 1 foot, if the angular point $A$ reaches the surface of the water? The upper half of this square gives the expenditure:

$Q=\frac{2}{5}\,b\,\sqrt{2gh^3}=\frac{2}{5}\,1\,.\,8{,}03\,\sqrt{\frac{1}{8}}=2{,}803\,.\,0{,}11=3{,}21\,.\,0{,}11=3531$ cubic feet, but the lower water expenditure:

$$Q_1=\frac{2\,b\,\sqrt{2\,g}}{15}\left(\frac{2h^{\frac{5}{2}}-5\,h\,h_1^{\frac{3}{2}}+3\,h_1^{\frac{5}{2}}}{h-h_1}\right)=\frac{2{,}803}{15}\left(\frac{2-5\,(\frac{1}{2})^{\frac{3}{2}}+3\,(\frac{1}{2})^{\frac{5}{2}}}{1-\frac{1}{2}}\right)$$

$=\frac{32{,}12}{15}(2-1{,}7678+0{,}5303)=\frac{32{,}12\,.\,0{,}7625}{15}=1{,}6309$ cubic feet; the discharge through the entire orifice is $Q=3531+1{,}6309=1{,}9840$ cubic feet.

§ 310. *Circular lateral orifices.*—The discharge through a circular orifice $AB$, Fig. 412, may be determined by an approximate formula in the following manner. Let us decompose the orifice by concentric circles into equally small annuli, and each annulus into very small elements, which may be regarded as parallelograms. If now $r$ is the radius of such an annulus, $b$ its breadth and $n$ the number of its elements, we have the magnitude of an element

FIG. 412.

$K_, = \frac{2\,\pi\,r\,b}{n}$. If $h$ is the depth $CG$ of the centre $C$ below the surface of water $HR$, and $\phi$ the angle $ACK$, by which an element $K$ is distant from the highest point $A$ of the annulus, we have then the head of water of this element:

$$KF = CG - CL = h - r\cos.\,\phi,$$

and hence the discharge of this element

$$= \frac{2\,\pi rb}{n}\sqrt{2g\,(h - r\cos.\,\phi)}.$$ It is now $\sqrt{h - r\cos.\,\phi}$

$$= \sqrt{h}\left[1 - \tfrac{1}{2}\frac{r}{h}\cos.\,\phi - \tfrac{1}{8}\left(\frac{r}{h}\right)^2\cos.\,\phi^2 + \ldots\right]$$

$$= \sqrt{h}\left[1 - \tfrac{1}{2}\frac{r}{h}\cos.\,\phi - \tfrac{1}{16}\left(\frac{r}{h}\right)^2(1 + \cos.\,2\,\phi) + \ldots\right],$$

hence the discharge of an element:

$$= \frac{2\,\pi rb}{n}\sqrt{2\,gh}\left[1 - \tfrac{1}{2}\,.\,\frac{r}{h}\cos.\,\phi - \tfrac{1}{16}\left(\frac{r}{h}\right)^2(1 + \cos.\,2\,\phi) - \ldots\right].$$

The discharge of an entire annulus is now known, if we put in the parenthesis for 1, $n\,.\,1 = n$, for $\cos.\,\phi$ the sum of all the cosines of $\phi$ from $\phi = 0$ to $\phi = 2\,\pi$, and for the cosine of $2\,\phi$, the sum of all the cosines of $2\,\phi$ from $2\,\phi = 0$ to $2\,\phi = 4\,\pi$. But as the sum of all the cosines of a complete circle is $= 0$, these cosines vanish, and the discharge for the annulus:

$$= \frac{2\,\pi rb}{n}\sqrt{2\,gh}\left[n - \tfrac{1}{16}\left(\frac{r}{h}\right)^2 .\, n - \ldots\right]$$

$$= 2\,\pi rb\,\sqrt{2\,gh}\left[1 - \tfrac{1}{16}\left(\frac{r}{h}\right)^2 - \ldots\right]$$

If now for $b$ we substitute $\frac{r}{m}$, and for $r$, $\frac{r}{m}, \frac{2\,r}{m}, \frac{3\,r}{m}$, to $\frac{mr}{m}$, we then obtain the discharge of all the annuli which make up the circular surface, and lastly, the quantity of efflux of the whole circle

$$Q = 2\pi r\sqrt{2gh}\left[\frac{r}{m^2}(1 + 2 + 3 + \ldots + m)\right.$$

$$\left. - \tfrac{1}{16}\,\frac{r^3}{m^4h^2}(1^3 + 2^3 + 3^3 + \ldots + m^3)\right]$$

$$= 2\,\pi r\sqrt{2\,gh}\,.\left(\frac{r}{m^2}\cdot\frac{m^2}{2} - \tfrac{1}{16}\cdot\frac{r^3}{m^4h^2}\cdot\frac{m^4}{4}\right)$$

$$=\pi r^2 \sqrt{2gh}\left[1-\tfrac{1}{32}\left(\frac{r}{h}\right)^2-\ldots\right],$$

or more accurately:

$$Q=\pi r^2 \sqrt{2gh}\left[1-\tfrac{1}{32}\left(\frac{r}{h}\right)^2-\tfrac{5}{1024}\left(\frac{r}{h}\right)^4-\ldots\right].$$

If the circle reaches the surface of the water, then

$$Q=\tfrac{987}{1024}\,\pi r^2 \sqrt{2gh}=0{,}964\,F\sqrt{2gh},$$

if $F$ represents the area of the circle.

It is besides easy to conceive that in all cases where the head of water at the centre is equal to or greater than the diameter, we may put the whole series $= 1$, and take $Q=F\sqrt{2gh}$. This rule may also be applied to other orifices, and, therefore, in all cases where the centre of gravity of an orifice lies at least as deep below the fluid surface as the figure is high, the depth $h$ of this point may be regarded as the head of water, and $Q$ put $= F\sqrt{2gh}$.

If we consider that the mean of all the cosines of the first quadrant $=\frac{\pi}{4}$, and that all the cosines of the second $=-\frac{\pi}{4}$, the mean of the first and of the second vanishes, we may then, after the method adopted above, find the discharge of the upper semicircle:

$$Q_1=\frac{\pi r^2}{2}\sqrt{2gh}\left[1-\frac{\pi}{12}\left(\frac{r}{h}\right)-\tfrac{1}{32}\left(\frac{r}{h}\right)^2\right],$$

and that of the lower:

$$Q_2=\frac{\pi r^2}{2}\sqrt{2gh}\left[1+\frac{\pi}{12}\left(\frac{r}{h}\right)-\tfrac{1}{32}\left(\frac{r}{h}\right)^2\ldots\right].$$

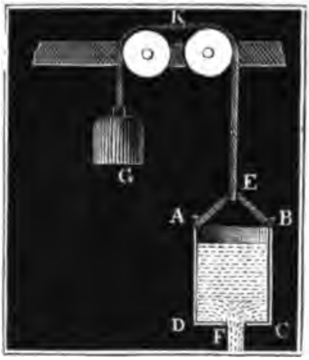
*Example.* What quantity of water flows hourly through a circular orifice 1 inch diameter, above which the fluid surface stands $\frac{7}{12}$ inch high?

$\frac{r}{h}=\frac{6}{7}$, hence $\left(\frac{r}{h}\right)^2=\frac{36}{49}=0{,}735$; further, $1-\frac{1}{32}\left(\frac{h}{r}\right)^2=1-0{,}023=0{,}977$, and consequently the discharge per second:

$Q=\frac{\pi\cdot 1^2}{4}\,12\,.\,8{,}03\sqrt{\frac{7}{144}}\,.\,0{,}977=\frac{\pi}{4}\,.\,8{,}03\,.\,0{,}977\sqrt{7}=16{,}26$ cubic inches per minute $=963$ per hour $=33\tfrac{1}{2}$ cubic feet.

§ 311. *Discharging vessels in motion.*—The velocity of efflux varies if a vessel previously at rest or in uniform motion changes its condition of motion, because in this case every particle acts by its own weight, as well as by its inertia against the surrounding medium.

FIG. 413.




If we move the vessel $AC$, Fig. 413, upwards with a vertical accelerating force, whilst the water flows through the bottom by the hole $F$, an increase takes place, and if it be moved downwards vertically by an accelerating force, a diminution of the velocity of efflux ensues. If $p$ is the accelerating force, each element of water $M$ presses not only by its own weight $Mg$, but also by its inertia $Mp$; consequently the force of each element in the one case, must be put $(g+p)\ M$, and in the other $(g-p)\ M$, therefore instead of $g$, $g \pm p$. From this it follows then that $\frac{v^2}{2} = (g \pm p)\ h$, and hence for the velocity:

$$v = \sqrt{2\ (g \pm p)\ h}.$$

If the vessel ascends with the accelerating force $g$, then is $v = \sqrt{2\ .\ 2\,gh} = 2\sqrt{gh}$, therefore the velocity of efflux 1,414 times that of a vessel at rest. If the vessel falls by its own weight, therefore with the accelerated motion $g$, there is $v = \sqrt{0} = 0$, no water therefore flows out. If the vessel moves uniformly up or down, there remains $v = \sqrt{2\,gh}$, but if it ascends with a retarded motion, then will $v = \sqrt{2\ (g - p)\ h}$, and if it descends with the same retardation, then $v = \sqrt{2\ (g + p)\ h}$.

If the vessel moves horizontally, or at an acute angle to the horizon, (§. 274) the fluid surface will be inclined to the horizon, and a change in the velocity of flow will take place.

By the rotation of a vessel $AC$, Fig. 414, about its vertical axis $X\overline{X}$, the concave surface forms a parabolic funnel $AOB$, hence there will be over the middle $F$ of the bottom a lesser head of water than at the edges, and hence the water will flow through the orifice $F$ in the axis more slowly than through any other orifice $K$ at the bottom. If $h$ represent the head of water in the middle, then the velocity of efflux at the middle will be $= \sqrt{2\,gh}$, if $y$ be the distance

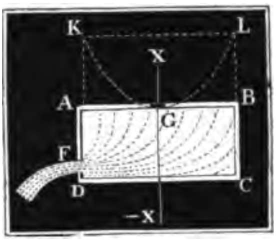
FIG. 414.

$FK = ME$ of any other orifice $K$ from the axis, and $\omega$ the angular velocity, we shall then have the corresponding elevation of the water above the middle:

$$OM = \tfrac{1}{2}\, TM = \tfrac{1}{2}\, ME \text{ cotang. } T = \tfrac{1}{2}\, y \cdot \frac{\omega^2 y}{g} = \frac{\omega^2 y^2}{2g} = \frac{w^2}{2g},$$

if $w$ be the velocity of rotation of the orifice $K$. Hence then the velocity of efflux for this is

FIG. 415.




$$v = \sqrt{2g\left(h + \frac{w^2}{2g}\right)} = \sqrt{2gh + w^2}.$$

This formula is true for every arbitrarily shaped vessel, and also for one closed above, as $AC$, Fig. 415, so that the funnel cannot be formed. Its application to wheels of reaction and to turbines will be found in the sequel.

*Examples.*—1. If a vessel full of water $AC$, Fig. 413, weighs 350 lbs., and by means of a rope passing over a roller $K$ is drawn by a weight $G$ of 450 lbs., it will ascend with an accelerating force $p = \frac{450-350}{450+350} \cdot g = \frac{100}{800}\, g = \frac{1}{8}\, g$, and hence the velocity of efflux will be $v = \sqrt{2\,(g+p)\,h} = \sqrt{2 \cdot \frac{9}{8} \cdot gh} = \sqrt{\frac{9}{4} gh}$. Were the head of water $h = 4$ feet, the velocity of efflux would be $v = 1 \sqrt{9 \cdot g} = 3 \sqrt{32,2} = 16,01$ feet.—2. If a vessel $AC$, Fig. 415, full of water revolves so that it makes 100 revolutions per minute, if the depth of the orifice $F$ below the surface of water in the middle amounts to 2 feet, and the distance from the axis $X\overline{X}$ 3 feet, then the velocity of efflux is

$$v = \sqrt{2\,g\,h + w^2} = \sqrt{64,4 \cdot 2 + \left(\frac{3 \cdot \pi \cdot 100}{30}\right)^2} = \sqrt{128,8 + 100 \cdot \pi^2}$$
$$= \sqrt{128 + 987} = 32,4 \text{ feet.}$$

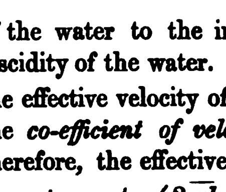
# CHAPTER II.

## ON THE CONTRACTION OF THE FLUID VEIN BY THE EFFLUX OF WATER THROUGH ORIFICES IN A THIN PLATE.

§ 312. *Co-efficient of velocity.*—The laws of efflux developed in the preceding chapter accord almost entirely with experiment, so long as the head of water is not small compared with the width of the orifice, and as long as the orifice gradually widens inwards without forming corners or edges, and is close at the bottom or sides of the vessel. The experiments made by Michelotti, by Eytelwein, and by the author on this subject with smoothly polished metallic mouth-pieces, have shewn that the effective discharge, or that which actually flows out, amounts to from 96 to 98 per cent of the theoretical quantity.

The mouth-piece $AD$, Fig. 416, represented in half its natural size, gave for a head of water of 10 feet, 97,5 per cent., for 5 ft. 96,9 per cent., and for 1 ft. 95,8 per cent.* Since for this efflux the fluid vein has the same transverse section as the orifice, we must then assume that this diminution of discharge is accompanied with a loss of velocity, which is caused by the friction or adhesion of the water to the inner circumference of the orifice, and by the viscidity of the water. In what follows, we shall call the ratio of the effective velocity of efflux to that of the theoretical $v = \sqrt{2gh}$, the *co-efficient of velocity*, and represent it by $\phi$. From this, therefore, the effective velocity of efflux in the most simple case is $v_1 = \phi\, v = \phi \sqrt{2gh}$, and the discharge:

FIG. 416.




$$Q = F v_1 = \phi F v = \phi F \sqrt{2gh}.$$

If we substitute for $\phi$ the mean value 0,97, we then obtain for the quantity in feet

* For experiments with larger orifices, *see* "Untersuchungen in dem Gebiete der Mechanick und Hydraulick," 2te. Abtheil.

$$Q = 0{,}97 \,.\, F\sqrt{2\,gh} = 0{,}97 \,.\, 8{,}03\, F\sqrt{h} = 7{,}789\, F\sqrt{h}.$$

A *vis viva* $\frac{Q\gamma}{g} \,.\, v_1^2$, is inherent in a discharge $Q$ issuing with the velocity $v_1$, by virtue of which it is capable of producing the mechanical effect $Q\gamma \,.\, \frac{v_1^2}{2g}$. But since by its descent from the height $h = \frac{v^2}{2g}$, the weight $Q\gamma$ produces the mechanical effect $Q\gamma \,.\, h = Q\gamma \frac{v^2}{2g}$, it follows that by the efflux of the water, this suffers a loss

$$Q\gamma\left(\frac{v^2}{2g} - \frac{v_1^2}{2g}\right) = Q\gamma \,.\, \frac{v^2}{2g}\,(1 - \phi^2) = (1 - 0{,}97^2)\; Q\gamma \,.\, \frac{v^2}{2g}$$

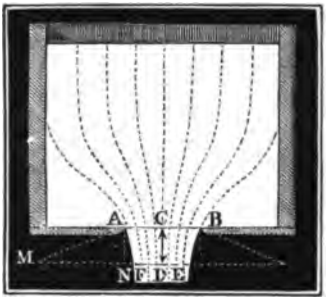
$$= 0{,}059\; Q\gamma \,.\, \frac{v^2}{2g}, \text{ or } 5{,}9 \text{ per cent.}$$

Therefore, the effluent water produces by its *vis viva* 5,9 per cent. less mechanical effect, than does its weight by falling from the height $h$.

§ 313. *Co-efficient of contraction.*—If water flows through an *orifice in a thin plate,* a considerable diminution of the discharge under otherwise similar circumstances takes place, whilst the particles of fluid rushing through the orifice move in convergent directions, and in this way give rise to a *contraction of the fluid vein.* The measurements of the vein made by many, and especially of late by the author, have shewn that the vein at a distance which is about equal to one half of the width of the orifice, has the greatest contraction, and a thickness equal to 0,8 that of the diameter of the orifice. If $F_1$ is the transverse section of the contracted vein, as also $F$ the transverse section of the orifice, we then have from this $F_1 = (0{,}8)^2\, F = 0{,}64\, F$. The ratio $\frac{F_1}{F}$ of these transverse sections is called the *co-efficient of contraction,* and is represented by $a$, and accordingly the mean value for the efflux of water through orifices in a thin plate may be put: $a = 0{,}64$.

As long as we possess no more accurate knowledge on the contraction of the fluid vein, we may assume that the stream flowing

FIG. 417.




through a circular orifice $AB$, Fig. 417, forms a body of rotation $ABEF$, whose envelope is generated by the revolution of a circular arc $AF$ about the axis $CD$ of the stream. Let the diameter $AB$ of the orifice $= d$, and the distance $CD$ of the contracted section $EF$, $= \frac{1}{2}\, d$, we then obtain the radius:
$MA = MF = r$ of the generating arc $AF$ by means of the equation:

$$\overline{AN}^2 = FN\,(2\,MF - FN), \text{ or}$$

$$\frac{d^2}{4} = \frac{d}{10}\left(2\,r - \frac{d}{10}\right),\ r = 1{,}3\ d.$$

Orifices made after this figure of the contracted vein give pretty nearly the velocity of discharge $v_1 = 0{,}97\ .\ v$.

The contraction of the fluid vein is caused by the water which lies directly above the orifice flowing out together with that which comes to it from the sides. There takes place, therefore, in the interior of the vessel a convergence of the filaments of water, similar to that represented in the figure, and the contraction of the fluid vein consists in a mere propagation of this convergence. We may convince ourselves of this motion of the water in the vicinity of the orifice by means of a glass apparatus of efflux; if we drop into the fluid minute substances which are either heavier or lighter than water, for example, such as oak saw-dust, bits of sealing wax, &c., and allow them to pass out with it from the orifice.

§ 314. *Contraction of the fluid vein.*—If water flows through triangular or quadrilateral orifices, and in a thin plate, the stream then assumes particular figures. The inversion of the jet, or the altered position of its transverse section with respect to that of the orifice, is very striking to the eye, in consequence of which a corner of this section comes to coincide with the middle of one side of the orifice.

Hence, from a triangular orifice $ABC$, Fig. 418, the section of the stream at a certain distance from the orifice forms a treble star-

like vein *DEF*, from a quadrilateral orifice *ABCD*, Fig. 419, a

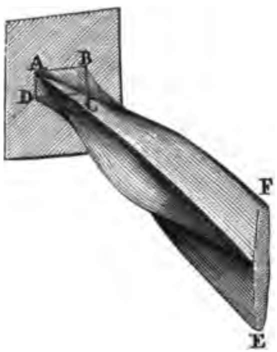
FIG. 418.


FIG. 419.


FIG. 420.


star of four veins *EFGH*, from a five sided orifice *ABCDE*, Fig. 420, a star *FGHKL*, consisting of five veins. These sections vary at different distances from the orifice: at a certain distance they diminish, and at a successive one again increase; hence the vein consists of plates or ribs of variable breadth, and thereby forms, when the efflux is observed under great pressure, bulges and nodes, similar to what is seen in the cactus. If the orifice *ABCD*, Fig. 421, is rectangular; at a lesser distance from the orifice, the section will then form a cross or star; and at a greater one, it will again assume the form of a rectangle *EF*.

FIG. 421.




Observations on various kinds of orifices have been made by Bidone, and accurate measurements of the vein from square apertures also by Poncelet and Lesbros.* The last measurements have led to a small co-efficient of contraction 0,563. The measurements of water issuing through lesser orifices, give us, however, greater co-efficients of contraction; they shew, moreover, that these are greater for elongated rectangles than for rectangles which approximate more to the square.

§ 315. *Co-efficient of efflux.*—If in the flow of water through orifices in thin plates, the effective velocity were equal to the theoretical $v = \sqrt{2gh}$, we should have the effective discharge:

$$Q = \alpha F \,.\, v = \alpha F \sqrt{2gh},$$

because $\alpha F$ represents the transverse section of the vein at the

* *See* Allgem. Maschinen-encyclopädie, article Ausfluss.

place of greatest contraction, where the particles of water move in parallel directions. But this is in no way the case: it is shewn rather by experience that $Q$ is smaller than $a\,F \cdot \sqrt{2\,gh}$, that we must therefore multiply the theoretical discharge $F\,\sqrt{2\,gh}$ by a co-efficient which is less than the co-efficient of contraction, in order to obtain the effective discharge. We must hence assume that for efflux from an orifice in a thin plate, a certain loss of velocity takes place, and therefore introduce a co-efficient of velocity $\phi$, and hence put the effective velocity of efflux $v_1 = \phi\,v = \varphi\,\sqrt{2\,gh}$. From this then we have the effective discharge: $Q_1 = F_1 \,.\, v_1 = a\,F \,.\, \phi\,v = a\,\phi\,F\,v = a\,\phi\,F\,\sqrt{2\,gh}$. Again, if we call the ratio of the effective discharge to the theoretical or hypothetical quantity, the *co-efficient of efflux*, and represent it in what follows by $\mu$, we then have:

$$Q_1 = \mu\,Q = \mu\,F\,v = \mu\,F\,\sqrt{2\,gh},$$

hence $\mu = a\,\phi$, i. e. ***the co-efficient of efflux is the product of the co-efficients of contraction and of velocity.***

Multiplied observations, but chiefly the measurements of the author, have led to this, that the co-efficient of efflux for orifices in thin plates is not constant; that for small orifices and for small velocities, it is greater than for large orifices and for great velocities; and that it is considerably greater for elongated and small orifices than for orifices which have a regular form, or which approximate to the circle.

For square orifices of from 1 to 9 square inches area, with from 7 to 21 feet head of water, according to the experiments of Bossut and Michelotti, the mean co-efficient of efflux is $\mu = 0{,}610$; for circular ones of from $\frac{1}{2}$ to 6 inches diameter, with from 4 to 21 feet head of water, $\mu = 0{,}615$, or about $\frac{8}{13}$. The single values observed by Bossut and Michelotti vary considerably from one another, but we cannot discover in them any dependance between the dimensions of the orifice and the magnitude of the head of water. From the author's experiments at a pressure of 24 inches, the co-efficient for an orifice of

| | | | | | | |
|---|---|---|---|---|---|---|
| .393 | inches or | 1 | centimetre | diameter | is $\mu$ | $= 0{,}628$ |
| .787 | „ | 2 | „ | „ | | $= 0{,}621$ |
| 1.181 | „ | 3 | „ | „ | | $= 0{,}614$ |
| 1.574 | „ | 4 | „ | „ | | $= 0{,}607.$ |

On the other hand, at a pressure of 10 inches for the round orifice of

| | | | |
|---|---|---|---|
| 1 centimetre | diameter | $\mu$ | = 0,637 |
| 2 | " | " | = 0,629 |
| 3 | " | " | = 0,622 |
| 4 | " | " | = 0,614. |

From these it is manifest that the co-efficient of efflux increases when the dimensions of the orifice and the head of water decrease.

If for $\mu$ we take the mean value = 0,615, and for $a$ = 0,64, we obtain the co-efficient of velocity for the efflux through orifices in a thin plate, $\phi = \frac{\mu}{a} = 0,96$, therefore, nearly as great as for efflux through rounded or conoidal orifices.

*Remark* 1. Buff's experiments (See Poggendorf's Ann. Band 46), show that the co-efficient of efflux for small orifices and for small heads of water or velocities is considerably greater than for large or mean orifices and velocities. An orifice of 2,084 lines diameter, gave for $1\frac{1}{3}$ inch pressure, $\mu$ = 0,692, for 35 inches $\mu$ = 0,644; on the other hand, an orifice of 4,848 lines for $4\frac{1}{3}$ inches pressure $\mu$ = 0,682, and for 29 inches $\mu$ = 0,653.

*Remark* 2. According to the author's experiments, the co-efficients for efflux under water are about $1\frac{1}{4}$ per cent less than for efflux in air.

§ 316. *Rectangular lateral orifices.*—The most accurate experiments on efflux through large rectangular lateral apertures are those made at Metz by Poncelet and Lesbros. The widths of these orifices were two decimeters, (nearly 8 inches); the depths, however, varied from one centimetre to two decimetres. In order to produce perfect contraction, a brass plate of four millimetres, =.136 inches, thickness was used for these orifices. From the results of their experiments, these experimenters have calculated by interpolation the tables at the end of this paragraph for the co-efficients which may be used for the measurement or calculation of the discharge.

If $b$ be the breadth of the orifice, and if $h_1$ and $h_2$ are the heads of water above the lowest, and above the uppermost horizontal edge of the orifice, we then have, from § 308, the discharge: $Q = \frac{2}{3}\, b\, \sqrt{2g}\,(h_1^{\frac{3}{2}} - h_2^{\frac{3}{2}})$. But if we substitute the height of the aperture $a$, and the mean head of water $h = \frac{h_1 + h_2}{2}$, we then have approximately $Q = \left(1 - \frac{a^2}{96\, h^2}\right) ab\, \sqrt{2gh}$, and hence the effective discharge $Q_1 = \mu\, Q = \left(1 - \frac{a^2}{96\, h^2}\right) \mu\, ab\, \sqrt{2gh}$. If, further, we put $\left(1 - \frac{a^2}{96\, h^2}\right) \mu = \mu_1$, we have then simply $Q_1 = \mu_1\, ab\, \sqrt{2gh}$, and

in order to allow of our calculating by this simple or general formula of efflux, not only the values of $\mu$, but also those of $\mu_1$ are given in the following tables.

Since the water in the vicinity of the orifice is in motion, it stands lower directly before the aperture, than at a greater distance from the plate in which the aperture is made; on this account two tables have been compiled, the one for heads of water measured at a greater distance from the orifice, and the other for those measured immediately at the side in which the orifice lies. It may be seen, moreover, from both tables, although with certain variations, that the co-efficients of efflux increase, the ~~lower~~ the orifice is and the less the head of water.

If the orifices have different breadths, there is nothing left, so long as we have no further experiments, but to use the co-efficients of these tables in like manner for the calculation of the discharge. If, further, the orifices are not rectangular, we must determine their mean breadth and mean depth, and introduce into the calculation the co-efficients corresponding to these dimensions. Lastly, it is always preferable to measure the head of water at a certain distance from the side in which the orifice lies, and to use the first table, than directly at the orifice where the surface of water is curved and less tranquil, than a little above it.

*Examples.*—1. What quantity of water flows through a rectangular aperture, 2 decimetres broad and 1 decimetre deep, if the surface of water is $1\frac{1}{2}$ metre above the upper edge? Here $b = 0,2$; $a = 0,1$, $h = \frac{h_1 + h_2}{2} = \frac{1,6 + 1,5}{2} = 1,55$; hence the theoretical discharge $Q = 0,1 \,.\, 0,2 \sqrt{2g} \,.\, \sqrt{1,55} = 0,02 \,.\, 4,429 \,.\, 1,245 = 0,1103$ cubic metre. But now Table I. gives for $a = 0,1$ and $h_2 = 1,5$, $\mu_1 = 0,611$, hence the effective discharge $Q_1 = 0,611 \,.\, 0,1103 = 0,0674$ cubic metre.—2. What discharge corresponds to a rectangular orifice in a thin plate of 8 inches breadth, 2 inches depth, with a 15 inches head of water above the upper edge? The theoretical discharge is $Q = \frac{2}{3} \,.\, \frac{1}{6} \,.\, 7,906 \sqrt{\frac{4}{3}} = 0,8784 \,.\, 1,1547 = 1,014$ cubic feet. But now 2 inches is about 0,05 metre, and 15 inches about 0,4 metre; hence, according to the table $a = 0,05$ and $h_2 = 0,4$, the corresponding co-efficient $\mu_1 = 0,628$ is to be taken, and the quantity of water sought is $Q_1 = 0,628 \,.\, 1,014 = 0,637$ cubic feet.—3. If the breadth $= 0,25$, the depth $= 0,15$, and the head of water $h_2 = 0,045$ metre, then is $Q = 0,25 \,.\, 0,15 \,.\, 4,429 \,.\, \sqrt{0,12} = 0,166 \,.\, 0,3464 = 0,0575$ cubic metre. To the height 0,15 corresponds for $h_2 = 0,04$, the mean value: $\mu_1 = \frac{0,582 + 0,603}{2} = 0,5925$, and $h_2 = 0,05$, $\mu_1 = \frac{0,585 + 0,605}{2} = 0,595$; but since $h_2$ is given $= 0,045$, we must then substitute the new mean $\frac{0,5925 + 0,595}{2} = 0,594$ for the co-efficient of efflux, and we therefore obtain the discharge sought: $Q_1 = 0,594 \,.\, 0,0575 = 0,03415$ cubic metre.

## TABLE I.

The co-efficients for the efflux through rectangular orifices in a thin vertical plate, from Poncelet and Lesbros. The heads of water are measured at a certain distance back from the orifice, or at a point where the water may be considered as still.

| Head of water, or distance of the surface of water from the upper side of the orifice in metres. | HEIGHT OF ORIFICE. | | | | | |
|---|---|---|---|---|---|---|
| | 0,20ᵐ or 8 inches. | 0,10ᵐ or 4 inches. | 0,05ᵐ or 2 inches. | 0,03ᵐ or 1.13 in. | 0,02ᵐ or .8 in. | 0,01ᵐ or .4 in. |
| 0,000 | ” | ” | ” | ” | ” | ” |
| 0,005 | ” | ” | ” | ” | ” | 0,705 |
| 0,010 | ” | ” | 0,607 | 0,630 | 0,660 | 0,701 |
| 0,015 | ” | 0,593 | 0,612 | 0,632 | 0,660 | 0,697 |
| 0,020 | 0,572 | 0,596 | 0,615 | 0,634 | 0,659 | 0,694 |
| 0,030 | 0,578 | 0,600 | 0,620 | 0,638 | 0,659 | 0,688 |
| 0,040 | 0,582 | 0,603 | 0,623 | 0,640 | 0,658 | 0,683 |
| 0,050 | 0,585 | 0,605 | 0,625 | 0,640 | 0,658 | 0,679 |
| 0,060 | 0,587 | 0,607 | 0,627 | 0,640 | 0,657 | 0,676 |
| 0,070 | 0,588 | 0,609 | 0,628 | 0,639 | 0,656 | 0,673 |
| 0,080 | 0,589 | 0,610 | 0,629 | 0,638 | 0,656 | 0,670 |
| 0,090 | 0,591 | 0,610 | 0,629 | 0,637 | 0,655 | 0,668 |
| 0,100 | 0,592 | 0,611 | 0,630 | 0,637 | 0,654 | 0,666 |
| 0,120 | 0,593 | 0,612 | 0,630 | 0,636 | 0,653 | 0,663 |
| 0,140 | 0,595 | 0,613 | 0,630 | 0,635 | 0,651 | 0,660 |
| 0,160 | 0,596 | 0,614 | 0,631 | 0,634 | 0,650 | 0,658 |
| 0,180 | 0,597 | 0,615 | 0,630 | 0,634 | 0,649 | 0,657 |
| 0,200 | 0,598 | 0,615 | 0,630 | 0,633 | 0,648 | 0,655 |
| 0,250 | 0,599 | 0,616 | 0,630 | 0,632 | 0,646 | 0,653 |
| 0,300 | 0,600 | 0,616 | 0,629 | 0,632 | 0,644 | 0,650 |
| 0,400 | 0,602 | 0,617 | 0,628 | 0,631 | 0,642 | 0,647 |
| 0,500 | 0,603 | 0,617 | 0,628 | 0,630 | 0,640 | 0,644 |
| 0,600 | 0,604 | 0,617 | 0,627 | 0,630 | 0,638 | 0,642 |
| 0,700 | 0,604 | 0,616 | 0,627 | 0,629 | 0,637 | 0,640 |
| 0,800 | 0,605 | 0,616 | 0,627 | 0,629 | 0,636 | 0,637 |
| 0,900 | 0,605 | 0,615 | 0,626 | 0,628 | 0,634 | 0,635 |
| 1,000 | 0,605 | 0,615 | 0,626 | 0,628 | 0,633 | 0,632 |
| 1,100 | 0,604 | 0,614 | 0,625 | 0,627 | 0,631 | 0,629 |
| 1,200 | 0,604 | 0,614 | 0,624 | 0,626 | 0,628 | 0,626 |
| 1,300 | 0,603 | 0,613 | 0,622 | 0,624 | 0,625 | 0,622 |
| 1,400 | 0,603 | 0,612 | 0,621 | 0,622 | 0,622 | 0,618 |
| 1,500 | 0,602 | 0,611 | 0,620 | 0,620 | 0,619 | 0,615 |
| 1,600 | 0,602 | 0,611 | 0,618 | 0,618 | 0,617 | 0,613 |
| 1,700 | 0,602 | 0,610 | 0,617 | 0,616 | 0,615 | 0,612 |
| 1,800 | 0,601 | 0,609 | 0,615 | 0,615 | 0,614 | 0,612 |
| 1,900 | 0,601 | 0,608 | 0,614 | 0,613 | 0,612 | 0,611 |
| 2,000 | 0,601 | 0,607 | 0,613 | 0,612 | 0,612 | 0,611 |
| 3,000 | 0,601 | 0,603 | 0,606 | 0,608 | 0,610 | 0,609 |

## TABLE II.

Co-efficients of efflux through rectangular orifices in a vertical plate, from Poncelet and Lesbros. The heads of water are measured directly at the orifice.

| Head of water, or distance of the surface of water from the upper side of the orifice in metres. | HEIGHT OF ORIFICE. | | | | | |
|---|---|---|---|---|---|---|
| | 0,20m or 8 inches. | 0,10m or 4 inches. | 0,05m or 2 inches. | 0,03m or 1.13 in. | 0,02m or .8 in. | 0,01m or .4 in. |
| 0,000 | 0,619 | 0,667 | 0,713 | 0,766 | 0,783 | 0,795 |
| 0,005 | 0,597 | 0,630 | 0,668 | 0,725 | 0,750 | 0,778 |
| 0,010 | 0,595 | 0.618 | 0,642 | 0,687 | 0,720 | 0,762 |
| 0,015 | 0,594 | 0,615 | 0,639 | 0,674 | 0,707 | 0,745 |
| 0,020 | 0,594 | 0,614 | 0,638 | 0,668 | 0,697 | 0,729 |
| 0,030 | 0,593 | 0,613 | 0,637 | 0,659 | 0,685 | 0,708 |
| 0,040 | 0,593 | 0,612 | 0,636 | 0,654 | 0,678 | 0,695 |
| 0,050 | 0,593 | 0,612 | 0,636 | 0,651 | 0,672 | 0,686 |
| 0,060 | 0,594 | 0,613 | 0,635 | 0,647 | 0,668 | 0,681 |
| 0,070 | 0,594 | 0,613 | 0,635 | 0,645 | 0,665 | 0,677 |
| 0,080 | 0,594 | 0,613 | 0,635 | 0,643 | 0,662 | 0,675 |
| 0,090 | 0,595 | 0,614 | 0,634 | 0,641 | 0,659 | 0,672 |
| 0,100 | 0,595 | 0,614 | 0,634 | 0,640 | 0,657 | 0,669 |
| 0,120 | 0,596 | 0,614 | 0,633 | 0,637 | 0,655 | 0,665 |
| 0,140 | 0,597 | 0,614 | 0,632 | 0,636 | 0,653 | 0,661 |
| 0,160 | 0,597 | 0,615 | 0,631 | 0,635 | 0,651 | 0,659 |
| 0,180 | 0,598 | 0,615 | 0,631 | 0,634 | 0,650 | 0,657 |
| 0,200 | 0,599 | 0,615 | 0,630 | 0,633 | 0,649 | 0,656 |
| 0,250 | 0,600 | 0,616 | 0,630 | 0,632 | 0,646 | 0,653 |
| 0,300 | 0,691 | 0,616 | 0,629 | 0,632 | 0,644 | 0,651 |
| 0,400 | 0,602 | 0,617 | 0,629 | 0,631 | 0,642 | 0,647 |
| 0,500 | 0,603 | 0,617 | 0,628 | 0,630 | 0,640 | 0,645 |
| 0,600 | 0,604 | 0,617 | 0,627 | 0,630 | 0,638 | 0,643 |
| 0,700 | 0,604 | 0,616 | 0,627 | 0,629 | 0,637 | 0,640 |
| 0,800 | 0,605 | 0,616 | 0,627 | 0,629 | 0,636 | 0,637 |
| 0,900 | 0,605 | 0,615 | 0,626 | 0,628 | 0,634 | 0,635 |
| 1,000 | 0,605 | 0,615 | 0,626 | 0,628 | 0,633 | 0,632 |
| 1,100 | 0,604 | 0,614 | 0,625 | 0,627 | 0,631 | 0,629 |
| 1,200 | 0,604 | 0,614 | 0,624 | 0,626 | 0,628 | 0,626 |
| 1,300 | 0,603 | 0,613 | 0,622 | 0,624 | 0,625 | 0,622 |
| 1,400 | 0,603 | 0,612 | 0,621 | 0,622 | 0,622 | 0,618 |
| 1,500 | 0,602 | 0,611 | 0,620 | 0,620 | 0,619 | 0,615 |
| 1,600 | 0,602 | 0,611 | 0,618 | 0,618 | 0,617 | 0,613 |
| 1,700 | 0,602 | 0,610 | 0,617 | 0,616 | 0,615 | 0.612 |
| 1,800 | 0,601 | 0,609 | 0,615 | 0,615 | 0,614 | 0,612 |
| 1,900 | 0.601 | 0,608 | 0,614 | 0,613 | 0,613 | 0,611 |
| 2,000 | 0.601 | 0.607 | 0,614 | 0,612 | 0,612 | 0,611 |
| 3,000 | 0,601 | 0,603 | 0,606 | 0,608 | 0,610 | 0,609 |

§ 817. *Wiers.* — If water flows through *wiers*, or through notches in a thin plate, as for example, $FB$, Fig. 422, the fluid vein then suffers a contraction on three sides, by which a diminution of the discharge is effected, since the quantity discharged from these orifices is $Q_1 = \frac{2}{3} \mu b h \sqrt{2gh}$. But here the head of water $EH = h$, or the head of water above the sill, of the wier must not be measured immediately at the sill, but at least two feet before the plate in which the orifice lies, because the fluid surface before the opening suffers a depression, which becomes greater and greater the nearer it is to the orifice, and in the plane of the orifice amounts to a quantity $GR$ of from 0,1 to 0,25 the head of water $FR$, so that the thickness $FG$ of the stream in this plane is only 0,9 to 0,75 of the head of water. Experiments instituted by many philosophers on the flow of water through notches in thin plates, have afforded a multiplicity of results, but not always of the desired accordance. The following short table contains the results of the experiments of Poncelet and Lesbros on wiers of two decimetres, or about 8 inches breadth.

FIG. 422.

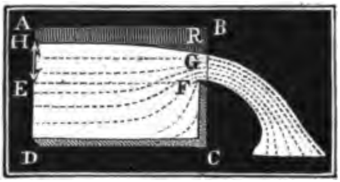



TABLE OF THE CO-EFFICIENTS OF EFFLUX FOR WIERS OF 2 DECIMETRES, = 7.87 INCHES BREADTH, ACCORDING TO PONCELET AND LESBROS.

| Head of water $h$. | metrs. 0,01 or 4 in. | metrs. 0,02 8 in. | metrs. 0,03 1.2 in. | metrs. 0,04 1.6 in. | metrs. 0,06 2.4 in. | metrs. 0,08 3.2 in. | metrs. 0,10 4 in. | metrs. 0,15 6 in. | metrs. 0,20 8 in. | metrs. 0,22 9 in. |
|---|---|---|---|---|---|---|---|---|---|---|
| Co-efficient of efflux $\mu_1 = \frac{2}{3}\mu$. | 0,424 | 0,417 | 0,412 | 0,407 | 0,401 | 0,397 | 0,395 | 0,393 | 0,390 | 0,385 |

From the average of determinations, we may here put $\mu_1 = 0,4$. Experiments on wiers of greater breadth gave Eytelwein the mean $\mu_1 = \frac{2}{3}$, $\mu = 0,42$, and Bidone $\mu_1 = \frac{2}{3} \cdot 0,62 = 0,41$, &c. The most extensive experiments are those of d'Aubuisson and Castel. From these, d'Aubuisson asserts that for wiers whose breadth is no more than the third part of the breadth of the canal or side in which the wier lies, the mean of $\mu$ is $= 0,60$, therefore we may put $\frac{2}{3} \mu = 0,40$; but, on the other hand, for wiers which extend over the whole side, or have the same breadth as the water-course :

**$\mu=0,665$, therefore $\mu_1=0,444$; lastly, for other relations between the breadth of the wier and that of the canal, the co-efficients**

[illegible] That the upper part of the [illegible] a basin a rectangular opening [illegible] with a horizontal base, the water [illegible] which we suppose kept constantly [illegible] flowed in the form of a sheet over [illegible] base or sill.

[illegible] we give the name of wier; [illegible] also extend the name to dams which [illegible] close [illegible] bed of a stream of water [illegible] manner that the water [illegible] is obliged to rise up [illegible] top or crown.

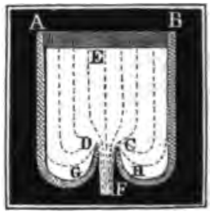
[illegible]

**convergence of the side which embraces the orifice, the lateral flow is entirely prevented, and a maximum if the side has a direction**

opposite to that of the fluid stream, so that certain particles of water must revolve 180° before arriving at the orifice. Both cases are represented in Figs. 425 and 426. In the first case, the co-

FIG. 425


FIG. 426.




efficient of efflux is about 1, viz. 0,96 to 0,97; and in the second, from the measurements of Borda, Bidone and the author, a mean of 0,53. Changes in the co-efficients of efflux through convergent sides very often present themselves in practice; they occur in dams which are inclined to the horizon, as in Fig. 427. Poncelet found for a similar opening the co-efficient of efflux $\mu = 0{,}80$, when the board was inclined 45°, and on the other hand, $\mu = 0{,}74$ only for an inclination of $63\frac{1}{2}$°, that is, for a slope of $\frac{1}{2}$. For similar wiers, Fig. 428, where, as in the Poncelet sluice-board, contraction

FIG. 427.

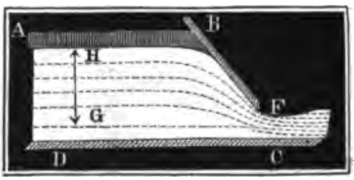
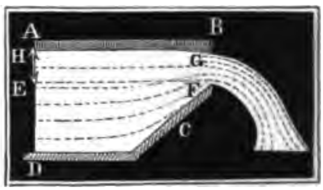



FIG. 428.




takes place at one side only, the author found $\mu = 0{,}70$, therefore, $\mu_1 = \frac{2}{3}\ \mu = 0{,}467$ for an inclination of 45°, and $\mu = 0{,}67$, therefore, $\mu_1 = 0{,}447$ for an inclination of $62\frac{1}{2}$°.

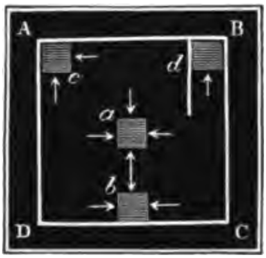
*Example.* If a sluice-board, inclined at an angle of 50°, which goes across a channel $2\frac{1}{4}$ feet broad, is drawn up $\frac{1}{2}$ foot high, and the surface of water stands 4 feet above the bottom of the channel, the height of the aperture may be put $a = \frac{1}{2}\ sin.\ 50° = 0{,}3830$ feet, the head of water $h = 4 - \frac{1}{2} \cdot 0{,}3830 = 3{,}8085$ feet, and the co-efficient of efflux $\mu = 0{,}78$; hence, the discharge $Q_1 = 0{,}78 \cdot 2{,}25 \cdot 0{,}3830 \cdot 7{,}906\ \sqrt{3{,}8085}$ $= 10{,}36$ cubic feet.

§ 319. *Partial contraction.*—We have only hitherto considered

those cases where the water flows from all sides towards the aperture, and forms a contracted vein around, and we must now investigate others, where the water flows from one or more sides to the aperture, and therefore produces a stream only partially contracted. To distinguish the circumstances of contraction, we will call the case, where the vein is contracted on all sides, *general;* and the case, where it is only contracted in one part of its circumference, *partial,* or *imperfect contraction.* Partial contraction is induced when an orifice in a plane thin plate is confined by other plates in the direction of the fluid stream at one or more sides.

In Fig. 429, are represented four orifices of equal size *a, b, c, d,* in the bottom *AC* of a vessel. The contraction by efflux through the orifice *a* in the middle of the bottom is general, because the water can flow to it from all sides; the contraction from the efflux through *b, c, d,* is partial, because the water can only flow to them from one, two, or three sides. Likewise, if a rectangular lateral aperture goes to the bottom of the vessel, the contraction is then partial, because it falls away at the bottom side, if further the aperture of the dam reaches the bottom in the lateral walls of the channel, there is then only a contraction on one side.

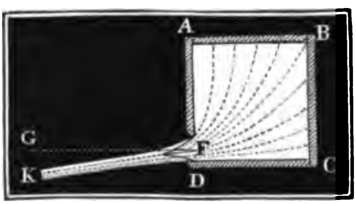
FIG. 429.




Partial contraction is remarkable in two respects; first, by giving an oblique direction to the stream; and secondly, by increasing the quantity of discharge.

If the lateral aperture *F,* Fig. 430, reaches a second side *CD,* so that no contraction takes place there, the axis *FK* of the fluid stream becomes deflected by an angle *KFG* of about 9° from the normal *FG* to the plane of the orifice. The obliquity of the stream is much greater if two adjacent sides of the orifice have projecting borders. If the orifice has borders in two oppositely situated sides, and contraction at these prevented, such a deviation

FIG. 430.



of course will not take place, but at the other side, the vein at some distance from the orifice, will spread out more than if the border were not there. Although a greater discharge is obtained by a partial contraction, we must, as a rule, endeavour to avoid this, because the fluid stream, in consequence, suffers a deviation in its direction and a greater extension in its breadth.

Experiments on the efflux of water with partial contraction have been made by Bidone and by the author. They allow us to assume that the co-efficients of efflux increase simultaneously with the ratio of the contracted part to the whole perimeter, though it is easy to perceive that this relation is different, if the perimeter is almost or entirely restricted, and the contraction almost or entirely suppressed. Let us put the ratio of this restriction to the entire perimeter $= n$, and let us represent by $\kappa$, any number deduced from experiment, we may then, although only approximately, put the ratio of the corresponding co-efficient of efflux $\mu_n$ of partial contraction to the co-efficient of efflux of perfect contraction:

$$\frac{\mu_n}{\mu_0} = 1 + \kappa n, \text{ and consequently } \mu_n = (1 + \kappa\, n)\, \mu_0.$$

Bidone's experiments give for circular orifices $\kappa = 0{,}128$, and for rectangular $\kappa = 0{,}152$; the author's, however, give for the last, $\kappa = 0{,}134$. Rectangular orifices with borders, are those which are most frequently met with in practice; we will assume for them the mean value $\kappa = 0{,}143$, and hence put $\mu_n = (1 + 0{,}143 \,.\, n)\, \mu_0$. For a rectangular lateral orifice of the depth $a$ and breadth $b$, $n = \frac{b}{2\,(a+b)}$, if the contraction on one side $b$ is suppressed; if, for instance, this side lies in the plane of the bottom; again, $n = \frac{1}{2}$, if a side $a$ and a side $b$ are bordered, and $n = \frac{2\,a + b}{2\,(a+b)}$, if on one side $b$, and both sides $a$, the contraction is prevented; if for example, the orifice takes up the whole breadth of the reservoir, and reaches the plane of the bottom.

*Example.* What quantity of water does a flow deliver through a 3 feet broad and 10 inch deep vertical aperture of a dam at a pressure of $1\frac{1}{2}$ feet above the upper side of the aperture, if the lower one coincides with the bottom of the channel, and hence there is no contraction at the bottom? The theoretical discharge is:

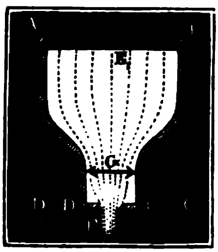
$$Q = \tfrac{10}{12} \,.\, 3 \,.\, 8{,}03 \sqrt{1{,}5 + \tfrac{5}{12}} = \tfrac{5}{2} \,.\, 8{,}03 \sqrt{1{,}9166\ldots} = 27{,}70 \text{ cubic feet.}$$

According to Poncelet's table for general contraction $\mu = 0{,}604$, we have, therefore, $n = \frac{3}{2\,(3 + \frac{10}{12})} = \frac{9}{18 + 5} = \frac{9}{23}$; hence, for the above case of partial contraction

$\mu_n \left(= 1 + 0{,}143 \, . \, \frac{2}{5}\right) \, . \, 0{,}604 = 1{,}056 \, . \, 0{,}604 = 0{,}638$, and the effective discharge is $Q_1 = 0{,}638 \, Q = 0{,}638 \, . \, 27{,}70 = 17{,}67$ cubic feet.

§ 320. *Imperfect contraction.*—The contraction of the fluid vein depends further upon whether the water before the orifice is tolerably still, or whether it arrives before it with a certain velocity. The quicker the water flows to the orifice, the less contracted does the vein become, and the greater is the discharge. The relations of contraction and efflux above given and investigated, have reference only to the case where the orifice lies in a large side, and it can only be assumed that the water flows to it with a small velocity; hence we must know the relations of contraction and efflux, when the transverse section of the orifice is not much less than that of the affluent water, and when, consequently, the water arrives at the orifice with a considerable velocity. In order to distinguish these two cases from one another, we shall call the contraction, where the superincumbent water is still, *perfect*; and that where it is in motion, *imperfect contraction.* The contraction, for example, is imperfect in the efflux from a vessel $AC$, Fig. 431, because the transverse section $F$ of the orifice is not much smaller than that $G$ of the arriving water, or the area of the side $CD$, in which this orifice lies. If, on the other hand, the vessel had the form $ABC_1D_1$, and, therefore, the area of the bottom surface $C_1D_1$ much greater than the transverse section $F$ of the orifice, the efflux would then go on with perfect contraction. The imperfectly contracted vein is besides distinguishable, not merely by its greater thickness from the perfectly contracted fluid vein, but also by its not having so transparent and crystalline an appearance.

FIG. 431.



If the ratio of the area of the orifice $F$, and the side containing the orifice $G$, therefore, $\frac{F}{G} = n$, the co-efficient of efflux for perfect contraction $= \mu_0$, and that for imperfect $= \mu_n$, we may with greater accuracy, according to the experiments and calculations made by the author, put:

1. For circular orifices:

$$\mu_n = \mu_0 \left[1 + 0{,}04564 \left(14{,}821^n - 1\right)\right], \text{ and}$$

2. For rectangular orifices:

$$\mu^{n} = \mu_0 \, [1 + 0{,}0760 \, (9^{n} - 1)].*$$

To render the calculation easier in cases of application, the corrections $\frac{\mu_n - \mu_0}{\mu_0}$ of the co-efficient of efflux on account of imperfect contraction are compiled in the following short tables.

TABLE I.

CORRECTIONS OF THE CO-EFFICIENTS OF EFFLUX FOR CIRCULAR ORIFICES.

| $n$ | 0,05 | 0,10 | 0,15 | 0,20 | 0,25 | 0,30 | 0,35 | 0,40 | 0,45 | 0,50 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{\mu_n - \mu_0}{\mu_0}$ | 0,007 | 0,014 | 0,023 | 0,034 | 0,045 | 0,059 | 0,075 | 0,092 | 0,112 | 0,134 |
| $n$ | 0,55 | 0,60 | 0,65 | 0,70 | 0,75 | 0,80 | 0,85 | 0,90 | 0,95 | 1,00 |
| $\frac{\mu_n - \mu_0}{\mu_0}$ | 0,161 | 0,189 | 0,223 | 0,260 | 0,303 | 0,351 | 0,408 | 0,471 | 0,546 | 0,613 |

TABLE II.

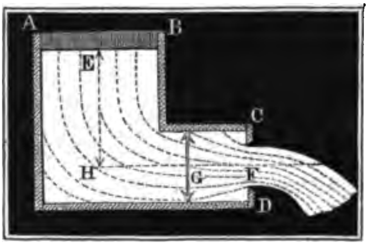
CORRECTIONS OF THE CO-EFFICIENTS OF EFFLUX FOR RECTANGULAR ORIFICES.

| $n$ | 0,05 | 0,10 | 0,15 | 0,20 | 0,25 | 0,30 | 0,35 | 0,40 | 0,45 | 0,50 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{\mu_n - \mu_0}{\mu_0}$ | 0,009 | 0,019 | 0,030 | 0,042 | 0,056 | 0,071 | 0,088 | 0,107 | 0,128 | 0,152 |
| $n$ | 0,55 | 0,60 | 0,65 | 0,70 | 0,75 | 0,80 | 0,85 | 0,90 | 0,95 | 1,00 |
| $\frac{\mu_n - \mu_0}{\mu_0}$ | 0,178 | 0,208 | 0,241 | 0,278 | 0,319 | 0,365 | 0,416 | 0,473 | 0,537 | 0,608 |

* Versuche über die unvollkommene Contraction des Wassers, u. s. w., Leipzig, 1843.

The different values of the ratio of the transverse sections $n = \frac{F}{G}$ stand above in these tables, and immediately below additions to the co-efficients of efflux, on account of imperfect contraction; for example, for the ratio of the transverse sections $n = 0,35$, *i. e.* for the case where the area of an orifice is 35 hundreds of the area of the whole side of the orifice, we have for circular orifices $\frac{\mu_n - \mu_0}{\mu_0} = 0,075$, and for rectangular orifices $= 0,088$; therefore, the co-efficient of efflux for perfect contraction in the first case is to be made about 75 thousandths, and in the second about 88 thousandths greater to obtain the corresponding co-efficients of efflux for imperfect contraction. Were the co-efficient of efflux $\mu_0 = 0,615$, we should have in the first case $\mu_{0\ 35} = 1,075 \cdot 0,615 = 0,661$, and in the second, $\mu_{0\ 35} = 1,088 \cdot 0,615 = 0,669$.

FIG. 432.




*Example.*—What discharge does a rectangular lateral aperture $F$, $1\frac{1}{4}$ feet broad and $\frac{1}{2}$ foot deep, give if it be cut in a rectangular wall $CD$, Fig. 432, 2 feet broad and 1 foot deep, and the head of water $EH = h$ in still water amounts to 2 feet? The theoretical discharge is $Q = 1,25 \cdot 0,5 \cdot 8,03\sqrt{2} = 5,018 \cdot 1,414 = 7,095$ cubic feet, and the co-efficient of efflux for perfect contraction is, according to Poncelet, $\mu_0 = 0,610$; but now the ratio of the transverse sections $n = \frac{F}{G} = \frac{1,25 \cdot 0,5}{2 \cdot 1} = 0,312$, and for $n = 0,312$, from Table II.,

$\frac{\mu_n - \mu_0}{\mu_0} = 0,071 + \frac{12}{50}(0,088 - 0,071) = 0,071 + 0,004 = 0,075$; hence it follows, that the co-efficient of efflux for the present case is $\mu_{0,312} = 1,075 \cdot \mu_0 = 1,075 \cdot 0,610 = 0,6557$, and the discharge $Q_1 = 0,6557 \cdot Q = 0,6557 \cdot$ ~~6,987~~ 7 $= 4,581$ cubic feet.

§ 321. *Efflux of water in motion.*—We have hitherto assumed that the head of water has been measured in still water; we must now, therefore, consider the case when only the head of water in motion, and flowing with a certain velocity towards the orifice, can be measured. Let us suppose the case of a rectangular lateral orifice, and represent its breadth by $b$, and the heads of water with respect to both horizontal edges $h_1$ and $h_2$, the height due to

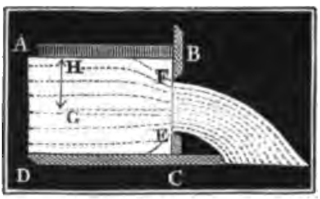
the velocity $c$ of the affluent water by $k$, we shall then have the theoretical discharge:

$$Q = \frac{2}{3}\, b \sqrt{2g}\, [(h_1+k)^{\frac{3}{2}} - (h_2+k)^{\frac{3}{2}}].$$

This formula is not directly applicable to the determination of the discharge, because the height due to the velocity:

$k = \frac{c^2}{2g} = \frac{1}{2g}\left(\frac{Q}{G}\right)^2$ is again dependant on $Q$, and further transformation leads to a complicated equation of a higher order, hence it is far simpler to put the effective discharge $Q_1 = \mu_1\, a\, b \sqrt{2gh}$, and understand by $\mu_1$ not the mere co-efficient of efflux, but one especially dependant on the ratios of the transverse sections. Most frequently, this case presents itself when the object is to measure water flowing in canals and courses, because it is seldom possible in this case to dam up the water so high by a transverse section $BC$, Fig. 433, containing the orifice of discharge, that the orifice $EF$ becomes only a small part, compared with the transverse section of the stream of water flowing to it; and, hence, the velocity of the last very small compared with the mean velocity.

FIG. 433.




From experiments made by the author on this subject with Poncelet orifices, where the head of water is measured one metre above the plane of the orifice, the expression: $\frac{\mu_n - \mu_0}{\mu_0} = 0{,}641 \left(\frac{F}{G}\right)^2 = 0{,}641 \,.\, n^2$, may be taken as tolerably accurate, when $n = \frac{F}{G}$ is the ratio of the transverse section, which, however, should not much exceed $\frac{1}{2}$; further, $\mu_0$ represents the co-efficient for general contraction, taken from Poncelet's table corresponding to the present case. If $b$ be the breadth, $a$ the depth of the orifice, $B$ the breadth and $A$ the depth of the fluid stream, and $h$ the depth of the upper side of the orifice below the surface of water, we have accordingly the effective discharge:

$$Q_1 = \left[1 + 0{,}641 \left(\frac{ab}{AB}\right)^2\right] \mu_0 \,.\, ab \sqrt{2g\left(h + \frac{a}{2}\right)}.$$

The following table serves for shortening the calculation in cases of application.

| $n$ | 0,05 | 0,10 | 0,15 | 0,20 | 0,25 | 0,30 | 0,35 | 0,40 | 0,45 | 0,50 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{\mu_n - \mu_0}{\mu_0}$ | 0,002 | 0,006 | 0,014 | 0,026 | 0,040 | 0,058 | 0,079 | 0,103 | 0,130 | 0,160 |

*Example.* To find the quantity of water conducted through a course 3 feet broad, a board is placed across, with a 2 feet wide and 1 foot deep rectangular orifice, and the water in this way is so dammed up, that it at last attains a height of 2¼ feet above the bottom, and 1¼ above the lower edge of the orifice. The theoretical discharge is $Q = ab\sqrt{2gh} = 1 \cdot 2 \cdot 8{,}03\sqrt{1{,}25} = 16{,}06 \cdot 1{,}118 = 17{,}95$ cubic feet; the co-efficient of efflux for perfect contraction may be put ~~0,680~~ 0,602, and the ratio of the transverse sections $n = \frac{F}{G} = \frac{ab}{AB} = \frac{1 \cdot 2}{2{,}25 \cdot 3} = 0{,}296$; hence it follows, that the co-efficient of efflux for the present ratio of discharge:
$= (1 + 0{,}641 \cdot 0{,}296^2)\,\mu_0 = 1{,}056 \cdot 0{,}602 = 0{,}6357$, and the effective quantity discharged $= 17{,}95 \cdot 0{,}6357 = 11{,}31$ cubic feet.

§ 322. Imperfect contraction very often occurs in the efflux through wiers, as in Fig. 422. Wiers may take up a part only of the breadth of the reservoir or canal, or the whole breadth. In the latter case, contraction at the sides of the aperture does not take place, and for this reason more water flows through them than through wiers of the first kind. The author has made experiments also on these circumstances of efflux, and deduced from the results formulas by which the corresponding co-efficients may be estimated with tolerable certainty with the assistance of the ratio of the sections $n = \frac{G}{V} = \frac{hb}{AB}$. If we retain the denominations of the former paragraph, we then have for the Poncelet wiers:

$$\frac{\mu_n - \mu_0}{\mu_0} = 1{,}718\left(\frac{F}{G}\right)^4 = 1{,}718 \cdot n^4,$$

and for wiers occupying the entire breadth of the canal:

$$\frac{\mu_n - \mu_0}{\mu_0} = 0{,}041 + 0{,}3693\, n^2,$$

hence, in the first case the discharge is:

$$Q_1 = \tfrac{2}{3}\left[1 + 1{,}718\left(\frac{hb}{AB}\right)^4\right]\mu_0 \cdot b\sqrt{2gh^3}.$$

And in the second:

$$Q_1 = \tfrac{2}{3}\left[1{,}041 + 0{,}3693\left(\frac{h}{A}\right)^2\right]\mu_0 \,.\, b\,\sqrt{2\,gh^3},$$

where $h$ represents the head of water $EH$ above its sill $F$, measured at about 3 feet 6 inches back from the wier.

In the following tables the corrections $\frac{\mu_n - \mu_0}{\mu_0}$, for the most simple values of $n$ are put down.

TABLE I.

CORRECTIONS FOR THE PONCELET WIERS.

| $n$ | 0,05 | 0,10 | 0,15 | 0,20 | 0,25 | 0,30 | 0,35 | 0,40 | 0,45 | 0,50 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{\mu_n - \mu_0}{\mu_0}$ | 0,000 | 0,000 | 0,001 | 0,003 | 0,007 | 0,014 | 0,026 | 0,044 | 0,070 | 0,107 |

TABLE II.

CORRECTIONS FOR WIERS OVER THE ENTIRE SIDE, OR WITHOUT ANY LATERAL CONTRACTION.

| $n$ | 0,00 | 0,05 | 0,10 | 0,15 | 0,20 | 0,25 | 0,30 | 0,35 | 0,40 | 0,45 | 0,50 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{\mu_n - \mu_0}{\mu_0}$ | 0,041 | 0,042 | 0,045 | 0,049 | 0,056 | 0,064 | 0,074 | 0,086 | 0,100 | 0,116 | 0,133 |

*Example.* To determine the quantity of water carried off by a canal 5 feet broad, a waste board is applied, with an outward sloping edge, over which the water is allowed to flow after it has ceased to rise; the head of water above the bottom of the canal is $3\frac{1}{2}$ feet, and above the edge $1\frac{1}{2}$ feet, hence the theoretical discharge is $Q = \frac{2}{3} \,.\, 5 \,.\, 8{,}03 \,.\, \left(\frac{3}{2}\right)^{\frac{3}{2}} = 48{,}18$ cubic feet. The co-efficient of efflux is, since $\frac{h}{A} = \frac{1{,}5}{3{,}5} = \frac{3}{7}$ and $\mu_0 = 0{,}577$,
$\mu_{\frac{3}{7}} = [1{,}041 + 0{,}3693 \,.\, (\frac{3}{7})^2] \,.\, 0{,}577 = 1{,}110 \,.\, 0{,}577 = 0{,}64$, hence the effective discharge $Q_1 = 0{,}64 \,.\, Q = 0{,}64 \,.\, 48{,}18 = 30{,}83$ cubic feet.

# CHAPTER III.

## ON THE EFFLUX OF WATER THROUGH TUBES.

§ 323. *Short tubes, or mouth-pieces.*—If water is allowed to flow through *short tubes*, or *mouth-pieces*, other relations take place than when it flows through orifices in a thin plate, or through outwardly sloping orifices in a thick plate. When the tube is prismatic, and its length $2\frac{1}{2}$ to 3 times that of its width, it then gives an uncontracted and opaque stream, which has a small distance of projection, and hence, also, a smaller velocity than that of a jet flowing, under otherwise similar circumstances, through an orifice in a thin plate. If, therefore, the tube *KL* has the same transverse section as the orifice *F*, Fig. 434; and if also the head of water of both is one and the same, we then obtain in *LR* a troubled and uncontracted, and, therefore, a thicker jet, and in *FH* a clear and contracted, and, therefore, thinner one; and, it may be observed, that the distance of the projection *ER*, is less than that of *DH*. This ratio of efflux only takes place

FIG. 434.

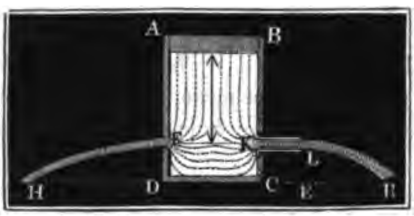
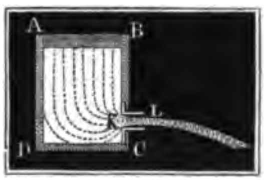



FIG. 435.

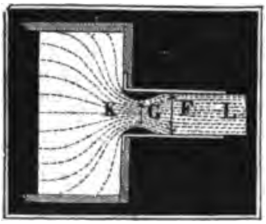



when the tube is of a given length; if it is shorter, or scarcely as long as it is broad, then the jet *KL*, Fig. 435, will not touch the sides of the tube, the tube will have no influence on the efflux, and the jet will be the same as through orifices in a thin plate.

Sometimes in tubes of greater length, the fluid stream does not entirely fill the tube, namely: when the water is not allowed to come into contact with the sides of the tube; but if in this case we close the outer orifice by the hand or by a board for a few moments, a stream will then be formed which will entirely fill the tube, and the so-called *full flow* will then take place. Con-

traction of the fluid vein takes place also in the flow through tubes, but the place of contraction is here in the interior of the tube. We may be convinced of this, if we avail ourselves of glass tubes, such as *KL*, Fig. 436, and colour the water, for in this case we shall remark, that there is progressive motion only in the middle of the transverse section *G* close behind the place of entrance *K*, but not at the outside of it, and that it is a sort of eddying motion which takes place. But it is the capillarity, or the adhesion of the water to the sides of the tube, which causes the fluid entirely to fill the end *FL* of the tube. The water flowing from the tube has only a pressure equal to that of the atmosphere, but the contracted section *G* is only $a$ times the size of the section *F* of the tube, and for this reason the velocity in it $\frac{1}{a}$ times as great as the velocity of efflux $v$; hence the pressure of the water in the vicinity of *G* is

FIG. 436.




$$\frac{\left(\frac{1}{a}v\right)^2}{2g} - \frac{v^2}{2g} = \left[\left(\frac{1}{a}\right)^2 - 1\right]\frac{v^2}{2g}$$ (§ 307) less than at its exit,

or than the atmospheric pressure. If we bore a narrow hole in the tube at *G*, no discharge will pass through it, but there will be an absorption of air rather; the full discharge and the action of the tube will at last entirely cease if the hole be made wider, or more holes bored.

§ 323. *Cylindrical tubes.*—Numerous experiments have been made on the flow of water through cylindrical additional tubes; but the results vary considerably from one another. The coefficients of Bossut are those, which from their smallness (0,785) have been found to vary most from others. From the experiments of Michelotti, with tubes from ½ to 3 inches width, and with a head of water of from 3 to 20 feet, the mean of this co-efficient is: $\mu = 0{,}813$. The experiments of Bidone, Eytelwein and d'Aubuisson vary very little from this. The mean, however, which may be adopted, and which corresponds particularly with the author's experiments on the discharge through short mouth-pieces $= 0{,}815$. As we have found this for

orifices in a thin plate 0,615, it follows that, under otherwise similar circumstances and relations, $\frac{815}{615} = 1{,}325$ times as much water flows through cylindrical additional tubes, as through round orifices in a thin plate. These co-efficients, moreover, increase as the width of tubes becomes less, and but slightly with the increase of the head of water or velocity of efflux. According to the author's experiments under a pressure of from 9 to 24 inches for tubes three times as long as broad :

| at | 1 or .4in. | 2 or. 8in. | 3 or 1.2 | 4 centimetres width. or 1.6 inches width. |
|---|---|---|---|---|
| $\mu =$ | 0,843 | 0,832 | 0,821 | 0,810 |

According to this table, therefore, the co-efficients increase considerably as the width of the tubes decreases. Buff found for tubes 2,79 lines wide, and 4,3 lines long, the co-efficients of efflux gradually to increase from 0,825 to 0,855, when the head of water sank from 33 to 1½ inches successively.

The author found a co-efficient of efflux of 0,819 for the flow of water through rectangular additional tubes.

If the additional tubes $KL$, Fig. 437, are on the inside partially confined ; if, for instance, one side is contiguous to the bottom, and if a partial contraction is produced thereby, then the co-efficient of efflux, from the author's experiments, does not perceptibly increase, but the water flows away at different parts of the section, with

FIG. 437.

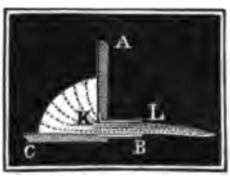
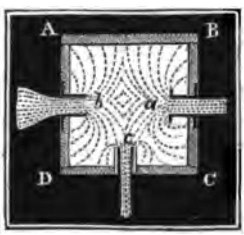



FIG. 438.




different velocities, and of course from the side $BC$ faster than from the side opposite to it. If the inner anterior surface of a tube does not coincide with the side surface, but projects, as $a$, $b$, $c$, Fig. 438, then this tube is called an *internal additional tube*. If the anterior surface of this tube is at least ¼th as broad as the tube is wide, as

for example $a$, then the co-efficient of efflux will remain the same as if this surface were in the plane of the side, but if the anterior surface be less, as $b$, $c$, the co-efficient will then be less. For a very small and almost vanishing anterior surface, according to the experiments of Bidone and the author, this amounts to 0,71 if the vein fills the tube, and 0,53 (compare § 318) if it does not quite fill the inner sides of the tube. In the first case ($b$) the fluid stream is broken and divergent, like a brush; and in the second ($c$) strongly contracted and quite crystalline.

§ 324. *Co-efficient of resistance.*—As water flows without contraction from prismatic additional tubes, it follows that, in its efflux through these-mouth pieces, the co-efficient of contraction = unity, and the co-efficient of velocity $\phi$ = the co-efficient of efflux $\mu$. A discharge $Q$ with the velocity $v$, possesses a *vis viva* $\frac{Q\gamma}{g} v^2$, and is capable of producing the mechanical effect $\frac{v^2}{2g} Q\gamma$ (§ 71). But now the theoretical velocity of efflux $= \frac{v}{\phi}$, hence the mechanical effect $\frac{1}{\phi^2} \cdot \frac{v^2}{2g} \cdot Q\gamma$ corresponds to the mass of water flowing out, and the discharge $Q$ accordingly loses by efflux the mechanical effect

$$\left(\frac{1}{\phi^2} \cdot \frac{v^2}{2g} - \frac{v^2}{2g}\right)Q\gamma = \left(\frac{1}{\phi^2} - 1\right)\frac{v^2}{2g} Q\gamma.$$

For efflux through orifices in a thin plate, the mean of $\phi = 0{,}97$, hence the loss of effect here amounts to

$$\left[\left(\frac{1}{0{,}97}\right)^2 - 1\right]\frac{v^2}{2g} Q\gamma = 0{,}063 \frac{v^2}{2g} Q\gamma;$$

for efflux through short cylindrical tubes $\phi = 0{,}815$, and the corresponding loss of effect

$$= \left[\left(\frac{1}{0{,}815}\right)^2 - 1\right]\frac{v^2}{2g} Q\gamma = 0{,}505 \frac{v^2}{2g} Q\gamma,$$

*i. e.* eight times as great as for efflux through orifices in a thin plate. In rendering available the *vis viva* of flowing water, it is consequently better to let the fluid flow through orifices in a thin plate, than through prismatic tubes. But if the inner edges in which the tube meets the sides of the cistern are rounded, and by this a gradual passage from the cisterns into the tube effected, the co-efficient of efflux will then rise to 0,96, and the loss of mechanical effect will be brought down to 8½ per cent. In shorter

adjutages, accurately rounded, having the form of the contracted fluid vein $\mu = \phi = 0{,}97$, and hence the loss of mechanical effect as for orifices in a thin plate = 6 per cent.

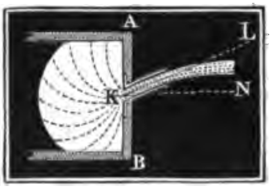
A head of water $\left(\frac{1}{\phi^2} - 1\right)\frac{v^2}{2g}$ $Q\gamma$ is due to the loss of mechanical effect $\left(\frac{1}{\phi^2} - 1\right)\frac{v^2}{2g}$ $Q\gamma$; hence we may also suppose that from the obstacles to the efflux, the head of water suffers the loss $\left(\frac{1}{\phi^2} - 1\right)\frac{v^2}{2g}$, and assume after deduction of this loss, that the residuary part of the head of water is expended in generating the velocity. We may call this loss $\left(\frac{1}{\phi^2} - 1\right)\frac{v^2}{2g}$, which increases with the square of the velocity of efflux, the *height due to the resistance*, and the co-efficient $\frac{1}{\phi^2} - 1$, with which the height due to the velocity is to be multiplied to obtain the height due to the resistance, the *co-efficient of resistance*. We shall represent in what follows, the co-efficient, expressing the ratio of the height of resistance to the head of water, by the letter ; therefore, the height due to the resistance itself may be expressed by $\zeta \cdot \frac{v^2}{2g}$. By the formula $\zeta = \frac{1}{\phi^2} - 1$ and $\phi = \frac{1}{\sqrt{1+\zeta}}$, the co-efficient of resistance may be calculated from the co-efficient of velocity, and *vice versâ*.

*Examples.*—1. What discharge will flow through a 2 inch wide tube, under a head of water of 3 feet, which corresponds to a co-efficient of resistance $\zeta = 0{,}4$? $\phi = \frac{1}{\sqrt{1{,}4}} = 0{,}845$; hence, $v = 0{,}845 \,.\, 8{,}03 \sqrt{3} = 12{,}06$ feet; further, $F = \left(\frac{1}{12}\right)^2 \pi = 0{,}02182$ square feet; consequently, the quantity of water sought is $Q = 0{,}263$ cubic feet.—2. If a tube of 2 inches width, under a pressure of 2 feet, deliver in a minute 10 cubic feet of water, its co-efficient of efflux, or of velocity, is then $\phi = \frac{Q}{F\sqrt{2gh}} = \frac{10}{60 \,.\, 0{,}02182 \,.\, 8{,}03 \,.\, \sqrt{2}} = \frac{1}{1{,}051\sqrt{2}} = 0{,}673$, the co-efficient of resistance $= \left(\frac{1}{0{,}673}\right)^2 - 1 = 1{,}23$; and lastly, the loss in head of water produced by the resistances of the tube:

$$= 1{,}23\,\frac{v^2}{2g} = 1{,}23 \,.\, 0{,}0155\left(\frac{Q}{F}\right)^2 = 0{,}0190 \,.\, \frac{1}{0 \,.\, (120)^2} = 1{,}319 \text{ feet.}$$

§ 325. *Oblique additional tubes.*—Obliquely attached or obliquely cut tubes give a smaller quantity of water than rectangularly attached, or rectangularly cut additional tubes, because the direction of the water in them becomes changed. Experiments conducted upon an extensive scale, have led the author to the following.

FIG. 439.



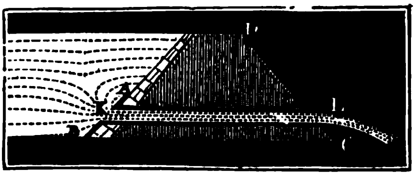

If $\delta$ be the angle which the axis of the tube $KL$, Fig. 439, makes with the normal $KN$ to the plane $AB$ of the inner orifice of the tube; and if $\zeta$ be the co-efficient of resistance for rectangularly cut tubes, we shall then have the co-efficient of resistance for the inclined tube: $\zeta_1 = \zeta + 0{,}303 \; sin. \; \delta + 0{,}226 \; sin. \; \delta^3$. Let us take for $\zeta$ the mean value 0,505, and we shall obtain:

| for $\delta^0$ = | 0 | 10 | 20 | 30 | 40 | 50 | 60° |
|---|---|---|---|---|---|---|---|
| The co-efficient of resistance $\zeta_1$ = | 0,505 | 0,565 | 0,635 | 0,713 | 0,794 | 0,870 | 0,937 |
| The co-efficient of efflux $\mu_1$ = | 0,815 | 0,799 | 0,782 | 0,764 | 0,747 | 0,731 | 0,719 |

From this, for example, the co-efficient of resistance of an additional tube by $20^0$ deviation from the axis is $\zeta = 0{,}635$, and the co-efficient of efflux $= \dfrac{1}{\sqrt{1{,}635}} = 0{,}782$, and for $35^0$ deviation, the first $= 0{,}753$, and the last $= 0{,}755$.

In general, these inclined and additional tubes are larger than we have hitherto assumed, and they should be longer too, because the water would not otherwise perfectly fill the tube. The preceding formula represents only that part of the resistance which is due to that portion of tube at the inner orifice, which is three times as long as the tube is wide. The resistance which the remaining portion of tube opposes to the motion of the water will be given subsequently.

FIG. 440.



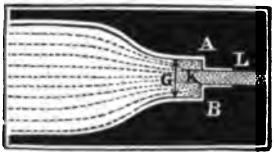

*Example.* If the plane of the inner orifice $AB$ of a horizontally lying pond sluice $KL$, Fig. 440, as likewise the interior surface of the pond dam, is inclined 40° to the horizon, then the axis of the pipe makes, with the normal to this plane, an angle of 50°, and hence the co-efficient of resistance for

efflux through the portion of the interior orifice of this tube is $= 0{,}870$; and if now the co-efficient of resistance 0,650 were due to the remaining and longer portion, the co-efficient of resistance of the entire tube would then be $= 0{,}870 + 0{,}650 = 1{,}520$, and hence the co-efficient of efflux $= \frac{1}{\sqrt{1 + 1{,}520}} = \frac{1}{\sqrt{2{,}520}} = 0{,}630$. For a 10 feet head of water and 1 foot width of tube, the following discharge would be given:

$$Q = 0{,}63 \cdot \frac{\pi}{4} \cdot 8{,}03\,\sqrt{10} = 12{,}46 \text{ cubic feet.}$$

§ 326. *Imperfect contraction.*—When a cylindrical additional tube $KL$, Fig. 441, is inserted into a plane wall $AB$, whose area $G$ does not much exceed the transverse section $F$ of the tube, the water then comes to the place of insertion with a velocity which must not be disregarded, and it then issues into the tube with imperfect contraction only, on which account the velocity of efflux is again greater than when the water before entrance into the tube is to be assumed as still.

FIG. 441.




Again if $\frac{F}{G} = n$ is the ratio of the section of the tube to that of the area of the side, and further, $\mu_0$ be the co-efficient of efflux for perfect contraction, where $\frac{F}{G}$ may be equated to 0, we shall have, according to the experiments of the author, to put the co-efficient of efflux for imperfect contraction, or for the ratio of the sections $n$:

$$\frac{\mu_n - \mu_0}{\mu_0} = 0{,}102\,n + 0{,}067\,n^2 + 0{,}046\,n^3, \text{ or}$$
$$\mu_n = \mu_0\,(1 + 0{,}102\,n + 0{,}067\,n^2 + 0{,}046\,n^3).$$

If the transverse section of the tube occupies the sixth part of the whole surface of the side, there is:

$$\mu_{\frac{1}{6}} = \mu_0\,(1 + 0{,}102 \cdot \tfrac{1}{6} + 0{,}067 \cdot \tfrac{1}{36} + 0{,}046 \cdot \tfrac{1}{216})$$
$$\mu =_0 (1 + 0{,}017 + 0{,}0019 + 0{,}0002) + 1{,}019\,\mu_0, \text{ or}$$
$$\mu_0 \text{ being put} = 0{,}815,\ \mu_{\frac{1}{6}} = 0{,}815 \cdot 1{,}019 = 0{,}830.$$

The following useful and convenient table gives somewhat more accurately the values for correction $\frac{\mu_n - \mu_0}{\mu_0}$

## TABLES

### OF CORRECTION FOR IMPERFECT CONTRACTION, BY EFFLUX THROUGH SHORT CYLINDRICAL TUBES.

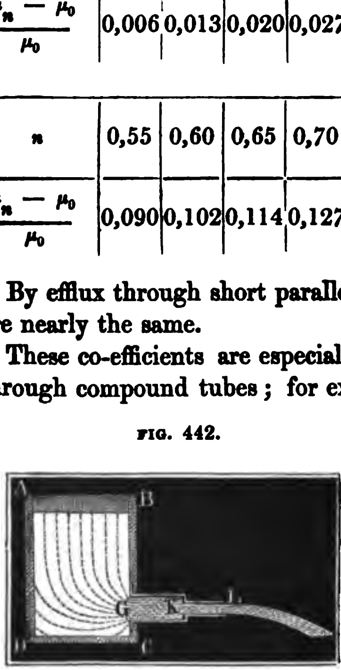
| $n$ | 0,05 | 0,10 | 0,15 | 0,20 | 0,25 | 0,30 | 0,35 | 0,40 | 0,45 | 0,50 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{\mu_n - \mu_0}{\mu_0}$ | 0,006 | 0,013 | 0,020 | 0,027 | 0,035 | 0,043 | 0,052 | 0,060 | 0,070 | 0,080 |

| $n$ | 0,55 | 0,60 | 0,65 | 0,70 | 0,75 | 0,80 | 0,85 | 0,90 | 0,95 | 1,00 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{\mu_n - \mu_0}{\mu_0}$ | 0,090 | 0,102 | 0,114 | 0,127 | 0,138 | 0,152 | 0,166 | 0,181 | 0,198 | 0,227 |

By efflux through short parallelepipedical tubes these corrections are nearly the same.

These co-efficients are especially applicable to the efflux of water through compound tubes; for example, in the case represented in Fig. 442, where the orifice of the short tube $KL$ enters the wider short tube $CK$, whose orifice again lies in the cistern $AC$. Imperfect contraction takes place at the entrance of the water from the wider into the narrower tube, and hence the co-efficient of efflux must be determined by the last rule. If we put the co-efficient of resistance corresponding to the co-efficient of efflux $= \zeta_1$, the co-efficient of resistance for the entrance into the wider tube from the cistern $= \zeta$, the head of water $= h$, the velocity of efflux $= v$, and the ratio $\frac{F}{G}$ of the section of the tubes $= n$, therefore, the velocity of the water in the wider tube $= nv$, then the formula gives:

FIG. 442.




$$h = \frac{v^2}{2g} + \zeta \cdot \frac{(nv)^2}{2g} + \zeta_1 \cdot \frac{v^2}{2g}, \text{ i. e. } h = (1 + n^2\zeta + \zeta_1)\frac{v^2}{2g},$$

And hence:

$$v = \frac{\sqrt{2gh}}{\sqrt{1 + n^2 \zeta + \zeta_1}}.$$

*Example.* What discharge will the apparatus delineated in Fig. 442 deliver, if the head of water $h = 4$ feet, the width of the narrower tube 2 inches, and that of the wider one 3 inches? $n = (\frac{2}{3})^2 = \frac{4}{9}$, hence $\mu_{\frac{4}{9}} = 1{,}069 \,.\, 0{,}815 = 0{,}871$, and the corresponding co-efficient of resistance $\zeta_1 = \left(\frac{1}{0{,}871}\right)^2 - 1 = 0{,}318$; but now we have $\zeta = 0{,}505$ and $n^2 \,.\, \zeta = \frac{16}{81} \,.\, 0{,}505 = 0{,}099$; hence it follows, that $1 + n^2\zeta + \zeta_1 = 1 + 0{,}099 + 0{,}318 = 1{,}417$, and the velocity of efflux $v = \frac{8{,}03 \,.\, \sqrt{4}}{\sqrt{1{,}417}} = \frac{16{,}06}{\sqrt{1{,}417}} = 14{,}36$ feet. Again, since the transverse section of the tube $= 0{,}02182$ square feet, the discharge will be $Q = 14{,}36 \,.\, 0{,}02182 = 0{,}313$ cubic feet.

§ 327. *Conical tubes.*—Additional conical tubes give a discharge different from that of prismatic or cylindrical tubes, they are either conically convergent, or conically divergent; in the first case the outer orifice is smaller, and in the second case larger than the inner orifice. The co-efficients of efflux for the first tubes are greater, and for the last, less than for cylindrical tubes. One and the same conical tube gives more water when the wider orifice is made the exit orifice, as $K$ in Fig. 443, than when it is turned inwards, as $L$ in the same figure, except that it does not give a greater quantity in proportion as the wider orifice exceeds the narrower. If many, as Venturi and Eytelwein, give for conically

FIG. 443.

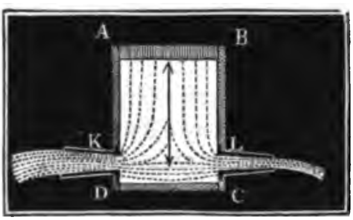



FIG. 444.


divergent tubes, a greater co-efficient of efflux, than for conically convergent tubes, it must always be borne in mind, that they take the narrower transverse section for the orifice. The following experiments instituted at pressures of from 0,25 to 3,3 metres, with a tube $AD$ 9 centimetres long, Fig. 444, bring before us the effect of conicalness in tubes. The width of these tubes at one extremity amounted to $DE = 2{,}468$, at the other $AB = 3{,}228$

centimetres, and the angle of convergence *i. e.* the angle $AOB$, which the oppositely situated sides $AE$ and $BD$ of a section in the direction of the longer axis include $= 4^0, 50'$. By efflux through the narrower orifice, the co-efficient was $= 0{,}920$, but by efflux through the wider it was $= 0{,}553$, and if in the calculation, we take the narrower entrance orifice for the transverse section, it will give $= 0{,}946$. In the first case, when the tube was applied as a conically convergent adjutage, the fluid vein was little contracted, thick and smooth; but in the second case, when the tube served as a conically divergent adjutage, it was strongly divergent, broken, and spouting. Venturi and Eytelwein have experimented further on efflux through conically divergent tubes. Both philosophers have applied these conical tubes to cylindrical and conoidal adjutages, made after the form of the contracted fluid vein. By such a connection as is represented in Fig. 445, where

FIG. 445.

the divergent portion $KL$ of the outer orifice is between 12 and $21\frac{1}{2}$ lines wide, and $8\frac{1}{1}\frac{3}{8}$ of an inch long, and the angle of convergence estimated at $5^0, 9'$. Eytelwein found $\mu = 1{,}5526$ when he took the narrow end for the orifice, and on the other hand, $\mu = 0{,}483$ for the wider end, in which he was right. Through this combined adjutage there certainly flows $\frac{1{,}5526}{0{,}615} = 2{,}5$ times as much as through a simple orifice in a thin plate, and $\frac{1{,}5526}{0{,}815} = 1{,}9$ times as much as through a short cylindrical tube. With small velocities and greater divergence, it is scarcely possible, even by previously closing the tubes, to bring about a full flow.

§ 328. The most ample experiments have been made by d'Aubuisson and Castel on efflux through conically convergent additional tubes. The tubes for this purpose were of great variety, of different lengths, widths, and angles of convergence. The most extensive experiments were those made with tubes of 1,55 centimetres width at the discharging orifice, and of from 2,6 times greater, *i. e.* of 4 centimetres in length, for which reason we will here communicate the results in the following table. The head of water was, throughout, 3 metres. The discharges were measured by a special gauge-cistern; but in order to obtain besides the co-efficients of efflux, those of the velocity and contraction, the amplitude of the jet, due to given heights, were

measured, and from these the velocity of efflux (*see* § 38, *Ex*. 2) calculated. The ratio $\frac{v}{\sqrt{2gh}}$ of the effective velocity $v$ to the theoretical $\sqrt{2gh}$ gave the co-efficient of velocity $\phi$, as also the ratio $\frac{Q}{F\sqrt{2gh}}$ of the effective discharge $Q$ to the theoretical $F\sqrt{2gh}$ gave the co-efficient of efflux $\mu$, and lastly, the ratio of both co-efficients, *i. e.* $\frac{\mu}{\phi}$, determined the co-efficient of contraction $\alpha$.

TABLE

OF THE CO-EFFICIENTS OF EFFLUX AND VELOCITY FOR EFFLUX THROUGH CONICALLY CONVERGENT TUBES.

| Angle of convergence | Co-efficient of efflux. | Co-efficient of velocity. | Angle of convergence | Co-efficient of efflux. | Co-efficient of velocity. |
|---|---|---|---|---|---|
| 0° 0′ | 0,829 | 0,829 | 13° 24′ | 0,946 | 0,963 |
| 1° 36′ | 0,866 | 0,867 | 14° 28′ | 0,941 | 0,966 |
| 3° 10′ | 0,895 | 0,894 | 16° 36′ | 0,938 | 0,971 |
| 4° 10′ | 0,912 | 0,910 | 19° 28′ | 0,924 | 0,970 |
| 5° 26′ | 0,924 | 0,919 | 21° 0′ | 0,919 | 0,972 |
| 7° 52′ | 0,930 | 0,932 | 23° 0′ | 0,914 | 0,974 |
| 8° 58′ | 0,934 | 0,942 | 29° 58′ | 0,895 | 0,975 |
| 10° 20′ | 0,938 | 0,951 | 40° 20′ | 0,870 | 0,980 |
| 12° 4′ | 0,942 | 0,955 | 48° 50′ | 0,847 | 0,984 |

From this table it is seen that the co-efficients of efflux attain their maximum 0,946 for a tube of 13½° lateral convergence; that, on the other hand, the co-efficients of velocity come out always greater and greater, the greater is the angle of convergence. The following example will show how this table may be used in those cases which present themselves in practice.

*Example.*—What discharge will a short conoidal tube of 1½ inches width at the outer orifice, and of 10° convergence, deliver under a pressure of 16 feet? According to the experiments of the author, a cylindrical tube of this width gives $\mu = 0{,}810$; d'Aubuisson's tube, however, gave $\mu = 0{,}829$, therefore about $0{,}829 - 0{,}810 = 0{,}019$ more. Now from the table for a tube of 10° convergence, $\mu$ is $= 0{,}937$; hence, for the given tube we may put $\mu = 0{,}937 - 0{,}019 = 0{,}918$, whence the discharge $Q = 0{,}918 \cdot \frac{\pi}{4 \cdot 8^2} \cdot 8{,}03\sqrt{4} = \frac{8{,}03\,\pi\,0{,}918}{64} = 0{,}3619$ cubic feet.

§ 329. *Resistance of friction.*—Long prismatic or cylindrical

tubes, the longer they are the more they retard the efflux; hence we must assume that the sides of the tubes offer resistance to the motion of the water by the friction, adhesion, or viscidity of the fluid against them. From reason and from numerous observations and measurements, we may assume that this resistance of friction is independent of the pressure, that it increases directly as the length $l$, and inversely as the width $d$, and therefore proportional to the ratio $\frac{l}{d}$. Moreover, it appears that this resistance is greater for great and lesser for less velocities, and that it very nearly increases with the square of the velocity $v$ of the water. If we measure this resistance by the height of a column of water, which must be deducted from the entire head $h$, in order to obtain the height requisite for the generation of the velocity, we may then put for this height, which we shall term *the height due to the resistance*, $h_1 = \zeta_1 \cdot \frac{l}{d} \cdot \frac{v^2}{2g}$, where $\zeta_1$ represents a number, from coefficient of experiment, which we may call the ~~height due to~~ *the resistance of friction*. There is a greater loss, therefore, of pressure or head of water from the friction of the water in the tube, the greater the ratio $\frac{l}{d}$ of the length to the width is, and the greater the height due to the velocity $\frac{v^2}{2g}$. From the discharge $Q$ and the transverse section of the tube $F = \frac{\pi d^2}{4}$ there follows the velocity $v = \frac{4Q}{\pi d^2}$, and hence the height due to the friction:

$$h_1 = \zeta_1 \cdot \frac{l}{d} \cdot \frac{1}{2g}\left(\frac{4Q}{\pi d^2}\right)^2 = \zeta_1 \cdot \frac{1}{2g} \cdot \left(\frac{4}{\pi}\right)^2 \cdot \frac{lQ^2}{d^5}.$$

In order to obtain the least possible loss of head of water, or fall, in leading a certain quantity of water $Q$, the pipe must be made as wide as possible, and not unnecessarily long. A double width requires, for instance, only $(\frac{1}{2})^5 = \frac{1}{32}$ of the fall that the single width does.

If the transverse section of a tube be rectangular, and of the depth $a$ and the breadth $b$, we must substitute for

$$\frac{1}{d} = \frac{1}{4} \cdot \frac{\pi d}{\frac{1}{4}\pi d^2} = \frac{1}{4} \cdot \frac{\text{circumference}}{\text{area}} = \frac{1}{4} \cdot \frac{2(a+b)}{ab} = \frac{a+b}{2ab},$$

whence it follows: $h_1 = \zeta \cdot \frac{l(a+b)}{2ab} \cdot \frac{v^2}{2g}$.

By means of this formula for the resistance of friction in pipes

we may also find the velocity and the quantity of efflux which a pipe, of given length and width, will conduct under a given pressure.

FIG. 446.


It is quite the same, whether the tube $KL$, Fig. 446, is horizontal, inclines, or ascends, if only by the head of water is understood the depth $RL$ of the middle point $L$ of the orifice of the tube below the surface of water $HO$ of the efflux reservoir. If $h$ is the head of water, $h_1$ the height due to the resistance for the orifice of entrance, and $h_2$ that for the remaining portion of the tube, we then have: $h - (h_1 + h_2) = \frac{v^2}{2g}$, or $h = h_1 + h_2 + \frac{v^2}{2g}$. If $\zeta$ represents the co-efficient of resistance for the portion of tube next the cistern, and $\zeta_1$ the co-efficient of the resistance of friction of the rest of the tube, we then have

$$h = \zeta \cdot \frac{v^2}{2g} + \zeta_1 \cdot \frac{l}{d} \cdot \frac{v^2}{2g} + \frac{v^2}{2g},$$

$$\text{or, 1.) } h = \left(1 + \zeta + \zeta_1 \frac{l}{d}\right) \frac{v^2}{2g},$$

$$\text{and 2.) } v = \frac{\sqrt{2gh}}{\sqrt{1 + \zeta + \zeta_1 \cdot \frac{l}{d}}}.$$

From the last formula the discharge $Q = Fv$ is given.

For very long tubes $1 + \zeta$ is small compared with $\zeta_1 \frac{l}{d}$, whence, simply, $h = \zeta_1 \frac{l}{d} \cdot \frac{v^2}{2g}$, and, inversely,

$$v = \sqrt{\frac{1}{\zeta_1} \cdot \frac{d}{l} \cdot 2gh}.$$

§. 330. The co-efficient of friction, like the co-efficient of efflux, is not quite constant; it is greater for small, and less for great velocities; *i. e.* the resistance of the friction of water in tubes does not increase exactly with the square of the velocity, but with some other power of it. Prony and Eytelwein have assumed, that the head of water lost by the resistance due to friction ought to increase as the simple velocity and as its square, and have given for it the expression $h_1 = (\alpha v + \beta v^2) \frac{l}{d}$, where $\alpha$ and $\beta$ are co-efficients deduced from experiment. To determine these co-effi-

cients, 51 experiments, which, at various times, were made by Couplet, Bossut and Du Buat, on the motion of water through long tubes, were made use of by these hydraulicians.

Prony found from his, that $h_1 = (0{,}0000693\, v + 0{,}0013932\, v^2)\, \frac{l}{d}$; Eytelwein $h_1 = (0{,}0000894\, v + 0{,}0011213\, v^2)\, \frac{l}{d}$; d'Aubuisson assumes $h_1 = (0{,}0000753\, v + 0{,}001370\, v^2)\, \frac{l}{d}$ metres.

A formula, discovered by the author, agrees more accurately with observation. It has the form

$$h_1 = \left(a + \frac{\beta}{\sqrt{v}}\right) \frac{l}{d}\, \frac{v^2}{2g},$$

and is based on the hypothesis, that the resistance of friction increases simultaneously as the square, and as the square root of the cube of the velocity. From this we have the co-efficient of resistance $\zeta_1 = a + \frac{\beta}{\sqrt{v}}$, and the height due to the resistance of friction $h_1 = \zeta_1 \cdot \frac{l}{d}\, \frac{v^2}{2g}$.

For the measurement of the co-efficient of resistance $\zeta_1$, or of the auxiliary constants $a$ and $\beta$, not only the determinations of Prony and Eytelwein from the 51 experiments of Couplet, Bossut, and Du Buat were used by the author, but also 11 experiments made by him, and 1 experiment by Gueymard in Grenoble. The older experiments extend only to velocities of from 0,043 to 1,930 metres; in the experiments of the author, however, the extreme limit of velocity reached to 4,648 metres. The widths of the tubes, in the older experiments, were 27 mm. = 1.06 in.; 36 mm. = 1.95 in.; 54 mm. = 2.12 in.; 135 mm. = 5.31 in.; and 490 mm. = 19.29 in.: later experiments were conducted with tubes of 33 mm. = 1.29 in.; 71 mm. = 2.79 in.; and 275 mm. = 5.31 in. By means of the method of least squares, it has been found from the 63 experiments laid down:

$$\zeta_1 = 0{,}01439 + \frac{0{,}0094711}{\sqrt{v}};\ \text{therefore,}$$

$$h_1 = \left(0{,}01439 + \frac{0{,}0094711}{\sqrt{v}}\right) \frac{l}{d} \cdot \frac{v^2}{2g}\ \text{metre};$$

for Prussian measure:

$$h_1 = \left(0{,}01439 + \frac{0{,}016921}{\sqrt{v}}\right) \frac{l}{d} \cdot \frac{v^2}{2g}\ \text{feet},$$

or for English measure:

$$h_1 = \left(0{,}01482 + \frac{0{,}017963}{\sqrt{v}}\right) \frac{l}{d} \cdot \frac{v^2}{2g} \text{ feet.}$$

§ 331. For facilitating the calculation, the following table of the co-efficients of resistance has been compiled. We see from this, that the variability of these co-efficients is not inconsiderable, as for 0,1 metre velocity it is = 0,0443, for 1 metre = 0,0239, and for 5 metres = 0,0186.

TABLE OF THE CO-EFFICIENTS OF FRICTION.

| $v$ ft. | $v$ Whole metres. | 10ths of a metre. 0 | 1 or 4 in. | 2 or 8 in. | 3 12 in. | 4 18 in. | 5 20 in. | 6 24 in. | 7 28 in. | 8 32 in. | 9 36 in. |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | ∞ | 0,0443 | 0,0356 | 0,0317 | 0,0294 | 0,0278 | 0,0266 | 0,0257 | 0,0250 | 0,0244 |
| 3.4 | 1 | 0230 | 0234 | 0230 | 0227 | 0224 | 0221 | 0219 | 0217 | 0215 | 0213 |
| 6.9 | 2 | 0211 | 0209 | 0208 | 0206 | 0205 | 0204 | 0203 | 0202 | 0201 | 0200 |
| 10.0 | 3 | 0199 | 0198 | 0197 | 0196 | 0195 | 0195 | 0194 | 0193 | 0193 | 0192 |
| 13.0 | 4 | 0191 | 0191 | 0190 | 0190 | 0189 | 0189 | 0188 | 0188 | 0187 | 0187 |

We find in this table the co-efficients of resistance due to a certain velocity, when we look for the whole metre in the vertical, and the tenths in the first horizontal column, then proceed from the first number horizontally, and from the last vertically to the place where both motions meet; for example, for $v$ = 1,3 metre $\zeta_1$ = 0,0227, for $v$ = 2,8, $\zeta_1$ = 0,0201.

For the Prussian measure we may put:

| $v$ | 0,1 | 0,2 | 0,3 | 0,4 | 0,5 | 0,6 | 0,7 | 0,8 | 0,9 ft. |
|---|---|---|---|---|---|---|---|---|---|
| $\zeta_1$ | 0,0679 | 0,0522 | 0,0453 | 0,0411 | 0,0383 | 0,0362 | 0,346 | 0,0333 | 0,0322 |

| $v$ | 1 | 1¼ | 1½ | 2 | 3 | 4 | 6 | 8 | 12 | 20 ft. |
|---|---|---|---|---|---|---|---|---|---|---|
| $\zeta_1$ | 0,0313 | 0,0296 | 0,0282 | 0,0263 | 0,0242 | 0,0229 | 0,0213 | 0,0204 | 0,0192 | 0,0182 |

*Remark.* More extensive and more convenient tables are given in the "Ingenieur."

§ 332. *Long tubes.*—With respect to the motion of water in long tubes or conducting pipes, the three following fundamental problems present themselves for solution.

1. The length $l$ and the width $d$ of the tube and the quantity of water $Q$ to be conducted are given, and the head of water is required. We have first to calculate the velocity $v = \frac{Q}{F} = \frac{4\,Q}{\pi\, d^2}$ $= 1{,}2732 \,.\, \frac{Q}{d^2}$, then to look for the co-efficient of friction $\zeta_1$ corresponding to this value, in one of the last tables, and, lastly, to substitute the values of $d, l, v, \zeta$ and $\zeta_1$, where $\zeta$ represents the co-efficient of friction for the portion of the interior orifice, in the last formula $h = \left(1 + \zeta + \zeta_1 \frac{l}{d}\right) \frac{v^2}{2g}$.

2. The length and width of the tube, as well as the head of water or fall, are given to determine the discharge. We must here find the velocity by the formula

$$v = \frac{\sqrt{2\,g\,h}}{\sqrt{1 + \zeta + \zeta_1 \,.\, \frac{l}{d}}};$$

but since the co-efficient of resistance is not quite constant, but varies somewhat with $v$, we must know $v$ approximately beforehand, in order to be able to find out $\zeta_1$.

From $v$ it then follows that $Q = \frac{\pi\, d^2}{4}\, v = 0{,}7854\, d^2\, v$.

3. The discharge, the head of water, and the length of the tube are given, to determine the requisite width of the tube.

$$\text{As } v = \frac{4Q}{\pi\, d^2}, \text{ therefore } v^2 = \left(\frac{4\,Q}{\pi}\right)^2 \,.\, \frac{1}{d^4}$$

we then have $2\,g\,h = \left(1 + \zeta + \zeta_1 \frac{l}{d}\right) \left(\frac{4\,Q}{\pi}\right)^2 \,.\, \frac{1}{d^4}$, or

$2\,g\,h \,.\, \left(\frac{\pi}{4\,Q}\right)^2 = (1 + \zeta)\, \frac{1}{d^4} + \zeta_1 \frac{l}{d^5}$, or

$2\,g\,h \,.\, \left(\frac{\pi}{4\,Q}\right)^2 d^5 = (1 + \zeta)\, d + \zeta_1\, l$, hence the width of the tube

$$d = \sqrt[5]{\frac{(1 + \zeta)\, d + \zeta_1\, l}{2\,g\,h} \,.\, \left(\frac{4\,Q}{\pi}\right)^2}.$$

But now $\left(\frac{4}{\pi}\right)^2 = 1,6212$, $1 + \zeta$ as a mean $= 1,505$ $\frac{1}{2g}$ $= 0,0155$, hence we may put:

$$d = 0,4787 \sqrt[5]{(1,505 \cdot d + \zeta_1 l) \frac{Q^2}{h}} \text{ feet.}$$

This formula can only be used as a formula of approximation, because the unknown quantity $d$, and also the co-efficient $\zeta_1$, dependant on $v = \frac{4}{\pi} \frac{Q}{d^2}$, appear in it.

*Examples.*—1. What head of water does a conducting pipe, of 150 feet length and 5 inches width, require, if it is to carry off 25 cubic feet of water per minute? Here $v = 1,2732 \cdot \frac{25 \cdot 12^2}{60 \cdot 5^2} = 3,056$ feet, hence we may put $\zeta_1 = 0,0242$, and the head of water or the entire fall of the pipe will be:

$$h = \left(1,505 + 0,0242 \cdot \frac{150 \cdot 12}{5}\right) \cdot 0,0155 \cdot 3,056^2$$
$$= (1,505 + 8,712)\ 0,0155 \cdot 9,339 = 1,479 \text{ feet.}$$

2. What discharge will a conducting pipe, 48 feet long and 2 inches wide, with a 5 feet head of water, deliver? It will be:

$$v = \frac{8,03 \sqrt{5}}{\sqrt{1,505 + \zeta_1 \frac{48 \cdot 12}{2}}} = \frac{17,90}{\sqrt{1,505 + 288 \cdot \zeta_1}}.$$

If we previously take $\zeta_1 = 0,020$, we shall obtain $v = \frac{17,90}{2,7} = 6,6$, but $v = 6,6$ gives more correctly $\zeta_1 = 0,0211$, hence we shall have more correctly: $v = \frac{17,90}{\sqrt{1,505 + 288 \cdot 0,0211}} = \frac{17,90}{\sqrt{7,582}} = 6,42$ feet, and the quantity of water $Q = 0,7584 \cdot \left(\frac{2}{12}\right)^2 \cdot 6,42 = 0,140$ cubic feet $= 242$ cubic inches.

3. What width must be given to a conducting pipe, 100 feet in length, which at a head of water of5 feet, delivers half a cubic foot of water per second? Here $d = 0,4817 \sqrt[5]{(1,505\, d + 100\, \zeta_1) \cdot \frac{1}{5} (\frac{1}{2})^2} = 0,4817 \sqrt[5]{0,075\, d + 5\, \zeta_1}$. If we set out with $\zeta_1 = 0,02$, we obtain $d$ $0,4787 \sqrt[5]{0,075\, d + 0,100}$, or approximately $= 0,4787 \sqrt[5]{0,100} = 0,30$, therefore more correctly,

$d = 0,4787 \sqrt[5]{0,0225 + 0,100} = 0,4787 \sqrt[5]{0,1225} = 0,3165$ feet $= 3,8$ inches. To this width corresponds the transverse section $F = 0,7854 \cdot 0,3165^2 = 0,0787$ square feet, the velocity $v = \frac{Q}{F} = \frac{0,5}{0,0787} = 6,35$ feet, and to this again the co-efficient of resistance $\zeta_1 = 0,0211$. If we substitute the last more accurate value, we then obtain $d = 0,4787 \sqrt[5]{0,1280} = 0,319$ feet.

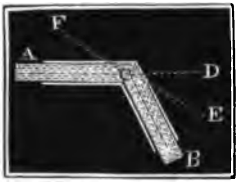
*Remark.* Experiments with $2\frac{1}{2}$ and $4\frac{1}{2}$ inch wide common wooden pipes have given the author the co-efficient of resistance 1,75 times as great as for metallic pipes, to which refer the values given in the table of the former §. Whilst, therefore, for example, for a velocity of 3 feet, $\zeta_1 = 0,0242$ for metallic pipes, we must put it for wooden pipes $= 0,0242 \cdot 1,75 = 0,04235$; whilst we have found in

*Example* 1, the head of water for a metallic pipe 150 feet long, 1,527 feet, it will amount, under the same circumstances, for a wooden one to:

$$= (1,505 + 0,04235 \,.\, 360) \,.\, 0,016 \,.\, 9,339 = 16,75 \,.\, 0,1494 = \cancel{2,50} \text{ feet.}$$

§ 333. *Bent tubes.*—Particular resistances are opposed to the motion of water in pipes when they are *bent* or *knee-shaped.* Experiments have been made by the author on both kinds of resistances, on which account it is necessary here to communicate the results.

FIG. 447.




If a pipe $ACB$, Fig. 447, forms a knee, or if, as it is termed, it be *angled*, a loss of pressure ensues, which increases uniformly with the height $\frac{v^2}{2g}$ due to the velocity, and further increases with half the angle of deflexion $ACF = BCE = \delta$. The height of water lost through the knee, or the height due to the resistance corresponding to its transit through the knee, may be given by the expression $h = \zeta_2 \frac{v^2}{2g}$, where $\zeta_2$ expresses the co-efficient of the knee resistance, dependant on the magnitude of the angle of deviation of the tube. Experiments made on different knees have led to the expression

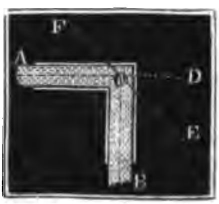
$$\zeta_2 = 0,9457 \; sin. \; \delta^2 + 2,047 \; sin. \; \delta^4,$$

and from this the following table has been calculated.

| $\delta^0$ | 10 | 20 | 30 | 40 | 45 | 50 | 55 | 60 | 65 | 70 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\zeta_2$ | 0,046 | 0,139 | 0,364 | 0,740 | 0,984 | 1,260 | 1,556 | 1,861 | 2,158 | 2,431 |

It is seen from this, that a considerable loss of *vis viva* is produced by knee tubes; for example, a rectangular knee $ACB$, Fig. 448, gives, since the angle of deviation, amounts to $45^0$, the loss of head $= 0,956 \frac{v^2}{2g}$, therefore, pretty nearly equal to the height due to the velocity; a knee of $125^0$, for which $\delta = 62\frac{1}{2}^0$, diminishes the head of water by so

FIG. 448.



much as double the height due to the velocity $2 \cdot \frac{v^2}{2g}$. By putting in the height due to the resistance of the knee $\zeta^2 \frac{v^2}{2g}$, we obtain the complete formula for the motion of water in tubes:

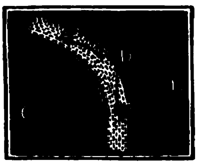
$$h = (1 + \zeta_1 \frac{l}{d} + \zeta_2) \frac{v^2}{2g}.$$

*Example.* If the conducting pipe in the first example of the former paragraph, 150 feet long and 5 inches wide, which is to deliver 25 cubic feet of water per minute, contains two rectangular knees (Fig. 451), we then have the required head of water:

$$h = (1{,}505 + 8{,}712 + 2 \cdot 0{,}956) \frac{v^2}{2g} = 12{,}129 \cdot 0{,}016 \cdot 9{,}339 = 12{,}129 \cdot 0{,}1494$$
$$= 1{,}812 \text{ feet.}$$

§ 334. *Curved tubes.*—Curved tubes $AB$, Fig. 449, offer under otherwise similar circumstances far less resistance than unrounded knee tubes. The height due to the resistance which measures this obstacle increases likewise as the square of the velocity, but at the same time also as the simple angle of deflexion or curvature $ACB = BDE = \beta$, and may be expressed, therefore, by the formula:

FIG. 449.



$$h = \zeta \cdot \frac{\beta^0}{180^0} \cdot \frac{v^2}{2g} = \zeta \cdot \frac{\beta}{\pi} \cdot \frac{v^2}{2g}.$$

The corresponding co-efficient of resistance is by no means constant, it depends much more on the ratio of the width of the tube to the radius of curvature of its axis, and is the less, the less is this ratio. An extensive series of experiments made by the author, and the well known experiments of Du Buat, have, by their combinations, led to the expression $\zeta = 0{,}131 + 1{,}847 \left(\frac{r}{R}\right)^{\frac{7}{2}}$, for tubes with circular transverse sections, and for tubes with quadrangular or rectangular transverse sections $\zeta = 0{,}124 + 3{,}104 \left(\frac{r}{R}\right)^{\frac{7}{2}}$, where $r$ represents half the width of the tube, and $R$ the radius of curvature of the axis.

The two following tables have been calculated accordingly.

## TABLE I.

### CO-EFFICIENTS OF THE RESISTANCE OF CURVATURE IN TUBES WITH CIRCULAR TRANSVERSE SECTIONS.

| $\frac{r}{R}$ | 0,1 | 0,2 | 0,3 | 0,4 | 0,5 | 0,6 | 0,7 | 0,8 | 0,9 | 1,0 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\zeta$ | 0,131 | 0,138 | 0,158 | 0,206 | 0,294 | 0,440 | 0,661 | 0,977 | 1,408 | 1,978 |

## TABLE II.

### CO-EFFICIENTS OF THE RESISTANCE OF CURVATURE IN TUBES WITH RECTANGULAR TRANSVERSE SECTIONS.

| $\frac{r}{R}$ | 0,1 | 0,2 | 0,3 | 0,4 | 0,5 | 0,6 | 0,7 | 0,8 | 0,9 | 1,0 |
|---|---|---|---|---|---|---|---|---|---|---|
| $\zeta$ | 0,124 | 0,135 | 0,180 | 0,250 | 0,398 | 0,643 | 1,015 | 1,546 | 2,271 | 3,228 |

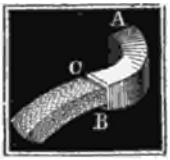
It is from hence seen, that for a round tube whose radius of curvature is twice as great as the radius of the tube, the co-efficient of resistance is $= 0{,}294$, and for a tube whose radius of curvature is at least ten times as great as the radius of the transverse section, this co-efficient $= 0{,}131$.

*Example.*—1. If the conducting pipe in the second example of § 332 has five small curves of 90° curvature, and of the ratio $\frac{r}{R} = \frac{1}{2}$, Fig. 452, what quantity of water will it deliver? The height due to the resistance of the one curve $= 0{,}294 \,.\, \frac{90^{\circ}}{180^{\circ}} \frac{v^2}{2g} = 0{,}147 \,.\, \frac{v^2}{2g}$; hence, for all four curvatures, it $= 5 \,.\, 0{,}147 \frac{v^2}{2g} = 0{,}735 \frac{v^2}{2g}$, and, accordingly, the velocity sought:

$v = \frac{17{,}678}{\sqrt{7{,}582 + 0{,}735}} = \frac{17{,}678}{\sqrt{8{,}317}} = 6{,}130$ feet, and the quantity of water:

$Q = 0{,}7854 \,.\, \frac{1}{36} \,.\, 6{,}130 = 0{,}1337$ cubic feet $= 231$ cubic inches.

FIG. 450.




2. If the curved buckets of a turbine form channels 12 inches long, 2 inches broad, and 2 inches deep, as *ABC*, Fig. 450, and if the water flows through them with a velocity of 50 feet, and the mean radius of curvature $R$ of this axis of the channels amounts to 8 inches, then is $\frac{r}{R} = \frac{1}{8}$, the co-efficient of the resistance of curvature = 0,127; further, $\frac{\beta}{\pi} = \frac{12}{\pi r} = \frac{12}{8\,\pi} = 0,4774$; and lastly, the height due to the resistance corresponding to the curvature of the scoop

$$= 0,127 \,.\, 0,4774 \frac{v^2}{2g} = 0,0606 \frac{v^2}{2g} = 0,0606 \,.\, 0,016 \,.\, 50^2 = 2,424 \text{ feet.}$$

Therefore, by the resistance of curvature, 2,424 feet in fall are lost.

*Remark.* The earlier formulæ given by Du Buat, Gerstner, and Navier for the resistance of curvature are quite useless. An extended account of the experiments of the author on this subject will be published in the third number of his "Investigations in Mechanics and Hydraulics."

§ 335. *Jets d'eau.*—A conducting tube either discharges into the air or under water. The discharge under water is applied when the tube at its outer orifice is so wide that the entrance of air may be feared. Here of course the head of water *RC*, Fig. 451, must be taken from the surface *H* of the upper water to that of *C* of the lower water. If the tube, for example, *KLM*, Fig. 452, discharges into the open air, it will give a stream of water *OR*, which when allowed to ascend is called a *jet d'eau*. We shall here consider what is most required for these jets. That a jet may ascend to the utmost possible height, it is necessary that the water should flow from the adjutage with great velocity; hence such adjutages must be applied which offer the fewest obstacles to the water in its passage, to which, therefore the greatest co-efficients of velocity are due. Orifices in a thin plate especially, and further, short tubes fashioned like the contracted fluid vein, and next, long and conically convergent ones, are those which give the greatest velocities of efflux. Orifices in

FIG. 451.

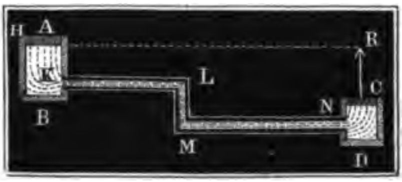
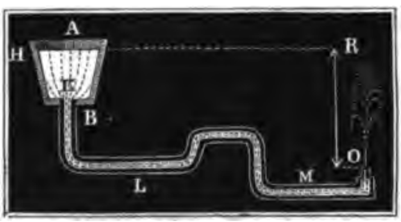
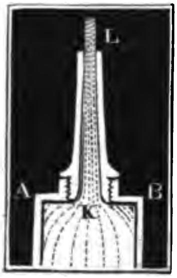



FIG. 452.



a thin plate are little suitable, because a jet formed by them presents nodes and bulgings, and, therefore, is sooner scattered by the external air than the prismatic jet. The same takes place in a certain degree with short mouth-pieces, shaped like the contracted vein. Hence for fountains and fire-engines, mostly long and slightly conically convergent, similar to those which d'Aubuisson used for his experiments, are very properly made use of. Sometimes entirely cylindrical jets are used. Where these mouth-pieces, as for example $KL$, Fig. 453, are screwed on to the the conducting tube $AB$, they should gradually widen, that no contraction may occur in passing into them. If these mouth-pieces or discharging ~~duties~~ are very long, like those of fire-engines, the tubes friction of the water in them will then cause a considerable loss of pressure, because the water has here a great velocity. For great velocities we may very well put the co-efficient of resistance $\zeta_1 = 0{,}016$, and, therefore, the loss of head of water $= 0{,}016 \, \frac{l}{d} \cdot \frac{v^2}{2g}$. If now the length of a hose is twenty times as great as the mean width we shall then obtain the height of the resistance due to friction

FIG. 453.




$$= 0{,}016 \cdot 20 \cdot \frac{v^2}{2g} = 0{,}32 \cdot \frac{v^2}{2g};$$

thus, from this, above 32 per cent of the height of ascent is lost. These tubes are generally much longer, hence this loss is greater.

The velocity with which water passes out of a mouth-piece or hose, and on which the jet or the height of ascent principally depends, may be estimated by means of the above principles. If we put this velocity of efflux $= v$, the width of the mouth-piece at the exit orifice $= d$, and the mean width of the conducting tube $= d_1$, we shall then obtain the velocity of the water in it $v_1 = \frac{d^2}{d_1^2} v$. If $\zeta$ represent the co-efficient of resistance at the inner orifice of the tube, $\zeta_1$ that of the resistance of friction in the pipe, and $\zeta_2$ the co-efficient for the knees or curvature of the pipe, the height due to the resistance for the motion of water in the conduit pipes will be:

$$h = (\zeta + \zeta_1 \frac{l}{d} + \zeta_2) \frac{v_1^2}{2g} = (\zeta + \zeta_1 \frac{l}{d} + \zeta_2) \frac{d^4}{d_1^4} \cdot \frac{v^2}{2g}.$$

It is seen from this, that the resistance to the water is less, the wider the conduit pipe is. It is hence an important rule, to employ as wide pipes and hoses as possible, for leading water to *jets d'eau* and for fire-engines.

If further we represent the co-efficient of resistance for the mouthpiece by $\zeta_3$, we have then the height due to the resistance for this $= \zeta_3 \frac{v^2}{2g}$, and the sum of all the heights due to the resistance is then:

$$= \left[\left(\zeta + \zeta_1 \frac{l}{d} + \zeta_2\right) \frac{d^4}{d_1^4} + \zeta_3\right] \frac{v^2}{2g}.$$

If, lastly, the height of pressure, *i. e.* the depth $RO$, Fig. 452, of the outer orifice $O$ below the surface of water $H$ in the reservoir $= h$, the formula

$$h = \left[1 + \zeta_3 + (\zeta + \zeta_1 \frac{l}{d} + \zeta_2) \frac{d^4}{d_1^4}\right] \frac{v^2}{2g},$$

holds true, and hence the velocity of efflux is:

$$v = \frac{\sqrt{2\,gh}}{\sqrt{1 + (\zeta + \zeta_1 \frac{l}{d} + \zeta_2) \frac{d^4}{d_1^4} + \zeta_3}}.$$

If the jet were to spring perpendicularly and in vacuo, the height of ascent would be:

$$s = \frac{v_2}{2g} = \frac{h}{1 + (\zeta + \zeta_1 \frac{l}{d} + \zeta_2) \frac{d^4}{d_1^4} + \zeta_3},$$

but because the air and the descending water offer impediments to the ascent, and to the direction of the jet, as is the case in fire-engines, the effective height of ascent is somewhat less. According to d'Aubuisson's conclusions from the experiments undertaken upon this subject by Mariotte and Bossut, the effective height of ascent is $s_1 = s - 0{,}01 \cdot s^2 = s\,(1 - 0{,}01 \cdot s)$ metres, or for our measure $= s\,(1 - 0{,}00305\, s)$ feet.

We see from this that in great ascents proportionately more height is lost than in small velocities. Thick jets ascend somewhat higher than thin ones. In order to diminish the resistance of the descending water, the jet must be directed a little inclined. As to the height and amplitude of oblique jets, see § 38.

*Example.* If the conduit pipe for a fountain be 350 feet long, and 2 inches diameter, if the co-efficient of resistance corresponding to the mouth-piece $= .32$, if the entrance orifice at the reservoir be sufficiently rounded, and the bends that occur have sufficient radii of curvature to allow of our neglecting the corresponding co-efficients of resistance, to what height will a jet ½ inch thick, under a head of water of 30 feet, spring? If we take the co-efficient $\zeta_1$ of friction $=$ 0,025, we shall then obtain the entire height due to the resistance:

$$h = \left(1 + 0{,}025 \,.\, \frac{250}{\frac{1}{6}} \,.\, \left(\frac{\frac{1}{2}}{2}\right)^4 + 0{,}32\right) \frac{v^2}{2g} = 1{,}47 \,.\, \frac{v^2}{2g};$$

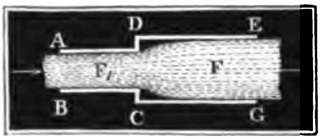
hence, the height due to the velocity $s = \frac{h}{1{,}47} = \frac{30}{1{,}47} = 20{,}41$ feet, and the effective height of ascent $s_1 = 20{,}41\,(1 - 0{,}00314 \,.\, 20{,}41) = 20{,}41 - 1{,}31 = 19{,}1$ feet.

# CHAPTER IV.

## ON THE RESISTANCES OF WATER IN PASSING THROUGH CONTRACTION.

§. 337. *Abrupt widening.*—Changes in the transverse section of a tube, or of any other reservoir of efflux, produce changes in the velocity of the water. The velocity is inversely proportional to the transverse section of the stream. The wider the vessel is, the the less is the velocity, and the narrower the vessel, the greater the velocity of the water flowing through. If the transverse section of a vessel be suddenly altered, as, for example, in the tube *ACE*, Fig. 454, there then ensues a sudden alteration of the velocity, and this is accompanied by a loss of *vis viva*, or connected with a corresponding diminution of pressure. This loss may be as accurately measured as the mechanical effect in the impact of inelastic bodies (§ 258). Every particle of water which passes from the narrower tube *BD* into the wider tube *DG*, strikes against the slowly moving mass of water in this tube, and, after impulse, joins itself to and proceeds onwards with it. It is exactly the same with the collision of solid and inelastic bodies; these bodies go on likewise after impact with a common velocity. Since we have already found that the loss of mechanical effect by the impact of these bodies is

FIG. 454.




$$L = \frac{(v_1 - v_2)^2}{2g} \,.\, \frac{G_1\, G_2}{G_1 + G_2},$$

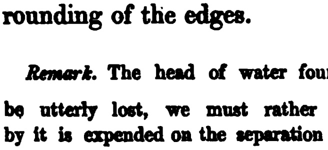
so we may here, as the impinging particle of water $G_1$ is indefinitely small compared with the impinged mass of water $G_2$, put: $L = \frac{(v_1 - v_2)^2}{2g} G_1$, and, consequently, the corresponding loss of head: $h = \frac{(v_1 - v_2)^2}{2g}$.

*There arises, therefore, from a sudden change of velocity a loss of head, which is measured by the height due to the velocity corresponding to this change.*

If now the transverse section of the one tube $AC$, $= F_1$, and that of the other $CE$, $= F$, the velocity of the water in the first tube $= v_1$, and that in the other $= v$, we then have $v_1 = \frac{Fv}{F_1}$, hence the loss in head of water in the passage from one tube to the other is $h_1 = \left(\frac{F}{F_1} - 1\right)^2 \cdot \frac{v^2}{2g}$, and the corresponding co-efficient of resistance $\zeta = \left(\frac{F}{F_1} - 1\right)^2$.

The experiments undertaken by the author on this subject accord well with theory. That the tube $DG$ may be filled with water, it is requisite that it be not very short, nor much wider than the tube $AC$. This loss vanishes, when, as represented in Fig. 455, a gradual passing from one tube into the other is accomplished by the rounding of the edges.

FIG. 455.

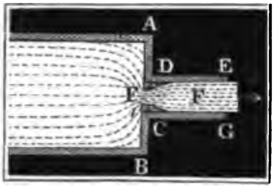



*Remark.* The head of water found $h_1 = \left(\frac{F}{F_1} - 1\right)^2 \frac{v^2}{2g}$ cannot, of course, be utterly lost, we must rather assume that the mechanical effect produced by it is expended on the separation of the previous continuity of the particles of water.

*Example.* If the diameter of a tube, of the construction in Fig. 454, is as great again as that of another tube, then is $\frac{F}{F_1} = \left(\frac{2}{1}\right)^2 = 4$, hence the co-efficient of resistance $\zeta = (4 - 1)^2 = 9$, and the corresponding height due to the resistance on passing from the narrow into the wide tube $= 9 \cdot \frac{v^2}{2g}$. If the velocity of the water in the latter tube $= 10$ feet, the height due to the resistance is then $= 9 \cdot 0{,}0155 \cdot 10^2 = 14{,}95$ feet.

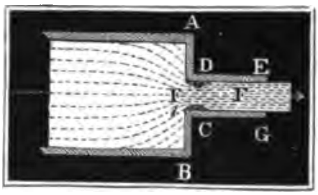
§ 388. *Abrupt contraction.*—A sudden change of velocity also occurs when water passes from a cistern $AB$, Fig. 456,

FIG. 456.




into a narrow tube *DG*, especially when there is a diaphragm at the place of entrance *CD*, whose orifice is less than the transverse section of the tube *DG*. If the area of the contraction is $= F_1$, and $a$ the co-efficient of contraction, we have then the transverse section $F_2$ of the contracted fluid vein $= a\,F_1$; and if, on the other hand, $F$ is the transverse section of the tube and $v$ the velocity of efflux, we then find the velocity at the contracted section $F_2$; $v_2 = \dfrac{F}{a\,F_1}\,v$, and hence the loss of head in passing from $F_2$ into $F$, or from $v_2$ into $v$: $h = \dfrac{(v_2 - v)^2}{2g} = \left(\dfrac{F}{a\,F_1} - 1\right)^2 \dfrac{v^2}{2g}$, and the corresponding co-efficient due to the resistance: $\zeta = \left(\dfrac{F}{a\,F_1}\right)^2$.

FIG. 457.




Without the diaphragm, we have a mere short tube, Fig. 457; hence, $F = F_1$ and $\zeta = \left(\dfrac{1}{a} - 1\right)^2$.

If we assume $a = 0{,}64$, we then obtain:

$$\zeta = \left(\frac{1 - 0{,}64}{0{,}64}\right)^2 = (\tfrac{9}{16})^2 = 0{,}316.$$

But the co-efficient due to the resistance for the transit through an orifice in a thin plate is about 0,07; hence, here, where the water flows out $\dfrac{1}{a}$ times as fast as from the contracted transverse section, the corresponding height due to the resistance

$$= 0{,}07 \cdot \left(\frac{v}{a}\right)^2 \cdot \frac{1}{2g} = 0{,}07 \cdot \frac{1}{a^2} \cdot \frac{v^2}{2g} = \frac{0{,}07}{0{,}41} \cdot \frac{v^2}{2g} = 0{,}171 \cdot \frac{v^2}{2g}.$$

By combining these two resistances, we obtain the entire height due to the resistance for efflux through a short tube:

$$= 0{,}316\,\frac{v^2}{2g} + 0{,}171\,\frac{v^2}{2g} = 0{,}49 \cdot \frac{v^2}{2g},$$

whilst we before found it $= 0{,}50\,\dfrac{v^2}{2g}$.

Experiments on the efflux of water through an additional tube, with a narrow inner orifice, as in Fig. 466, have led the author to the following results. The co-efficient of resistance for transit

through a diaphragm, and for a contraction at the wider tube, may be expressed by the formula $\zeta = \left(\frac{F}{a F_1} - 1\right)^2$, but there must be put:

| for $\frac{F_1}{F}$ | 0,1 | 0,2 | 0,3 | 0,4 | 0,5 | 0,6 | 0,7 | 0,8 | 0,9 | 1,0 |
|---|---|---|---|---|---|---|---|---|---|---|
| $a$ | 0,616 | 0,614 | 0,612 | 0,610 | 0,607 | 0,605 | 0,603 | 0,601 | 0,598 | 0,596 |
| and it follows that: | | | | | | | | | | |
| $\zeta$ | 231,7 | 50,99 | 19,78 | 9,612 | 5,256 | 3,077 | 1,876 | 1,169 | 0,734 | 0,480 |

From this, for example, the co-efficient of resistance in the case where the narrow transverse section is half as great as that of the tube, is $\zeta = 5{,}256$, *i. e.* for transit through this contraction a head of water is required which is $5\frac{1}{4}$ times as great as the height due to the velocity.

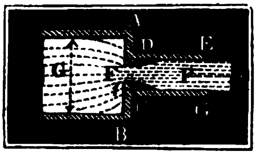
*Example.* What discharge will the apparatus delineated in Fig. 456 give, if the head of water is $1\frac{1}{2}$ feet, the width of the circular contraction $1\frac{1}{2}$, and that of the tube 2 inches? We have here $\frac{F_1}{F} = \left(\frac{1\frac{1}{2}}{2}\right)^2 = (\frac{3}{4})^2 = \frac{9}{16} = 0{,}56$, hence $a = 0{,}606$, and $\zeta = \left(\frac{16}{9 \cdot 0{,}606} - 1\right)^2 = \left(\frac{16 - 5{,}454}{5{,}454}\right)^2 = \left(\frac{10{,}546}{5{,}454}\right)^2 = 3{,}74$. If now we put $h = (1 + \zeta)\frac{v^2}{2g}$, we shall then obtain the velocity of efflux:

$v = \frac{\sqrt{2gh}}{\sqrt{1+\zeta}} = \frac{8{,}03\sqrt{1{,}5}}{\sqrt{4{,}74}} = 4{,}47$ feet, and consequently the quantity discharged:

$Q = \frac{\pi d^2}{4} v = \frac{\pi}{4} \cdot 4 \cdot 12 \cdot 4{,}45 = 53{,}4 \cdot \pi = 168$ cubic inches.

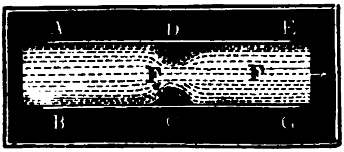
§ 339. *Effect of imperfect contraction.*—In the case considered in the last paragraph, the water issues from a large cistern, hence the contraction may be regarded as perfect; but if the transverse section of the cistern or of the stream of fluid arriving at the narrow part is not great with respect to the transverse section $F_1$, Fig. 458, of this contracted part; the contraction is then imperfect, and hence also the corresponding co-efficient of resistance is less than in the case above investigated. If we retain the former denominations, we have then also here the

FIG. 458.



height due to the resistance, or the head of water expended by the transit through $F_1$, $h = \left(\frac{F}{a F_1} - 1\right)^2 \frac{v^2}{2g}$, only, for $a$, we must substitute variable numbers, greater the greater the ratio $\frac{F_1}{G}$ between the transverse section of the contraction and that of the conducting tube $AB$. If the diaphragm $CD$ lies in a uniform tube $AG$, Fig. 459, then the same condition holds, only here the co-efficient $a$ depends on $\frac{F_1}{F}$.

FIG. 459.



From experiments undertaken by the author, we must put into the formula $\zeta = \left(\frac{F}{a F_1} - 1\right)^2$ for the co-efficient of resistance,

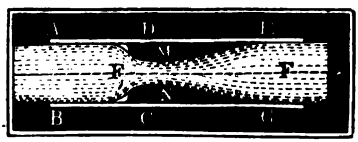
| for $\frac{F_1}{F}$ = | 0,1 | 0,2 | 0,3 | 0,4 | 0,5 | 0,6 | 0,7 | 0,8 | 0,9 | 1,0 |
|---|---|---|---|---|---|---|---|---|---|---|
| $a$ | 0,624 | 0,632 | 0,643 | 0,659 | 0,681 | 0,712 | 0,755 | 0,813 | 0,892 | 1,000 |

and it follows:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\zeta$ | 225,9 | 47,77 | 17,50 | 7,801 | 3,753 | 1,796 | 0,797 | 0,290 | 0,060 | 0,000 |

These losses are diminished when, by the rounding off the edges, the contraction is diminished or counteracted, and they may be entirely neglected, if, as is represented in Fig. 460, a gradually widening tube $MN$ is put on.

FIG. 460.

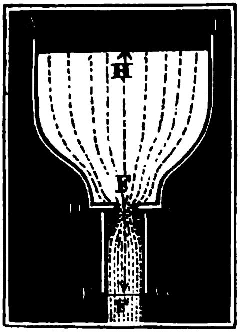


*Example.* What head of water is requisite that the apparatus in Fig. 461 may deliver 8 cubic feet of water per minute? If the width of the diaphragm $F_1 = 1\frac{1}{2}$, the width of the efflux tube $DG$ = 2 inches, and the lower width of the afflux tube $AC$ = 3 inches, we shall then have $\frac{F_1}{G} = \left(\frac{1\frac{1}{2}}{3}\right)^2 = \frac{1}{4}$, hence $a = 0,637$; further, $\frac{F}{F_1} = \left(\frac{2}{1\frac{1}{2}}\right)^2 = \left(\frac{4}{3}\right)^2 = \frac{16}{9}$, and the co-efficient of resistance:

$$\zeta = \left(\frac{16}{9 \,.\, 0,637} - 1\right)^2 = \left(\frac{10,267}{5,733}\right)^2 = 3,207.$$ The velocity of efflux is now:

$v = \frac{4\,Q}{\pi\,d^2} = \frac{4\,.\,8}{60\,.\,\pi(\frac{1}{2})^2} = \frac{19{,}2}{\pi} = 6{,}112$ feet; hence, the head of water in question is $h = (1 + \zeta)\frac{v^2}{2g} = 4{,}207\,.\,0{,}0155\,.\,6{,}112^2 = 2{,}51$ feet.

FIG. 461.



§ 340. *Slides, cocks, valves.* — For regulating the flow of water from pipes and cisterns, slides, cocks, valves, &c., are used, by which contractions are produced which offer obstacles to the passage of the water, and these may be determined in a manner similar to the loss estimated in the last paragraph. But since the water here undergoes further changes of direction, divisions, &c., the co-efficients $\alpha$ and $\zeta$ cannot be determined directly, but special experiments are necessary for this purpose. Such experiments have been also made,* and their principal results are communicated in the following tables.

TABLE I.

THE CO-EFFICIENTS OF RESISTANCE TO THE PASSAGE OF WATER THROUGH SLIDING VALVES IN RECTANGULAR TUBES.

| Rules of transverse section $\frac{F_1}{F}$ | 1,0 | 0,9 | 0,8 | 0,7 | 0,6 | 0,5 | 0,4 | 0,3 | 0,2 | 0,1 |
|---|---|---|---|---|---|---|---|---|---|---|
| Co-efficient of resistance $\zeta$ | 0,00 | 0,09 | 0,39 | 0,95 | 2,08 | 4,02 | 8,12 | 17,8 | 44,5 | 193 |

TABLE II.

THE CO-EFFICIENTS OF RESISTANCE TO THE PASSAGE OF WATER THROUGH SLIDES IN CYLINDRICAL TUBES.

| Height of place $s$ . . | 0 | $\frac{1}{8}$ | $\frac{2}{8}$ | $\frac{3}{8}$ | $\frac{4}{8}$ | $\frac{5}{8}$ | $\frac{6}{8}$ | $\frac{7}{8}$ |
|---|---|---|---|---|---|---|---|---|
| Ratio of transverse section | 1,000 | 0,948 | 0,856 | 0,740 | 0,609 | 0,466 | 0,315 | 0,159 |
| Co-efficient of resistance $\zeta$ | 0,00 | 0,07 | 0,26 | 0,81 | 2,06 | 5,52 | 17,0 | 97,8 |

* Experiments on the efflux of water through valves, slides, &c., undertaken and calculated by Julius Weisbach, under the title "Untersuchungen im Gebiete der Mechanik und Hydraulik," &c. Leipzig, 1842.

## TABLE III.

### THE CO-EFFICIENTS OF RESISTANCE FOR THE PASSAGE OF WATER THROUGH A COCK IN A RECTANGULAR TUBE.

| Angle of position. | 5° | 10° | 15° | 20° | 25° | 30° | 35° | 40° | 45° | 50° | 55° | 66⅔ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Ratio of transverse section. | 0,926 | 0,849 | 0,769 | 0,687 | 0,604 | 0,520 | 0,436 | 0,352 | 0,269 | 0,188 | 0,110 | 0 |
| Co-efficient of resistance. | 0,05 | 0,31 | 0,88 | 1,84 | 3,45 | 6,15 | 11,2 | 20,7 | 41,0 | 95,3 | 275 | ∞ |

## TABLE IV.

### THE CO-EFFICIENTS OF RESISTANCE FOR THE PASSAGE OF WATER THROUGH A COCK IN A CYLINDRICAL TUBE.

| Angle of position. | 5° | 10° | 15° | 20° | 25° | 30° | 35° |
|---|---|---|---|---|---|---|---|
| Ratio of transverse section. | 0,926 | 0,850 | 0,772 | 0,692 | 0,613 | 0,535 | 0,458 |
| Co-efficient of resistance. | 0,05 | 0,29 | 0,75 | 1,56 | 3,10 | 5,47 | 9,68 |

| Angle of position. | 40° | 45° | 50° | 55° | 60° | 65° | 82⅛° |
|---|---|---|---|---|---|---|---|
| Ratio of transverse section. | 0,385 | 0,315 | 0,250 | 0,190 | 0,137 | 0,091 | 0 |
| Co-efficient of resistance. | 17,3 | 31,2 | 52,6 | 106 | 206 | 486 | ∞ |

## TABLE V.

### THE CO-EFFICIENTS OF RESISTANCE FOR THE PASSAGE OF WATER THROUGH THROTTLE VALVES IN RECTANGULAR TUBES.

| Angle of position. | 5° | 10° | 15° | 20° | 25° | 30° | 35° |
|---|---|---|---|---|---|---|---|
| Ratio of transverse section. | 0,913 | 0,826 | 0,741 | 0,658 | 0,577 | 0,500 | 0,426 |
| Co-efficient of resistance. | 0,28 | 0,45 | 0,77 | 1,34 | 2,16 | 3,54 | 5,72 |

| Angle of position. | 40° | 45° | 50° | 55° | 60° | 65° | 70° | 90° |
|---|---|---|---|---|---|---|---|---|
| Ratio of transverse section. | 0,357 | 0,293 | 0,234 | 0,181 | 0,134 | 0,094 | 0,060 | 0 |
| Co-efficient of resistance. | 9,27 | 15,07 | 24,9 | 42,7 | 77,4 | 158 | 368 | ∞ |

## TABLE VI.

### THE CO-EFFICIENTS OF RESISTANCE FOR THE PASSAGE OF WATER THROUGH THROTTLE VALVES IN CYLINDRICAL TUBES.

| Angle of position. | 5° | 10° | 15° | 20° | 25° | 30° | 35° |
|---|---|---|---|---|---|---|---|
| Ratio of transverse section. | 0,913 | 0,826 | 0,741 | 0,658 | 0,577 | 0,500 | 0,426 |
| Co-efficient of resistance. | 0,24 | 0,52 | 0,90 | 1,54 | 2,51 | 3,91 | 6,22 |

| Angle of position. | 40° | 45° | 50° | 55° | 60° | 65° | 70° | 90° |
|---|---|---|---|---|---|---|---|---|
| Ratio of transverse section. | 0,357 | 0,293 | 0,234 | 0,181 | 0,134 | 0,094 | 0,060 | 0 |
| Co-efficient of resistance. | 10,8 | 18,7 | 32,6 | 58,8 | 118 | 256 | 751 | ∞ |

§ 341. By means of the co-efficients derived from the above tables, we may not only assign the loss of pressure corresponding to a certain slide, cock, or position of a valve, but also deduce what position is to be given to this apparatus that the velocity of efflux or the resistance may be of a certain amount. Such a determination is of course the more to be relied on the more the regulating arrangements are like to those used in the experiments. The numerical values given in the tables are only true for the case where the water, after its transit through the contractions produced by means of this apparatus, again fills this tube. That this full flow may take place for small contractions, the tube should have a considerable length. The transverse sections of the rectangular tubes were 5 centimetres broad and 2½ deep. The transverse sections of the cylindrical tubes had, however, a width of 4 centimetres. By the slide, Fig. 462, there is a simple contraction, whose transverse section forms in the one tube a mere rectangle $F_1$, Fig. 463; in the second, however, a lune, $F_1$, Fig. 464. In the case of

FIG. 462.

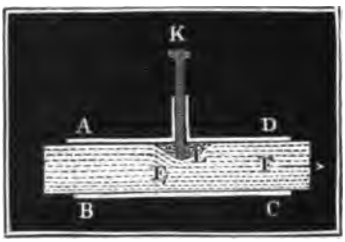



FIG. 463.


FIG. 464.


cocks, two contractions present themselves, and also two changes of direction; on this account the resistances are also very considerable. The transverse sections of the greatest contractions have very peculiar figures. The stream in throttle valves, Fig. 466, divides itself into two portions, each of which passes through a contraction. The transverse sections of these con-

FIG. 465.

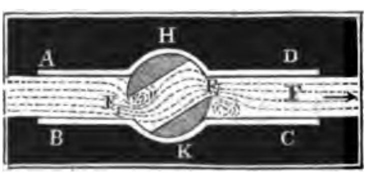
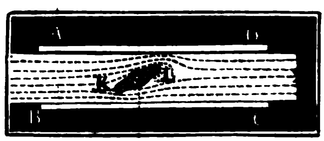



FIG. 466.



tractions are, in the throttle valve of rectangular tubes, rect-

angular, and in cylindrical ones lunar-shaped. The following examples will suffice to show the application of the above tables.

*Examples.*—1. If a sliding valve is applied to a cylindrical conducting pipe 500 feet long and 3 inches wide, and this be drawn up to $\frac{1}{4}$ of its entire height, and therefore close $\frac{3}{4}$ of the pipe, what discharge will it deliver under a pressure of 4 feet? The co-efficient of resistance $\zeta$ for entrance into the pipe may be put from the above at 0,505, and that of the resistance of the slide from Table II. $= 5{,}52$, hence the velocity of the flow $v = \frac{8{,}03 \cdot \sqrt{4}}{\sqrt{1{,}505 + 5{,}52 + \zeta_2 \frac{l}{d}}} = \frac{8{,}03 \cdot 2}{\sqrt{7{,}025 + 500 \cdot 4\, \zeta_2}}$ $= \frac{18{,}06}{\sqrt{7{,}025 + 2000\, \zeta_2}}$. If we put the co-efficient of friction $\zeta_2 = 0{,}025$, we shall then obtain $v = \frac{18{,}06}{\sqrt{57{,}025}} = 2{,}52$ feet. But the velocity $v = 2{,}1$ feet, gives more correctly $\zeta_2 = 0{,}026$; hence, more accurately, $v = \frac{18{,}06}{\sqrt{59{,}025}} = 2{,}35$ feet, and the discharge per second $= \frac{\pi}{4} \cdot 9 \cdot 12 \cdot 2{,}35 = 63{,}45 \cdot \pi = 199$ cubic inches.— 2. A conducting pipe, 4 inches wide, delivers, under a head of water of 5 feet, 10 cubic feet of water per minute; what position must be given to the throttle-valve applied, that it may afterwards deliver only 8 cubic feet? The velocity at the beginning is $= \frac{10 \cdot 4}{60 \cdot \pi (\frac{1}{3})^2} = \frac{6}{\pi} = 1{,}91$ feet, and after putting on the valve $\frac{8}{10} \cdot 191 = 1{,}528$ feet. The co-efficient of efflux is $\frac{v}{\sqrt{2 g h}} = \frac{1{,}91}{8{,}03 \sqrt{5}} = 0{,}106$, hence the co-efficient of resistance $= \frac{1}{\mu^2} - 1 = \frac{1}{0{,}106^2} - 1 = 87{,}9$; the co-efficient of efflux for the second case is $= \frac{8}{10} \cdot 0{,}108 = 0{,}0848$; hence the co-efficient of resistance $= \frac{1}{0{,}0848^2} - 1 = 138{,}0$, and consequently to produce the co-efficient of resistance of the throttle-valve: $\zeta = 138 - 87{,}9 = 58{,}2$. But now, from Table VI., the angle of position $\alpha = 50°$, $\zeta = 32{,}6$, and the angle of position $\alpha = 55°$, $\zeta = 58{,}8$; hence we may be allowed to assume, for a position of $50° + \frac{15{,}7}{26{,}2} \cdot 5° = 53°$ the desired quantity of discharge may be obtained. If we consider, further, for a change of velocity of 1,91 feet to 1,528 feet, the co-efficient of resistance passes from 0,0266 into 0,0281; then, more correctly, $\zeta = 138{,}0 - 87{,}9 \cdot \frac{281}{266} = 138{,}0 - 92{,}9 = 45{,}1$, and, accordingly, the angle of position $= 50° + \frac{10{,}9}{26{,}2} \cdot 5° = 52°$.

§ 342. *Valves.*—The knowledge of the resistance produced by valves is of great importance. Experiments have been made by the author on this subject. The *conical* and *clack*, or *flap valves*, Figs. 467 and 468, are those which most frequently are met with in practice. In both, the water passes through the aperture formed by a ring *RG*; the conical valve *KL* has a guide rod,

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by which it is fixed in guides, and admits of an outward push only in the direction of the axis; the clack valve *KL* opens by

FIG. 467.

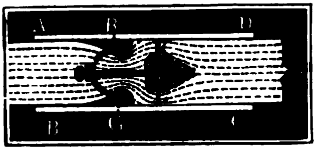
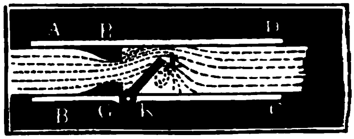



FIG. 468.




turning like a door. It is easily seen in both apparatus that a resistance is opposed to the water, not only by the valve ring, but also by the valve plate.

For the conical valve with which the experiments were undertaken, the ratio of the aperture in the valve ring, to the transverse section of the whole tube was 0,356, and, on the other hand, the ratio of the surface of the ring for the opened valve to the transverse section of the tube 0,406; hence, for the mean, we may put $\frac{F_1}{F} = 0,381$. Whilst the efflux in different positions of the valve was observed, it was found that the co-efficient of resistance diminished when the valve slide was greater, and that this diminution was almost insignificant when it exceeded half the width of the aperture. Its amount was in this case $= 11$, therefore, the height due to the resistance or the loss of head of water $= 11 \cdot \frac{v^2}{2g}$, $v$ being the velocity of the water in the full tube. This number may be also used for determining the co-efficients of resistance corresponding to the other ratios of the transverse sections. Let generally $\zeta = \left(\frac{F}{a F_1} - 1\right)^2$, we then obtain for the observed case $\frac{F_1}{F} = 0,381$, $\zeta = 11$, and $11 = \left(\frac{1}{0,381\, a} - 1\right)^2$, hence

$$a = \frac{1}{0,381\,(1 + \sqrt{11})} = \frac{1}{4,317 \,.\, 0,381} = 0,608,$$

and, lastly, in general:

$$\zeta = \left(\frac{F}{0,608\, F_1} - 1\right)^2 = \left(1,645 \,.\, \frac{F}{F_1} - 1\right)^2.$$

If, for example, the transverse section of the aperture is one half that of the tube, the co-efficient of resistance will accordingly $= (1,645 \,.\, 2 - 1)^2 = 2,29^2 = 5,24$.

For the clack, or trap valve, the ratio of the transverse section of the aperture to the tube was $\frac{F_1}{F}$, $= 0,535$, but the following table shews in what degree the co-efficient of resistance diminishes with the size of the aperture.

TABLE

OF THE CO-EFFICIENTS OF RESISTANCE FOR TRAP VALVES.

| Angle of aperture. | 15° | 20° | 25° | 30° | 35° | 40° | 45° | 50° | 55° | 60° | 65° | 70° |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Co-efficient of resistance. | 90 | 62 | 42 | 30 | 20 | 14 | 9,5 | 6,6 | 4,6 | 3,2 | 2,3 | 1,7 |

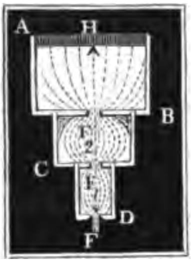
The co-efficients of resistance of these valves may be calculated approximately with the help of this table, even when the ratio of the transverse section is any other. The same method must be adopted as that followed for conical valves.

*Example.* A forcing-pump delivers by each descent of the piston, 5 cubic feet of water in 4 seconds, the width of the tube of ascent, in which lies the valve opening upwards, is 6 inches, the aperture of the valve-ring $3\frac{1}{2}$ inches, and the greatest diameter of the valve $4\frac{1}{2}$ inches; what resistance has the water in its passage through the valve to overcome? The ratio of the transverse section for the aperture is $\left(\frac{3,5}{6}\right)^2 = (\frac{7}{12})^2 = 0,34$, and that of the annular contraction to the transverse section of the tube $= 1 - \left(\frac{4,5}{6}\right)^2 = 1 - (\frac{3}{4})^2 = 0,44$, hence the mean ratio of section $= \frac{0,34 + 0,44}{2} = 0,39$ and the corresponding co-efficient of resistance $= \left(\frac{1,645\,.}{0,39} - 1\right)^2 = 3,22^2 = 10,4$. The velocity of the water is:

$$v = \frac{5}{4\,.\,\frac{\pi}{4}\,.\,(\frac{1}{2})^2} = \frac{20}{\pi} = 6,37 \text{ feet},$$ the height due to the velocity $= 0,649$ feet; and, consequently, the resistance due to the height $= 10,4\,.\,0,649 = 6,75$ feet. The quantity forced up in one second weighs $\frac{5}{4}\,.\,62,5 = 77,6$ lbs.; hence the mechanical effect which by the transit of the water through the valve is consumed in this time $= 523,8$ ft. lbs.

§ 343. *Compound vessels.*—The principles already laid down on the resistance of water in its passage through contractions, find their application in the efflux of water through compound vessels The apparatus *AD* represented in Fig. 469, is divided by two partition walls containing the orifices $F_1$ and $F_2$, and on this account

FIG. 469.




forms three vessels of communication. Were there no partition walls, and the edges, where one vessel passes into the other, rounded off, we should then have as for a single vessel the velocity of flow through $F$: $v = \frac{\sqrt{2gh}}{\sqrt{1+\zeta}}$, $h$ representing the depth $FH$ below the surface of water, and $\zeta$ the co-efficient of resistance for the passage through the orifice $F$. But since obstacles are to be overcome on passing through $F_1$ and $F_2$, we then have to put $v = \sqrt{\frac{2gh}{1+\zeta+\zeta_1+\zeta_2}}$, and to substitute for $\zeta_1$ and $\zeta_2$, the co-efficients of resistance corresponding to the contractions $F_1$ and $F_2$. If we represent the transverse sections of the vessels $CD$, $BC$ and $AB$, by $G$, $G_1$ and $G_2$, we may further put (§ 338):

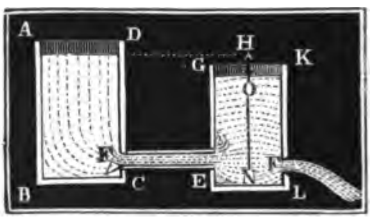
$$\zeta_1 = \left(\frac{G}{aF_1} - 1\right)^2, \text{ and } \zeta_2 = \left(\frac{G_1}{aF_2} - 1\right)^2,$$

hence also:

$$v = \frac{\sqrt{2gh}}{\sqrt{1+\zeta+\left(\frac{G}{aF_1} - 1\right)^2 + \left(\frac{G_1}{aF_2} - 1\right)^2}}.$$

Exactly the same relations take place in the compound apparatus of discharge represented in Fig. 470, except only that the friction of the water in the tube of communication $CE$ has perhaps to be taken into account. If $l$ is the length, and $d$ the width of this tube, but $\zeta_1$ the co-efficient of friction, and $v_1$ the velocity of the water in the tube of communication, we then have the height which the water loses in passing from $AC$ to $GL$:

FIG. 470.




$$h_1 = \left[1 + \left(\frac{1}{a} - 1\right)^2 + \zeta_1 \frac{l}{d}\right] \frac{v_1^2}{2g},$$

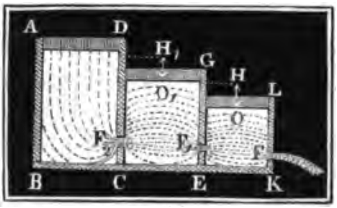
or, since the velocity is to be put:

$$v_1=\frac{a\,F}{F_1}\,v,\ h_1=\left[1+\left(\frac{1}{a}-1\right)^2+\zeta_1\frac{l}{d}\right]\left(\frac{a\,F}{F_1}\right)^2\frac{v^2}{2g}.$$

If this height be deducted from the whole head of water $h$, there will remain the head of water in the second vessel $h_2=h-h_1$, and hence the velocity of efflux:

$$v=\frac{\sqrt{2gh_2}}{\sqrt{1+\zeta}}=\frac{\sqrt{2gh}}{\sqrt{1+\zeta+\left[1+\left(\frac{1}{a}-1\right)^2+\zeta_1\frac{l}{d}\right]\left(\frac{a\,F}{F_1}\right)^2}}.$$

FIG. 471.




This determination is very simple with the apparatus represented in Fig. 471, because the transverse sections $G$, $G_1$, $G_2$, of the cisterns may be made indefinitely great with respect to the transverse sections of the orifices $F$, $F_1$, $F_2$. Hence the first difference of level $OH$, or height due to the resistance in passing through:

$$F_1 \text{ is } h_1 = \frac{1}{2g}\left(\frac{v_1}{a_1}\right)^2=\left(\frac{a\,F}{a_1\,F_1}\right)^2\cdot\frac{v^2}{2g},$$

and likewise the second difference of level $O_1\,H_1$, or the height due to the resistance in passing through

$$F_2 \text{ is } h_2 = \left(\frac{a\,F}{a_2\,F_2}\right)^2\cdot\frac{v^2}{2g},$$

where $a$, $a_1$, $a_2$, represent the co-efficients of contraction for the orifices $F$, $F_1$ and $F_2$. It accordingly follows that:

$$v=\frac{\sqrt{2gh}}{\sqrt{1+\left(\frac{a\,F}{a_1\,F_1}\right)^2+\left(\frac{a\,F}{a_2\,F_2}\right)^2}},$$

and the quantity of discharge:

$$Q=\frac{a\,F\,\sqrt{2gh}}{\sqrt{1+\left(\frac{a\,F}{a_1\,F_1}\right)^2+\left(\frac{a\,F}{a_2\,F_2}\right)^2}}$$

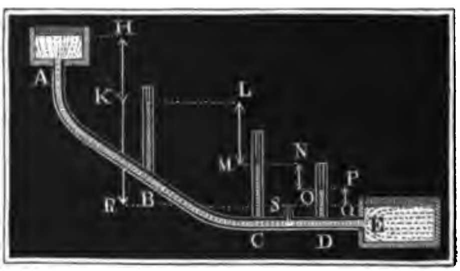
$$=\frac{\sqrt{2gh}}{\sqrt{\left(\frac{1}{a\,F}\right)^2+\left(\frac{1}{a_1\,F_1}\right)^2+\left(\frac{1}{a_2\,F_2}\right)^2}}.$$

It is easy to perceive that compound reservoirs of efflux deliver less water, under otherwise similar circumstances, than simple ones.

*Example.* If in the apparatus, Fig. 470, the whole head of water or depth of the centre of the orifice $F$ below the surface of water of the first cistern is 6 feet, the orifice 8 inches broad and 4 inches deep, the tube connecting both reservoirs 10 feet long, 12 inches broad, and 6 inches deep, what discharge will this reservoir give? The mean width of the tube $= \frac{4 \,.\, 1 \,.\, 0,5}{2 \,.\, 1,5} = \frac{2}{3}$ ft., hence $\frac{l}{d} = \frac{3 \,.\, 10}{2} = 15$; let us now put the co-efficient of friction $\zeta_1 = 0,025$, and it follows that $\zeta_1 \,.\, \frac{l}{d} = 0,025 \,.\, 15 = 0,375$; if the co-efficient of resistance for entrance into prismatic tubes be here put 0,505, we obtain $1 + \left(\frac{1}{a_1} - 1\right)^2 + \zeta_1 \frac{l}{d} = 1 + 0,505 + 0,375$ $= 1,88$. As $\frac{aF}{F_1} = \frac{0,64 \,.\, 8 \,.\, 4}{12 \,.\, 6} = 0,2845$, the co-efficient of resistance for the entire connecting tube $= 1,88 \,.\, 0,2845^2 = 0,152$, and the co-efficient of resistance for the transit through $F$, $= 0,07$, we then obtain the velocity of efflux: $v = \frac{8,03 \sqrt{6}}{\sqrt{1,07 + 0,159}} = \frac{8,03 \sqrt{6}}{\sqrt{1,222}} = 17,66$ feet. The contracted section is $0,64 \,.\, 1 \,.\, \frac{1}{2} = 0,32$ square feet, hence the discharge $= 0,32 \,.\, 17,66 = 5,65$ cubic feet.

§ 344. *Piëzometers.* — The loss of pressure which water suffers in conduit pipes from contractions, friction, &c., may be measured by columns of water, which are sustained in vertically placed tubes, which when used for this purpose are called *piëzometers.*

FIG. 472.




If $v$ is the velocity of the water at a place $B$, Fig. 472, where a piëzometer is applied, $l$ the length, $d$ the width of the portion of tube $AB$, $h$ the head of water or the depth of the point $B$ below the surface of water; if, further, $\zeta$ is the co-efficient of resistance for entrance from the reservoir into the tube, and $\zeta_1$ the co-efficient of friction, we then have for the height of the piëzometer measuring the pressure at $B$.

$$z = h - \left(1 + \zeta + \zeta_1 \frac{l}{d}\right) \frac{v^2}{2g}.$$

If the length of a portion of the tube $BC = l_1$, and its fall $= h_1$, we then have the height of the piëzometer at $C$:

$$z_1 = h + h_1 - \left(1 + \zeta + \zeta_1 \frac{l}{d} + \zeta_1 \frac{l_1}{d}\right) \frac{v^2}{2g},$$

and hence the difference of these two heights:

$$z_1 - z = h_1 - \zeta_1 \frac{l_1}{d} \cdot \frac{v^2}{2g},$$

hence, inversely, the height of the portion of tube $BC$, due to the resistance:

$$\zeta_1 \frac{l}{d} \cdot \frac{v^2}{2g} = h_1 + z - z_1 =$$ *the fall of the portion of tube plus the difference of the heights of the piëzometers.*

From this it is seen that piëzometers are applicable to the measurement of the resistance which the water has to overcome in conduit pipes. If a particular impediment is found in the tubes; if, for instance, some small body is found fixed there, this will immediately be shown by the falling of the piëzometer, and the amount of the resistance produced, expressed. The resistances which are caused by regulating apparatus, such as cocks, slides, &c., may be likewise expressed by the height of the piëzometer. The piëzometer, for example, stands lower at $D$ than at $C$, not only in consequence of the friction of the water in the portion of water $CD$, but also in consequence of the contraction which the slide $S$ produces in this tube. If for a perfectly opened slide the difference $NO$ of the height of the piëzometer $= h_1$, and for the slide partly closed $= h_2$, the new difference or depression $h_2 - h_1$ gives the height due to the resistance which corresponds to the passage of the water through the slide. Lastly, the velocity of efflux may be also estimated by the height of the piëzometer. If the height of the piëzometer $PQ = z$, the length of the last portion of tube $DE = l$, and its width $= d$, we then have:

$$z = \zeta_1 \frac{l}{d} \cdot \frac{v^2}{2g}, \text{ and hence } v = \sqrt{\frac{2gz}{\zeta_1 \frac{l}{d}}} = \sqrt{\frac{d}{l} \cdot \frac{2gz}{\zeta_1}}.$$

*Example.* If the height of the piëzometer $PQ = z$, Fig. 471, $\frac{3}{4}$ foot, and the length of the tube $DE$, measured from the piëzometer to the discharging orifice, $= 150$ feet, the width of the tube $3\frac{1}{2}$ inches, the velocity of efflux then follows:

$v = 8{,}03 \cdot \sqrt{\frac{3{,}5}{150 \cdot 12} \cdot \frac{0{,}75}{00{,}25}} = 8{,}03 \cdot 0{,}2415 = 1{,}939$ feet, and the discharge $Q = \frac{\pi}{4} \cdot \left(\frac{3{,}5}{12}\right)^2 \cdot 1{,}91 = 0{,}146$ cubic feet.

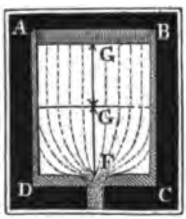
# CHAPTER V.

## ON THE EFFLUX OF WATER UNDER VARIABLE PRESSURE.

§ 845. *Prismatic vessels.*—If a cistern from which water flows through an orifice at the side or bottom, has no influx to it from any other side, a gradual sinking of the surface of water will take place, and the cistern at last empty itself. If further, the quantity of influx $Q$, be greater or less than the quantity of efflux $\mu F \sqrt{2gh}$, the surface of water will then rise or fall until the head of water $h = \frac{1}{2g}\left(\frac{Q}{\mu F}\right)^2$, and after this the head of water and the velocity of efflux will remain unaltered. Our problem then, is to find how the time, the rise and fall of the water, and the emptying of vessels of given form and dimensions, depend on each other.

The efflux from a prismatic vessel presents the most simple case when it takes place through an opening in the bottom, and when there is no efflux from above or below. If $x$ is the variable head of water $FG_1$, $F$ the area of the orifice, and $G$ the transverse section of the vessel $AC$, Fig. 473, we have then the theoretical

FIG. 473.




velocity of efflux $v = \sqrt{2gx}$, the theoretical velocity of the falling surface of the water $= \frac{F}{G} v = \frac{F}{G} \sqrt{2gx}$, and the effective velocity $v_1 = \frac{\mu F}{G} \sqrt{2gx}$. At the commencement: $x = FG = h$, and at the end of the efflux $x = 0$, therefore, the initial velocity is:

$c = \frac{\mu F}{G} \sqrt{2gh}$, and the final velocity $c_1 = 0$. It is seen from the formula $v_1 = \sqrt{2\left(\frac{\mu F}{G}\right)^2 g\, x}$, that the motion of the surface is uniformly retarded, and the measure of the retardation $p = \left(\frac{\mu F}{G}\right)^2 g$, hence we also know (§ 14), that this velocity $= 0$, and the discharge ceases, when

$$t = \frac{c}{p} = \frac{\mu F}{G} \sqrt{2gh} : \left(\frac{\mu F}{G}\right)^2 g = \frac{G}{\mu F} \sqrt{\frac{2gh}{g^2}}, \text{ i. e. } t = \frac{2\,G\,\sqrt{h}}{\mu\,F\,\sqrt{2g}}.$$

We may also put:

$$t = \frac{2\,G\,h}{\mu\,F\,\sqrt{2\,gh}} = \frac{2\,G\,h}{Q},$$

and, according to this, assume that double the time is required for the efflux of the discharge $Gh$ through the orifice at the bottom $F$, under a head of water decreasing from $h$ to 0, than under a uniform pressure.

As the co-efficient of efflux $\mu$ is not quite constant, but is greater for a diminution of pressure, we must, therefore, in calculations of this kind, substitute a mean value of this co-efficient.

*Example.* In what time will a rectangular cistern, of 14 square feet section, empty itself through a round orifice at the bottom, of 2 inches width, if the original head of water amount to 4 feet? The time of efflux would be theoretically:

$$t = \frac{2\,.\,14\,\sqrt{4}}{8{,}03\,.\,\frac{\pi}{4}\,(\frac{1}{6})^2} = \frac{2\,.\,14\,.\,144\,.\,2}{8{,}03\,.\,\pi} = \frac{8064}{8{,}03\,.\,\pi} = 318''\,3 = 5 \text{ min. } 3 \text{ sec.}$$

At the end of half the time of efflux, the head of water will be $= (\frac{1}{2})^2\,.\,h = \frac{1}{4}\,.\,4 = 1$ ft. Now the co-efficient of efflux, which corresponds to the head of water = 1 foot is 0,613, hence the effective time of discharge will be $= \frac{318'',3}{0{,}613} = 519''\,2$ = 8 minutes, 39 seconds:

§ 346. *Vessels of communication.*—Since for an initial head of water $h_1$, the time of efflux $t_1 = \frac{2\,G\,\sqrt{h_1}}{\mu\,F\,\sqrt{2g}}$, and for an initial head of water $h_2$ this time $t_2 = \frac{2\,G\,\sqrt{h_2}}{\mu\,F\,.\,\sqrt{2g}}$, it then follows by subtraction, that the time within which the head of water passes from $h_1$ to $h_2$, and the surface of water sinks $h_1 - h_2$ is:

$$t = \frac{2\,G}{\mu\,F\,.\,\sqrt{2g}}\,(\sqrt{h_1} - \sqrt{h_2}), \text{ or for the foot measure:}$$

$$t = 0{,}248\,\frac{G}{\mu F}\,(\sqrt{h_1} - \sqrt{h_2}).$$

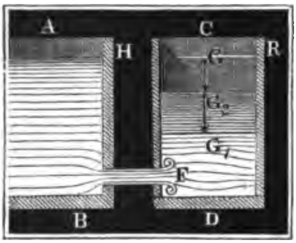
Inversely, the depression of the surface corresponding to a given time of efflux is $s = h_1 - h_2$, and is given by the formula:

$$h_2 = \left(\sqrt{h_1} - \frac{\mu\,\sqrt{2g}\,.\,F}{2\,G}\,t\right)^2, \text{ or,}$$

$$s = \frac{\mu\sqrt{2g}\,.\,F t}{G}\left(\sqrt{h_1} - \frac{\mu\sqrt{2g}}{4\,G}\,.\,t\right).$$

The same formulæ are further applicable, when a vessel *CD*, Fig. 474, is filled by another *AB* in which the water maintains a uniform height. If the transverse section of the tube of communication, or of the orifice $= F$, the transverse section of the vessel to be filled $= G$, and the original level $G\,G_1$ of the two surfaces of water $= h$, we have then, since here the surface of water $G_1$ in the second vessel is uniformly retarded, the time of filling likewise, or the time within which the second surface of water comes to the level *HR* of the first:

FIG. 474.




$$t = \frac{2\,G\sqrt{h}}{\mu F\,.\,\sqrt{2g}},$$

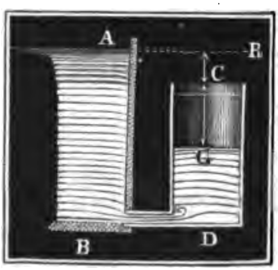
and likewise the time in which the height of level $h_1$ passes into $h_2$, and, therefore, the surface of water ascends to:

$$GG_1 = s = h_1 - h_2.$$

$$t = \frac{2\,G}{\mu F\,.\,\sqrt{2g}}\,(\sqrt{h_1} - \sqrt{h_2}).$$

*Examples.* 1. How much will the surface of water in the vessel of the last example sink in two minutes? $h_1 = 4$, $t = 2\,.\,60 = 120$, $\frac{F}{G} = \frac{\pi\,.}{14\,.\,144}$, and if we assume, further, $\mu = 0{,}605$, it follows then $h_2 = \left(\sqrt{h_1} - \mu\,.\,\sqrt{2g}\,.\,\frac{Ft}{2\,G}\right)^2$ $= \left(2 - \frac{0{,}605\,.\,8{,}03\,.\,\pi\,.\,120}{2\,.\,14\,.\,144}\right)^2 = \left(2 - 0{,}605\,.\,8{,}03\,.\,\frac{5\,.\,\pi}{168}\right)^2 = 2{,}390$ feet, and the depression sought is $s = 4 - 2{,}39 = 1{,}61$ feet.

FIG. 475.




2. What time does the water in the 18 inch wide tube *CD*, Fig. 475, require to run over if it communicates with a vessel *AB* by a short 1½ inch wide tube, and the rising surface of water *G* stands, at the beginning, 6 feet below the uniform surface of water *A*, and 4½ feet below the head *C* of the tube? It is:

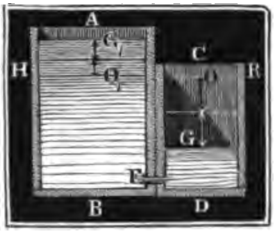
$$t = \frac{2\,G}{\mu\,\sqrt{2g}\,.\,F}\,(\sqrt{h_1} - \sqrt{h_2}),$$

$h_1 = 6$, $h_2 = 6 - 4{,}5 = 1{,}5$, $\frac{G}{F} = \left(\frac{18}{1{,}5}\right)^2 = 144$ and $\mu = 0{,}81$, whence it follows that:

$$t = \frac{2\,.\,144}{0{,}81\,.\,8{,}03}\,(\sqrt{6} - \sqrt{1{,}5}) = \frac{288\,.\,1{,}2248}{81\,.\,8{,}03} = 54{,}26 \text{ sec.}$$

If the first vessel $AB$, Fig. 476, from which the water runs into the other, has no influx, and its section $G_1$ also not to be considered as indefinitely great compared with the section $G$ of the subsequent vessel $CD$, we have then to modify the condition. If the variable distance $G_1 O_1$ of the first surface of water from the level $HR$ at which both surfaces stand at the end of the efflux $= x$, and the distance $GO$ of the second surface of water from this same plane $=y$, we have then the variable head of water $= x+y$, and the corresponding velocity of efflux: $v = \sqrt{2g\,(x+y)}$, and the quantity of water:

FIG. 476.




$$G_1 x = Gy,\ v = \sqrt{2g\left(1+\frac{G}{G_1}\right) y}.$$

The velocity with which the surface of water in the second vessel ascends is now:

$$v_1 = \frac{\mu F}{G} v = \frac{\mu F}{G}\sqrt{2g\left(1+\frac{G}{G_1}\right) y},$$

consequently the retardation: p $\frac{v^2}{2g}$

$$p = \left(\frac{\mu F}{G}\right)^2 \left(1+\frac{G}{G_1}\right) g,$$

and the time of efflux:

$$t = \frac{\mu F}{G}\sqrt{2g\left(1+\frac{G}{G_1}\right) y} : \left(\frac{\mu F}{G}\right)^2 \left(1+\frac{G}{G_1}\right) g$$

$$= \frac{2\,G\sqrt{y}}{\mu F\sqrt{2g\left(1+\frac{G}{G_1}\right)}}.$$

Let us substitute for $x$ and $y$, the initial height of level $h$, and therefore put:

$$x + y = h, \text{ or } \left(1+\frac{G}{G_1}\right) y = h,$$

and we then obtain:

$y = \dfrac{h}{1+\dfrac{G}{G_1}}$, and the time in which the two surfaces of water come to a level:

$$t = \frac{2\,G\,\sqrt{h}}{\mu F\left(1+\frac{G}{G_1}\right)\sqrt{2g}} = \frac{2\,G G_1\,\sqrt{h}}{\mu\,F\,(G+G_1)\,\sqrt{2g}}.$$

The time within which the level falls from $h$ to $h_1$, is, on the other hand:

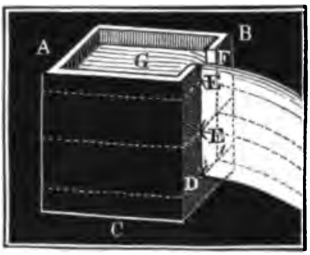
$$t = \frac{2\,G\,G_1\,(\sqrt{h}-\sqrt{h_1})}{\mu\,\sqrt{2g}\,F\,(G+G_1)}.$$

*Example.* If the section of a cistern from which water flows is 10 square feet, and the section $G$ of the recipient cistern 4 square feet; if, further, the initial level $h$ of the two surfaces amounts to 3 feet, and the cylindrical tube of communication is 1 inch wide, then the time in which the water comes in both vessels to the same level is:

$$t = \frac{2\,.\,10\,.\,4\,.\,\sqrt{3}}{0{,}82\,.\,8{,}03\,.\,\frac{\pi}{4}\,.\,\frac{14}{144}} = \frac{320\,.\,72\,.\,\sqrt{3}}{0{,}82\,.\,8{,}03\,.\,7\,\pi} = 269 \text{ sec.}$$

§ 348. *Notches in a side.*—If water flows through the notch or cut $DE$ of a prismatic cistern $ABC$, Fig. 477, to which there is no influx, the time of efflux may then be estimated in the following manner. Let us represent the transverse section of the cistern by $G$, the breadth $EF$ of the notch by $b$, and the depth $DE$ by $h$, and divide the whole aperture of efflux by horizontal lines into small slices, each of the breadth $b$ and depth $\frac{h}{n}$. At a constant pressure the discharge per second will be, $Q = \frac{2}{3}\,\mu\,b\,\sqrt{2\,gh^3}$, if we divide this into the area $\frac{G\,h}{n}$ of a stratum of water, we shall then obtain the time of efflux $\tau = \frac{Gh}{\frac{2}{3}\,\mu\,nb\,\sqrt{2\,g\,h^3}}$, which we may write: $\frac{3\,Gh}{2\,\mu\,n\,b\,\sqrt{2\,g}}\,.\,h^{-\frac{3}{2}}$.

FIG. 477.




Now, to obtain the time of efflux $t$ for a quantity of water $G\,(h - h_1)$, or to determine the time in which the head of water above the line $DE = h$ sinks to $DE_1 = h_1$, let us make $h_1 = \frac{m}{h}\,h$,

and therefore $h_1$ to consist of $m$ parts, and let us now substitute for $h^{-\frac{3}{2}}$, successively:

$G$ = Surface of water = $f x$   velocity = $\sqrt{2gx}$

elementary discharge $G\,dx$ also $F \mu \sqrt{2gx}\cdot dt$

$\mu F \sqrt{2gx}$ = discharge in unit of time.

$\therefore\ dt = \frac{G\,dx}{\mu F \sqrt{2gx}}\ \therefore\ t = \int \frac{G\,dx}{\mu F \sqrt{2gx}}$.

prismatic vessels $G = f x$ = const.

$t = \int_0^h \frac{G x^{-\frac{1}{2}} dx}{\mu F \sqrt{2g}} = \frac{2 G h^{\frac{1}{2}}}{\mu F \sqrt{2g}} = \frac{2Gh}{\mu F \sqrt{2gh}}$

$Q = \mu F \sqrt{2gh}\ \therefore\ t = \frac{2Gh}{Q}$

Notch breadth ... in depth.

... section ... $Q = \frac{2}{3}\mu b \sqrt{2g}\, x^{\frac{3}{2}}$

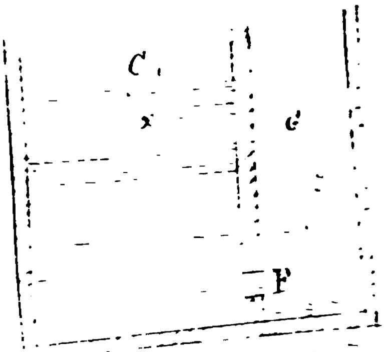
... $dt = \frac{G\,dh}{\frac{2}{3}\mu b \sqrt{2g}\, x^{\frac{3}{2}}}$

$\therefore\ t = \frac{G}{\frac{2}{3}\mu b \sqrt{2g}} \int_h x^{-\frac{3}{2}} dx = \frac{3G}{\mu b \sqrt{2g}}\left(\frac{1}{\sqrt{h}} - \ldots\right)$

Ans.

$$= \frac{19800}{8{,}03\,\mu}(1{,}4142 - 0{,}8944) = \frac{19800\,.\,0{,}5198}{8{,}03\,\mu} = \frac{1281}{\mu}\ \text{sec.}$$

$$t = \frac{2\,G\,\sqrt{h}}{\mu F\left(1+\frac{G}{G_1}\right)\sqrt{2g}} = \frac{2\,GG_1\,\sqrt{h}}{\mu\,F\,(G+G_1)\,\sqrt{2g}}.$$



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above the line $DE = h$ sinks to $DE_1 = h_1$, let us make $h_1 = \frac{m}{h}\,h$,

and therefore $h_1$ to consist of $m$ parts, and let us now substitute for $h^{-\frac{3}{2}}$, successively:

$$\left(\frac{m}{n}h\right)^{-\frac{3}{2}}, \left(\frac{m+1}{n}h\right)^{-\frac{3}{2}}, \left(\frac{m+2}{n}h\right)^{-\frac{3}{2}} \ldots \left(\frac{nh}{n}\right)^{-\frac{3}{2}},$$

and finally add the results obtained. In this manner we shall obtain the time required:

$$t = \frac{3\,Gh}{2\,\mu\, n\, b\,\sqrt{2\,g}}\left[\left(\frac{mh}{n}\right)^{-\frac{3}{2}} + \left(\frac{m+1}{n}h\right)^{-\frac{3}{2}} + \ldots + \left(\frac{nh}{n}\right)^{-\frac{3}{2}}\right]$$

$$= \frac{3\,Gh}{2\,\mu\, n\, b\,\sqrt{2g}} \cdot \frac{h^{-\frac{3}{2}}}{n^{-\frac{3}{2}}}\left(m^{-\frac{3}{2}} + (m+1)^{-\frac{3}{2}} + \ldots + n^{-\frac{3}{2}}\right)$$

$$= \frac{3\,Gh^{-\frac{1}{2}}}{2\,\mu n^{-\frac{1}{2}}\, b\,\sqrt{2g}}\Big[\left(1^{-\frac{3}{2}} + 2^{-\frac{3}{2}} + 3^{-\frac{3}{2}} + \ldots + n^{-\frac{3}{2}}\right) - \left(1^{-\frac{3}{2}} + 2^{-\frac{3}{2}} + 3^{-\frac{3}{2}} + \ldots m^{-\frac{3}{2}}\right)\Big],$$

or, from the "Ingenieur," Arithmetic, § 28:

$$t = \frac{3\,Gh^{-\frac{1}{2}}}{2\mu n^{-\frac{1}{2}}\, b\,\sqrt{2g}}\left(\frac{n^{-\frac{3}{2}-1}}{-\frac{3}{2}+1} - \frac{m^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}\right)$$

$$= \frac{3\,Gh^{\frac{1}{2}}}{2\,\mu\, b\,\sqrt{2\,gh}} \cdot 2\left(m^{-\frac{1}{2}} - n^{-\frac{1}{2}}\right)$$

$$= \frac{3\,G}{\mu\, b\,\sqrt{2\,g\,h}}\left[\left(\frac{m}{n}\right)^{-\frac{1}{2}} - 1\right]$$

$$= \frac{3G}{\mu b\sqrt{2g}}\left[\left(\frac{m}{n}h\right)^{-\frac{1}{2}} - h^{-\frac{1}{2}}\right] = \frac{3\,G}{\mu b\sqrt{2g}}\left(\frac{1}{\sqrt{h_1}} - \frac{1}{\sqrt{h}}\right).$$

Let $h_1 = 0$, we have then $\frac{1}{\sqrt{h_1}}$, and therefore also $t = \infty$; an indefinite time, therefore, is required for the water to run down to the sill.

*Example.* If the water flows through a notch in a side, of 8 inches in breadth, from a reservoir 110 feet long and 40 feet broad, what time will it require to pass from a head of water of 15 inches to one of 6 inches?

$$t = \frac{3\,.\,110\,.\,40}{\mu\,.\,\frac{2}{3}\,.\,8{,}03}\left(\frac{1}{\sqrt{0{,}5}} - \frac{1}{\sqrt{1{,}25}}\right) = \frac{19800}{\mu\,.\,8{,}03}\left(\sqrt{2} - \sqrt{\tfrac{4}{5}}\right)$$

$$= \frac{19800}{8{,}03\,\mu}(1{,}4142 - 0{,}8944) = \frac{19800\,.\,0{,}5198}{8{,}03\,\mu} = \frac{1281}{\mu}\text{ sec.}$$

If we assume the co-efficient $\mu = 0{,}60$, the effective time of efflux will be $t = \frac{1281}{0{,}6} = 2135$ sec. $= 35$ min. $58$ sec.

*Remark.* We may put for a rectangular lateral opening, approximatively:

$$t = \frac{2\,G}{\mu\,F\,\sqrt{2\,g}}\left[\left(\sqrt{h_1} - \sqrt{h_2}\right) - \frac{a^2}{288}\left(\sqrt{h_1^{-3}} - \sqrt{h_2^{-3}}\right)\right],$$

and $F$ and $G$ represent the transverse sections of the opening and of the vessel, $a$ the depth of the opening, $h_1$ the head of water at the commencement, $h_2$ that at the end of the efflux. If $h_2 = \frac{a}{2}$, the opening becomes a notch, and we must then apply the proper formula.

§ 349. *Wedge and pyramidal-shaped vessels.*—If the cistern of discharge $ABF$, Fig. 478, forms a horizontal triangular prism, the time of efflux may be found in the following manner. Let us divide the height $CE = h$ into $n$ equal parts, and carry horizontal planes through the points of division, let us then decompose the whole quantity of water into equally thick strata of equal length $AD = l$, and of breadths diminishing downwards. If the breadth of the upper stratum $BD = b$, we have then the breadth of another stratum $D_1B_1$, which stands about $CE_1 = x$ above the orifice $F$, lying at the lower edge,

FIG. 478.


$b_1 = \frac{x}{h}\,b$, and its volume $= b_1\,l\,.\,\frac{h}{n} = \frac{b\,l\,x}{n}$. But now the discharge referred to a unit of time is: $Q = \mu\,F\,\sqrt{2\,g\,x}$, hence then the small time in which the surface of water sinks about $\frac{h}{n}$ is $\tau = \frac{bl}{n}\,x : \mu\,F\,\sqrt{2\,g\,x} = \frac{b\,l}{n\mu F\sqrt{2g}}\,.\,\sqrt{x}$. Finally, since the sum of all $x^{\frac{1}{2}}$ from $x = \frac{h}{n}$ to $x = \frac{nh}{n} = \left(\frac{h}{n}\right)^{\frac{1}{2}}\,.\,\frac{n^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}\,n\,h^{\frac{1}{2}}$, we have the time for the discharge of the entire prism of water:

$$t = \frac{b\,l}{n\mu\,F\sqrt{2g}}\,.\,\tfrac{2}{3}\,n\,h^{\frac{1}{2}} = \tfrac{2}{3}\,\frac{b\,l}{\mu\,F\sqrt{2\,g}}\,.\,h^{\frac{1}{2}} = \tfrac{4}{3}\,\frac{\frac{1}{2}\,b\,l\,h}{\mu\,F\,\sqrt{2\,g\,h}}$$

$= \frac{4}{3}\,.\,\frac{V}{\mu\,F\,c}$, if $V$ represents the whole quantity of water and

§ 341

$$y = \frac{bx}{l}$$

$$= \frac{4}{3} \frac{V}{\mu F c}$$

$$= \frac{2}{3} \frac{\lambda y^2 x}{F} = \frac{4}{3} \frac{V}{\mu F c}$$

$$y = \frac{x^2}{h^2}$$

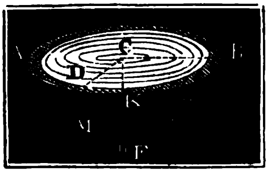
$$= \frac{6}{5} \frac{V}{\mu F c}$$

$$(l^2 \cdot r x^2 - x^3)\,dx$$

$$x = \frac{2}{3}$$

$$\frac{2}{15} \lambda \frac{(10 r - 3h)\, h}{\mu F}$$

FIG. 479.



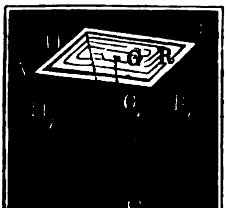
$c$ the initial velocity of efflux. Here the water, therefore requires $\frac{1}{3}$ more time than if the velocity of efflux $c$ were uniform.

If the vessel $ABF$, Fig. 479, forms an erect paraboloid, we then have for the ratio of the radii $KM = y$ and $CD = b : \frac{y}{b} = \frac{\sqrt{x}}{\sqrt{h}}$, and hence the ratio of the principal sections $\frac{G_1}{G} = \frac{y^2}{b^2} = \frac{x}{h}$, consequently $G_1 = \frac{Gx}{h}$ and the contents of a stratum of water $= G_1 \cdot \frac{h}{n} = \frac{Gx}{n}$. The perfect accordance of this expression with that found for the triangular prism admits of our here putting $t = \frac{4}{3} \cdot \frac{\frac{1}{2} Gh}{\mu F \sqrt{2gh}}$, or, as $V = \frac{1}{2} Gh$ (§ 118), also $t = \frac{4}{3} \cdot \frac{V}{\mu F c}$.

The formula may be used in many other cases for the approximative determination of the time of efflux, especially for that of the emptying of reservoirs. It is especially true in all cases where the horizontal sections increase as the distances from the bottom.

If, lastly, a vessel $ABF$ be prismatic, Fig. 480, then $G_1 : G = x^2 : h^2$, and hence $G_1 = \frac{Gx^2}{h^2}$ further the contents of the stratum

FIG. 480.



$H_1 R_1 : \frac{G_1 h}{n} = \frac{Gx^2}{nh}$, and the time for its discharge:

$$\tau = \frac{Gx^2}{nh} : \mu F \sqrt{2gx} = \frac{G}{n \mu Fh \sqrt{2g}} \cdot x^{\frac{3}{2}}.$$

But as the sum of all the $x^{\frac{3}{2}}$ taken from $x = \frac{h}{n}$ to $x = \frac{nh}{n} = \left(\frac{h}{n}\right)^{\frac{3}{2}} \cdot \frac{n^{\frac{5}{2}}}{\frac{5}{2}} = \frac{2}{5} n h^{\frac{3}{2}}$,

it follows that the time for the emptying of the whole pyramid is:

$$t = \frac{G}{n \mu F h \sqrt{2g}} \cdot \tfrac{2}{3}\, n h^{\frac{3}{2}} = \tfrac{2}{3} \cdot \frac{G h^{\frac{1}{2}}}{\mu F \sqrt{2g}} = \tfrac{4}{3} \cdot \frac{\tfrac{1}{2}\, G h}{\mu F \sqrt{2gh}},$$

or $\frac{1}{2}\, Gh$ put $= V$ $t = \frac{4}{3} \cdot \frac{V}{\mu F c}$.

As in this efflux the initial velocity of flow decreases gradually from $c$ to zero, the time of efflux is then $\frac{1}{3}$th greater than if the velocity $c$ remained uniform.

*Example.* In what time will a pond, whose surface has an area of 765000 square feet, empty itself, if there be a conduit 15 feet below the surface, and at the deepest place, which forms a channel 15 inches wide and 50 feet long? Theoretically, the time of efflux is $t = \frac{4}{3} \cdot \frac{V}{F\sqrt{2gh}} = \frac{4}{3} \cdot \frac{1}{2} \cdot \frac{765000 \,.\, 15}{\frac{\pi}{4} \cdot \left(\frac{5}{4}\right)^2 \,.\, 8{,}03 \sqrt{15}}$

$$= \frac{19584000}{\pi \,.\, 8{,}03\sqrt{15}} = 197762 \text{ sec.}$$

But now the co-efficient of resistance for entrance into the channel, inclined about 45°, is, $\zeta = 0{,}505 + 0{,}327$ (*see* § 323) $= 0{,}832$, and the resistance of the conduit due to friction $= 0{,}025\, \frac{l}{d} \cdot \frac{v^2}{2g} = 0{,}025 \,.\, \frac{50}{\frac{5}{4}} \cdot \frac{v^2}{2g} = \frac{v^2}{2g}$; hence, the complete co-efficient of efflux for the channel is:

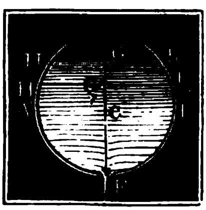
$\mu = \frac{1}{\sqrt{1 + 0{,}832 + 1}} = \frac{1}{\sqrt{2{,}832}} = 0{,}594$, and the time of efflux demanded:

$t = 197762 : 0{,}594 = 33029{,}3 = 91$ hours, 45 minutes, 52 seconds.

§ 350. *Spherical and obelisk-shaped vessels.*—By means of the formula of the last paragraph, we may now find the times of efflux for many other vessels, such as spherical, pontoon-shaped, pyramidal, &c. For the emptying of a spherical segment $AB$, Fig. 481, we obtain:

$$t = \tfrac{4}{3}\frac{\pi r h^2}{\mu F\sqrt{2gh}} - \tfrac{2}{5} \cdot \frac{\pi h^3}{\mu F \sqrt{2gh}}$$

FIG. 481.



$$= \tfrac{2}{15}\, \pi\, \frac{(10\, r - 3\, h)\, h^{\frac{3}{2}}}{\mu F \sqrt{2g}},$$

therefore, for the emptying of a full sphere, where $h = 2\, r$,

$$t = \frac{16\, \pi\, r^2 \sqrt{2\, r}}{15\, \mu F \sqrt{2g}},$$

and for that of half a sphere where:

$$h = r,\; t = \frac{14\, \pi\, r^2 \sqrt{r}}{15\, \mu F \sqrt{2g}}.$$

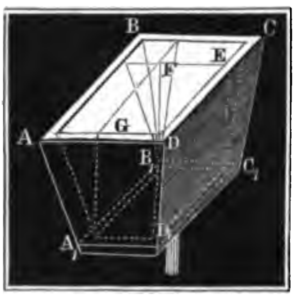
Here the horizontal stratum $H_1R_1$ corresponding to the depth $FG_1 = x = G_1 = \pi\, x\, (2\, r - x) \cdot \frac{h}{n} = \frac{2\pi r h x}{n} - \frac{\pi h x^2}{n}$, therefore:

$$\tau = \frac{2\pi r h}{n\mu F\sqrt{2g}} \cdot x^{\frac{1}{2}} - \frac{\pi h}{n\mu F\sqrt{2g}} \cdot x^{\frac{3}{2}};$$

as the first part of this expression agrees with the formula for the emptying of a prismatic, and the second part for the emptying of a pyramidal vessel, if we put first $2\pi r h$ in place of $bl$, and secondly $\pi h^2$ in place of $G$, we shall obtain by means of the difference of the times of emptying of a prismatic and pyramidal vessel, found in the former paragraph:

$$t = \tfrac{2}{3} \cdot \frac{b l h}{\mu F\sqrt{2gh}}, \text{ and } t = \tfrac{2}{5} \cdot \frac{G h}{\mu F\sqrt{2gh}},$$

FIG. 482.




the time also of the emptying of a spherical segment.

The above formula may be likewise applied to the case of an obelisk or pontoon-shaped vessel $ACD_1$, Fig. 482, since this is composed of a parallelepiped, two prisms, and a pyramid. Let $b$ be the breadth at top $AD$, and $b_1$ the breadth $A_1D_1$ at bottom, $l$ the length at top $AB$, and $l_1$ the length at bottom $A_1B_1$, and lastly, $h$ the height of the vessel, we have then for the area of the surface $AC: bl = b_1l_1 + b_1(l - l_1) + l_1(b - b_1) + (l - l_1)(b - b_1)$, which of $b_1l_1$ belongs to the parallelepiped $A_1C_1EG$, $b_1(l - l_1) + l_1(b - b_1)$ to the two prisms $CFB_1C_1$, and $AFB_1A_1$ and $(l - l_1)(b - b_1)$ to the pyramid $BFB_1$. But now the time of efflux for the parallelepiped, whose base is $b_1l_1: t_1 = \frac{2b_1l_1\sqrt{h}}{\mu F\sqrt{2g}}$; further, that for the two triangular prisms

$$t_2 = \tfrac{2}{3}\frac{[b_1(l - l_1) + l_1(b - b_1)]\sqrt{h}}{\mu F\sqrt{2g}},$$

and, lastly, for the pyramid:

$$t_3 = \tfrac{2}{5}\frac{(l - l_1)(b - b_1)\sqrt{h}}{\mu F\sqrt{2g}};$$

hence the time of discharge for the whole vessel is:

$$t = t_1 + t_2 + t_3$$

$$= [30 b_1 l_1 + 10 b_1 (l - l_1) + 10 l_1 (b - b_1) + 6 (l - l_1)(b - b_1)] \frac{\sqrt{h}}{15 \mu F \sqrt{2g}}$$

$$= [3\, bl + 8 b_1 l_1 + 2(b l_1 + b_1 l)] \frac{2\sqrt{h}}{15\, \mu F \sqrt{2g}}.$$

If $\frac{b_1}{l_1} = \frac{b}{l}$, we have then a truncated pyramid to consider. Let the one base $bl = G$ and the other $b_1 l_1 = G_1$, we then obtain:

$$t = (3\, G + 8\, G_1 + 4\, \sqrt{G G_1})\, \frac{2\, \sqrt{h}}{15\, \mu F \sqrt{2g}}.$$

It would be easy to show that this formula holds true also for every trilateral or multilateral pyramid.

*Example.* An obelisk-shaped water-cask is 5 feet long, and 3 feet broad at top, and at the depth of 4 feet, that is, at the level of a short horizontal discharge-tube, 1 inch in width and 3 inches in length, it is 4 feet long and 2 feet broad, what time will be required for the water in the full cask to sink $2\frac{1}{2}$ feet? The time for emptying is, $\mu$ being taken $= 0{,}815$:

$$t = [8 \cdot 4 \cdot 2 + 3 \cdot 5 \cdot 3 + 2\,(3 \cdot 4 + 5 \cdot 2)]\, \frac{2\, \sqrt{4}}{15 \cdot 0{,}815 \cdot \frac{\pi}{4} \cdot \left(\frac{1}{12}\right)^2 \cdot 8{,}03}$$

$$= \frac{153 \cdot 4 \cdot 4 \cdot 144}{15 \cdot 0{,}815 \cdot 8{,}03 \cdot \pi} = 153 \cdot \frac{2304}{12{,}225 \cdot 8{,}03\, \pi} = 153 \cdot 7{,}470 = 1078 \text{ sec.}$$

As the level $4 - 2\frac{1}{2} = 1\frac{1}{2}$ feet above the tube $l = l_1 + \frac{3}{8} = 4\frac{3}{8}$ and $b = b_1 + \frac{3}{8} = 2\frac{3}{8}$ feet, hence the time for emptying if the vessel be filled only up to this level, is:

$$t_1 = [8 \cdot 4 \cdot 2 + 3 \cdot \tfrac{35}{8} \cdot \tfrac{19}{8} + 2\,(2 \cdot \tfrac{35}{8} + 4 \cdot \tfrac{19}{8})] \cdot \frac{1152\, \sqrt{1{,}5}}{15 \cdot 0{,}815 \cdot 8{,}03\, \pi}$$

$= 131{,}672 \cdot 4{,}5749 = 602{,}38$ sec. The difference of the times found gives the time in which the surface of water originally at the top of the vessel sinks $2\frac{1}{2}$ feet.

§ 351. *Irregular vessels.*—When we have to find the time of efflux for an irregularly formed vessel *HFR*, Fig. 483, we must

FIG. 483.


apply Simpson's rule as a method of approximation. If we divide the whole mass of water into four equally thick strata, and the heads of water $G_0$, $G_1$, $G_2$, $G_3$, $G_4$, corresponding to the horizontal

slices, represented by $h_0, h_1, h_2, h_3, h_4$, the time of efflux will be given by Simpson's rule.

$$t = \frac{h_0 - h_4}{12\,\mu F \sqrt{2g}} \left(\frac{G_0}{\sqrt{h_0}} + \frac{4\,G_1}{\sqrt{h_1}} + \frac{2\,G_2}{\sqrt{h_2}} + \frac{4\,G_3}{\sqrt{h_3}} + \frac{G_4}{\sqrt{h_4}}\right).$$

In assuming six strata:

$$t = \frac{h_0 - h_6}{18\,\mu F \sqrt{2g}} \left(\frac{G_0}{\sqrt{h_0}} + \frac{4\,G_1}{\sqrt{h_1}} + \frac{2\,G_2}{\sqrt{h_2}} + \frac{4\,G_3}{\sqrt{h_3}} + \frac{2\,G_4}{\sqrt{h_4}} + \frac{4\,G_5}{\sqrt{h_5}} + \frac{G_6}{\sqrt{h_6}}\right).$$

The discharge in the first case is:

$Q = \frac{h_0 - h_4}{12}\,(G_0 + 4\,G_1 + 2\,G_2 + 4\,G_3 + G_4)$, in the second:

$$Q = \frac{h_0 - h_6}{18}\,(G_0 + 4\,G_1 + 2\,G_2 + 4\,G_3 + 2\,G_4 + 4\,G_5 + G_6).$$

When the form and dimensions of the vessel of efflux are not known, we may then very well calculate the discharge by the heads of water noted in equal intervals of time. Let $t$ be one such interval, we have then for apertures at the bottom and sides:

$$Q = \frac{\mu F t \sqrt{2g}}{3}\,(\sqrt{h_0} + 4\sqrt{h_1} + 2\sqrt{h_2} + 4\sqrt{h_3} + \sqrt{h_4}),$$

and for divisions or notches in a side.

$$Q = \tfrac{2}{9}\,\mu\, b\, t\, \sqrt{2g}\,(\sqrt{h^3_0} + 4\sqrt{h^3_1} + 2\sqrt{h^3_2} + 4\sqrt{h^3_3} + \sqrt{h^3_4}).$$

*Example.* In what time will the surface of water in a pond sink 6 feet, if the sluice forms a half cylinder, 18 inches wide, 9 inches deep, and 60 feet long, and the surfaces of water have the following areas?

| | | | |
|---|---|---|---|
| $G_0$, at 20 | feet head of water, | = 600000 | square feet. |
| $G_1$, „ 18,5 | „ „ | = 495000 | „ |
| $G_2$, „ 17,0 | „ „ | = 410000 | „ |
| $G_3$, „ 15,5 | „ „ | = 325000 | „ |
| $G_4$, „ 14,0 | „ „ | = 265000 | „ |

$F = \frac{\pi}{8} \cdot (\frac{3}{2})^2 = \frac{9\,\pi}{32} = 0{,}8836$ square feet. Let the co-efficient of resistance for the entrance $= 0{,}832$, and that for the friction:

$= 0{,}025 \cdot \frac{l}{d} = 0{,}025 \cdot 60 \cdot 1{,}091 = 1{,}6356$, then is the co-efficient of efflux

$$\mu = \frac{1}{\sqrt{1 + 0{,}832 + 1{,}6365}} = \frac{1}{\sqrt{3{,}4685}} = 0{,}537,$$

and $\mu F \sqrt{2g} = 0{,}537 \cdot 0{,}8836 \cdot 8{,}03 = 3{,}8091$. Now

$\frac{G_0}{\sqrt{h_0}} = \frac{600000}{\sqrt{20}} = 134170$, $\frac{G_1}{\sqrt{h_1}} = \frac{495000}{\sqrt{18{,}5}} = 115090$,

$\frac{G_2}{\sqrt{h_2}} = \frac{410000}{\sqrt{17}} = 99440$, $\frac{G_3}{\sqrt{h_3}} = \frac{325000}{\sqrt{15{,}5}} = 82550$,

$\frac{G_4}{\sqrt{h_4}} = \frac{265000}{\sqrt{14}} = 70830$; hence, then, the time of efflux follows:

$$t = \frac{6}{12 \cdot 3{,}7518} (134170 + 4 \cdot 115090 + 2 \cdot 99440 + 4 \cdot 82550 + 70830)$$
$$= \frac{1194440}{7{,}6182} = 157718 \text{ sec.} = 43 \text{ hours, } 28 \text{ min. } 3 \text{ sec.}$$

The discharge is:

$$Q = \tfrac{6}{12} (600000 + 4 \cdot 495000 + 2 \cdot 410000 + 4 \cdot 325000 + 265000)$$
$$= \frac{4965000}{2} = 2482500 \text{ cubic feet.}$$

§ 352. *Influx and efflux.*—If the vessel during the efflux from below has an influx to it from above, the determination of the time in which the surface of water rises or falls a certain height becomes more complicated, so that we must be satisfied generally with but an approximate determination. If the ~~discharge~~ influx per second $Q_1$ is $> \mu F \sqrt{2gh}$, then there is a rise, and if $Q_1 < \mu F \sqrt{2gh}$, a fall of the surface. Moreover, a state of permanency occurs every time that the head of water is increased or decreased by $k = \frac{1}{2g}\left(\frac{Q_1}{\mu F}\right)^2$. The time $\tau$, in which the variable head of water $x$ increases by the small amount $\xi$, is given by the equation

$$G_1 \xi = Q_1 \tau - \mu F \sqrt{2gx} \cdot \tau,$$

and, on the other hand, the time in which it sinks the height $\xi$, by

$$G_1 \xi = \mu F \sqrt{2gx} \cdot \tau - Q_1 \tau.$$

Hence we have in the first case $\tau = \frac{G_1 \xi}{Q_1 - \mu F \sqrt{2gx}}$, and in the second $\tau = \frac{G_1 \xi}{\mu F \sqrt{2gx} - Q_1}$. By the application of Simpson's rule we then obtain the time of efflux, during which the lowering surface passes from $G_0$ to $G_1, G_2 \ldots$, and the head of water from $h_0$ to $h_1, h_2 \ldots$

$$t = \frac{h_0 - h_4}{12}\left[\frac{G_0}{\mu F \sqrt{2gh_0} - Q_1} + \frac{4\,G_1}{\mu F \sqrt{2gh_1} - Q_1} + \frac{2\,G_2}{\mu F \sqrt{2gh_2} - Q_1} + \right.$$
$$\left. + \frac{4\,G_3}{\mu F \sqrt{2gh_3} - Q_1} + \frac{G_4}{\mu F \sqrt{2gh_4} - Q_1}\right],$$

or, more simply, if we represent $\frac{Q_1}{\mu F \sqrt{2g}}$ by $\sqrt{k}$,

$$= \frac{h_0 - h_4}{12 \mu F \sqrt{2g}}\left[\frac{G_0}{\sqrt{h_0} - \sqrt{k}} + \frac{4\,G_1}{\sqrt{h_1} - \sqrt{k}} + \frac{2\,G_2}{\sqrt{h_2} - \sqrt{k}} + \right.$$
$$\left. + \frac{4\,G_3}{\sqrt{h_3} - \sqrt{k}} + \frac{G_4}{\sqrt{h_4} - \sqrt{k}}\right].$$

§ 352 $G\,dx = Q_1\,dt - Q\,dt$

$$Q_1 = \tfrac{2}{3}\mu b \sqrt{2g}\,k^{\frac{3}{2}},\quad Q = \tfrac{2}{3}\mu b \sqrt{2g}\,x^{\frac{3}{2}}$$

$$G\,dx = dt\left(\tfrac{2}{3}\mu b\sqrt{2gk^3} - \tfrac{2}{3}\mu b\sqrt{2gx^3}\right)$$

$$dt = \frac{G}{\frac{2}{3}\mu b\sqrt{2g}}\cdot\frac{dx}{\sqrt{k^3}-\sqrt{x^3}} \qquad \text{put } x = z^2 \therefore dx = 2z\,dz$$

$$\sqrt{x^3} = z^3 \qquad dt = \frac{G}{\frac{2}{3}\mu b\sqrt{2g}}\cdot\frac{2z\,dz}{\sqrt{k^3}-z^3}$$

if $z^3 = \sqrt{k^3}$, $z = \sqrt{k}$ hence the factors of the denom. are $(\sqrt{k}-z)(z^2+z\sqrt{k}+k)$, $z^2+z\sqrt{k}+k = (z^2+z\sqrt{k}+\frac{k}{4}) + \frac{3}{4}k = (z+\frac{\sqrt{k}}{2})^2 + \frac{3}{4}k$. By Church's Calculus ...

and $\dfrac{2z}{\sqrt{k^3}-z^3} = \dfrac{A}{\sqrt{k}-z} + \dfrac{Mz+N}{(z+\frac{\sqrt{k}}{2})^2+\frac{3k}{4}}$, whence

$$2z = Az^2 + Az\sqrt{k} + Ak + Mz\sqrt{k} - Mz^2 + N\sqrt{k} - Nz$$

$$2 = (M+A)\sqrt{k} - N,\quad 0 = A - M,\quad 0 = Ak + N\sqrt{k}$$

hence $A = M$, $2 = 2A\sqrt{k} - N$, $N = 2A\sqrt{k} - 2$

$\therefore -Ak = 2Ak - 2\sqrt{k}$, $A = M = \frac{2}{3\sqrt{k}}$, $N = \frac{4}{3} - 2 = -\frac{2}{3}$

$$\frac{2z\,dz}{\sqrt{k^3}-z^3} = \frac{2\,dz}{3k-3z\sqrt{k}} + \frac{\frac{2}{3}k^{-\frac{1}{2}}(z-\frac{2}{3})dz}{(z+\frac{\sqrt{k}}{2})^2+\frac{3}{4}k}$$

put $(z+\frac{\sqrt{k}}{2}) = y$, $dz = dy$, $z = y - \frac{\sqrt{k}}{2}$

$$\frac{2z\,dz}{\sqrt{k^3}-z^3} = \frac{2\,dz}{3k-3z\sqrt{k}} + \frac{\frac{2}{3}k^{-\frac{1}{2}}y\,dy}{y^2+\frac{3k}{4}} - \frac{dy}{y^2+\frac{3k}{4}}$$

$$\int\frac{2z\,dz}{\sqrt{k^3}-z^3} = -\frac{2}{3\sqrt{k}}\,l(3k-3z\sqrt{k}) + \frac{1}{3\sqrt{k}}\,l\left(y^2+\frac{3k}{4}\right)$$

$$-\frac{2}{\sqrt{3k}}\tan^{-1}\frac{2y}{\sqrt{3k}} = \frac{1}{3\sqrt{k}}\,l\,\frac{y^2+\frac{3}{4}k}{(3k-3z\sqrt{k})^2}$$

$$-\frac{2}{\sqrt{3k}}\tan^{-1}\frac{2y}{\sqrt{3k}}$$

put ... z its value $\sqrt{x}$ & for $y \sqrt{x}$ $+\frac{1}{2}\sqrt{k}$

$$\int \frac{z^2 dz}{\sqrt{k^3} - z^3} = \frac{1}{3\sqrt{k}}\, l\left\{\frac{\lambda \ldots k + k}{(3k - 3\sqrt{k\lambda})^2} - \frac{2}{\sqrt{3k}} \tan^{-1}\left(\frac{2 \ldots}{\ldots}\right.\right.$$

substitute for $\frac{2}{3}\mu b \sqrt{2gk^3}$ its value $Q_1$

take the integral between the l...

of $h$ & $h_1$

$$t = -\frac{l}{\frac{2}{3}\mu b\sqrt{2g}} \int_h^{h_1} \frac{z^2 dz}{\sqrt{k^3} - z^3} = \frac{lk}{3Q_1}\left[l\left(\frac{h_1 + \sqrt{kh_1} + k}{(3k - 3\sqrt{h_1 k})^2}\right) - \right.$$

$$\left. - l\left(\frac{h + \sqrt{hk} + k}{(3k - 3\sqrt{hk})^2}\right) - 2\sqrt{3}\, \tan^{-1}\left(\frac{2\sqrt{h_1} + \sqrt{k}}{\sqrt{3k}}\right) + \ldots \tan^{-1}\frac{2\sqrt{h} + \sqrt{k}}{\sqrt{3k}}\right] =$$

$$\frac{lk}{3Q_1}\left[l\left(\frac{h_1 + \sqrt{kh_1} + k}{h + \sqrt{kh} + k}\cdot\frac{(\sqrt{k} - \sqrt{h})^2}{(\sqrt{k} - \sqrt{h_1})^2}\right) + \sqrt{12} \ldots\right.$$

$$\left. - \tan^{-1}\left(\frac{2\sqrt{h_1} + \sqrt{k}}{\sqrt{3k}}\right)\right]\Big]$$ to reduce the

two terms we have given the an[gles]

of l[ar]ges, and it is required to fin[d the]

tangent of their difference by [the]

formula $\tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b}$

substitute for $a$ & $b$ in this formula

$\frac{2\sqrt{h} + \sqrt{k}}{\sqrt{3k}}$ & $\frac{2\sqrt{h_1} + \sqrt{k}}{\sqrt{3k}}$ and we have.

$$\tan(a - b) = \frac{\frac{2(\sqrt{h} - \sqrt{h_1})}{\sqrt{3k}}}{1 + \frac{(2\sqrt{h_1} + \sqrt{k})(2\sqrt{h} + \sqrt{k})}{3k}} = \frac{(\sqrt{h} - \ldots)}{3k + 2 \ldots}$$

hence the last two terms in the ...

for $t$ reduces to $\sqrt{12}(a-b) =$

$$\sqrt{12}\ \cancel{\tan}^{-1}\frac{(\sqrt{h}-\sqrt{h_1})\sqrt{12k}}{3k+(2\sqrt{h}+\sqrt{k})(2\sqrt{h_1}+\sqrt{k})} \text{" is m.}$$

$$t = \frac{Sk}{3Q_1}\left[\text{hyp log}\frac{(\sqrt{h}-\sqrt{k})^2(h_1-\sqrt{h_1k}+k)}{(\sqrt{h_1}-k)^2(h-\sqrt{hk}+k)}\right.$$
$$\left. +\sqrt{12}\cdot\text{arc}\left(\tan = \frac{(\sqrt{h}-\sqrt{h_1})\sqrt{12k}}{3k+(2\sqrt{h}+\sqrt{k})(2\sqrt{h_1}+\sqrt{k})}\right)\right]\text{"}$$

---

$Q$ = Efflux } unit of time

$Q_1$ = Influx } | $k$ head by influx

$Sdx = Qdt - Q_1dt.$ const. |

Prismatic $\quad Q = \mu F\sqrt{2gx}$ }

$\quad Q_1 = \mu F\sqrt{2gk}$ }

$dt = \frac{S\,dx}{\mu F\sqrt{2g}(\sqrt{x}-\sqrt{k})} = C\frac{dx}{\sqrt{x}-\sqrt{k}}$ put $\sqrt{x}-\sqrt{k} = z$,

$dx = 2\frac{dz}{x^{-\frac{1}{2}}} = 2dz\,x^{\frac{1}{2}} \therefore dx = 2dz(z+\sqrt{k})$

$dt = C\left(2dz + \frac{2\sqrt{k}\,dz}{z}\right)$

$\therefore t = 2C\left[\sqrt{x}-\sqrt{k}+\sqrt{k}\log(\sqrt{x}-\sqrt{k})\right]_{h_1}^{h}$

$= C\left[\sqrt{h}-\sqrt{h_1}+\sqrt{k}\log\frac{\sqrt{h}-\sqrt{k}}{\sqrt{h_1}-\sqrt{k}}\right]$

$\therefore t = \frac{2S}{\mu F\sqrt{2g}}\left[\sqrt{h}-\sqrt{h_1}+\sqrt{k}\log\frac{\sqrt{h}-\sqrt{k}}{\sqrt{h_1}-\sqrt{k}}\right]$

If the vessel is prismatic, and has a uniform transverse section $G$, we then have:

$$t = \frac{2\ G}{\mu\ F\ \sqrt{2g}}\left(\sqrt{h} - \sqrt{h_1} + \sqrt{k}\ .\ hyp.\ log.\left(\frac{\sqrt{h}-\sqrt{k}}{\sqrt{h_1}-\sqrt{k}}\right)\right),$$

the time in which the head of water passes from $h$ to $h_1$. Since for:

$$h_1 = k,\ \frac{\sqrt{h}-\sqrt{k}}{\sqrt{h_1}-\sqrt{k}} = \frac{\sqrt{h}-\sqrt{k}}{0} = \infty,$$

it follows that the condition of permanency takes place indefinitely late.

The following formula is the result of investigation for a wier or notch in a side.

$$t = \frac{G\ k}{3Q_1}\Big[\ hyp.\ log.\ \frac{(\sqrt{h}-\sqrt{k})^2(h_1+\sqrt{h_1 k}+k)}{(\sqrt{h_1}-\sqrt{k})^2(h+\sqrt{hk}+k)}$$

$$+\ \sqrt{12}\ .\ arc.\ \left(tang. = \frac{(\sqrt{h}-\sqrt{h_1})\ \sqrt{12k}}{3k+(2\sqrt{h}+\sqrt{k})(2\sqrt{h_1}+\sqrt{k})}\right)\Big],$$

where $k = \left(\frac{Q_1}{\frac{2}{3}\ \mu\ b\ \sqrt{2g}}\right)^{\frac{2}{3}}$, *hyp. log.* represents the hyperbolic logarithm, and *arc.* (*tang.* $= y$) the arc whose tangent $= y$.

According as $k$ is $\lessgtr h$, and the inflowing quantity of water:

$Q_1 \gtrless \frac{2}{3}\ \mu\ b\ \sqrt{2gh^3}$, there is a rise or fall of the fluid surface.

The condition of permanency occurs, when $h_1 = k$, and the time corresponding becomes $\infty$.

*Example.* In what time will the water in a 12 feet long and 6 feet broad rectangular tank rise from 0 to 2 feet above the edge of a notch $\frac{1}{2}$ foot broad, if 5 cubic feet of water flows in per second? We have here $h = 0$; hence, more simply:

$$t = \frac{G k}{3\ Q_1}\Big[hyp.\ log.\ \frac{h_1+\sqrt{h_1 k}+k}{(\sqrt{h_1}\sqrt{k})^2} + \sqrt{12}\ arc\ \left(tang. = \frac{-\sqrt{3\ h_1}}{2\ \sqrt{k}+\sqrt{h_1}}\right)\Big].$$

Now $G = 12\ .\ 6 = 72,\ Q_1 = 5,\ h_1 = 2,\ b = \frac{1}{2}$, and $\mu = 0{,}6$,

$k = \left(\frac{5}{\frac{2}{3}\ .\ 0{,}6\frac{1}{2}\ .\ 8{,}03}\right)^{\frac{2}{3}} = 2{,}1321$, and the time sought is:

$$t = \frac{72\ .\ 2{,}1321}{3\ .\ 5}\Big[\ hyp.\ log.\ \frac{4{,}1544+\sqrt{4{,}3088}}{(1{,}4142-1{,}4678)^2} - \sqrt{12}.arc\left(tang. = \frac{\sqrt{6}}{1{,}4142+2{,}9357}\right)\Big]$$

$$= 10{,}234\ \Big[\ hyp.\ log.\ \frac{6{,}2302}{0{,}002873} - \sqrt{12}\ .\ arc\ \left(tang. = \frac{\sqrt{6}}{4{,}3499}\right)\Big]$$

$= 10{,}234\ (7{,}682 - 1{,}778) = 10{,}234\ .\ 5{,}90 = 61{,}38$ sec.

§ 353. *Locks.*—A very useful application of the doctrines hitherto treated of may be made to the filling and emptying of canal locks. We distinguish two kinds of locks (navigation locks) single and double. The single lock, Fig. 484, consists of a chamber $B$, which is separated by the upper gate $HF$ from the upper water $A$, and by the lower gate $RS$ from the lower water $C$. The double lock, Fig. 485, on the other hand, consists of two chambers, with the upper gate $KL$, the middle one $HF$ and the lower one $RS$.

FIG. 484.

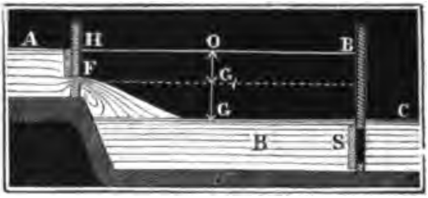
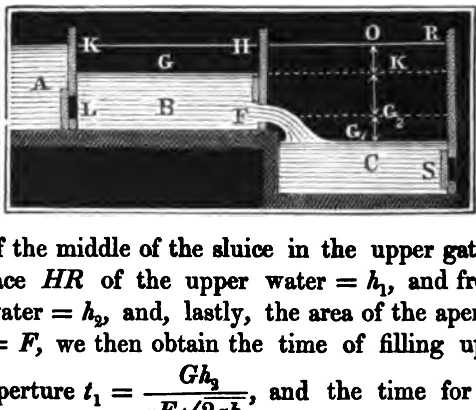



FIG. 485.




Let the mean horizontal transverse section of a simple lock chamber $= G$, the distance of the middle of the sluice in the upper gate from the upper surface $HR$ of the upper water $= h_1$, and from that of the lower water $= h_2$, and, lastly, the area of the aperture or sluice opening $= F$, we then obtain the time of filling up to the middle of the aperture $t_1 = \frac{G h_2}{\mu F \sqrt{2 g h_1}}$, and the time for filling the remaining space, where a gradual diminution of the head of water takes place, $t_2 = \frac{2\, G h_1}{\mu\, F\, \sqrt{2 g h_1}}$; consequently the time for filling the single sluice is:

$$t = t_1 + t_2 = \frac{(h_2 + 2\, h_1)\ G}{\mu\, F\, \sqrt{2 g h_1}}.$$

If the aperture in the lower gate is entirely under water, then while emptying, the head of water gradually decreases from $h_1 + h_2$ to zero, hence the time for emptying or running off is:

$$t = \frac{2\, G\, \sqrt{h_1 + h_2}}{\mu\, F\, \sqrt{2\, g}}.$$

If, on the other hand, a part of the aperture stands above the lower water, we then have two discharges to take into account; the

one flowing above and the other below the water. Let the height of the part of the aperture above the water $= a_1$, and that under the water $= a_2$, the breadth of the aperture $= b$, we then obtain the time of efflux from the expression:

$$t = \frac{2\,G\,(h_1 + h_2)}{\mu\, b\, \sqrt{2g}\left(a_1 \sqrt{h_1 + h_2 - \frac{a_1}{2}} + a_2\, \sqrt{h_1 + h_2}\right)}.$$

In double locks, the head of water gradually decreases in the chamber which is closed by the upper water during the discharge into the second chamber. If $G$ is the horizontal transverse section of the first chamber, and the original head of water $h_1$ in this chamber sinks to $x$, whilst the water in the second chamber rises to the middle of the aperture of the sluice, we have then the corresponding time $t_1 = \frac{2\,G}{\mu\, F\, \sqrt{2g}}(\sqrt{h_1} - \sqrt{x})$. Now the quantity of water $G\,(h_1 - x) = G_1 h_2$, hence $x = h_1 - \frac{G_1}{G}\, h_2$, and

$$t_1 = \frac{2\,G}{\mu F \sqrt{2g}}\left(\sqrt{h_1} - \sqrt{h_1 - \frac{G_1 h_2}{G}}\right) = \frac{2\sqrt{G}}{\mu F \sqrt{2g}}(\sqrt{G h_1} - \sqrt{G h_1 - G_1 h_2}).$$

The time in which the water rises as high in the second as in the first chamber, and in which, therefore, it comes to the same level in both, may be found from § 347:

$$t_2 = \frac{2\,G G_1 \sqrt{x}}{\mu F (G + G_1) \sqrt{2\,g}} = \frac{2\,G_1 \sqrt{G}\, \sqrt{G h_1 - G_1 h_2}}{\mu F\,(G + G_1)\, \sqrt{2g}},$$

and the whole time for filling:

$$t = t_1 + t_2 = \frac{2\sqrt{G}}{\mu F \sqrt{2\,g}}\left(\sqrt{G h_1} - \frac{G}{G + G_1}\sqrt{G h_1 - G_1 h_2}\right).$$

*Example.* What time is required for the filling and running off of the following single lock chamber? The mean length of the lock = 200 feet, mean breadth = 24 feet, therefore $G = 200 \cdot 24 = 4800$ square feet, distance of the centre of the aperture of the sluice in the upper gate from the two surfaces of water 5 feet, breadth of both apertures $2\frac{1}{2}$ feet, height of the aperture in the upper gate 4 feet, and of that in the lower gate (entirely under water) 5 feet. Let $t = \frac{(2\,h_1 + h_2)\,G}{\mu\, F\, \sqrt{2\,g\,h_1}}$, $h_1 = 5$, $h_2 = 5$, $G = 4800$, $\mu = 0{,}615$, $F = 4 \cdot 2\frac{1}{2} = 10$, $\sqrt{2g} = 8{,}03$, we then obtain the time of filling:

$t = \frac{3 \,.\, 5 \,.\, 4800}{6,15 \,.\, 8,03 \sqrt{5}} = \frac{14400}{1,23 \,.\, 8,03 \sqrt{5}} = 653,83$ seconds. If we substitute in the formula $t = \frac{2\, G \sqrt{h_1 + h_2}}{\mu F \sqrt{2g}}$, $G = 4800$, $h_1 + h_2 = 10$, $F = 5 \,.\, 2\frac{1}{2} = 12,5$, we then obtain the time for emptying of the sluice:

$$t = \frac{2 \,.\, 4800 \sqrt{10}}{0,615 \,.\, 12,5 \,.\, 8,03} = 491,78 \text{ sec.} = 8 \text{ min. } 21,78 \text{ sec.}$$

# CHAPTER VI.

## ON THE EFFLUX OF AIR FROM VESSELS AND TUBES.

§ 354. *Efflux of still air.*—Condensed air does not flow from vessels quite in accordance with the law which regulates the flow of water, because an expansion takes place during its discharge, which is not the case with water. But in order to discover a similar law for air and other gases, let us make the mechanical effect $Q\gamma\frac{v^2}{2g}$, which a quantity of air $Q$ of the density $\gamma$ requires to pass from a state of rest into that of the velocity $v$, equal to the mechanical effect $Q\, p$ *hyp. log.* $\left(\frac{p_1}{p}\right)$ found in § 298, which the same quantity of air produces when it passes from a greater pressure $p_1$ to a less $p$. If, therefore, $p_1$ be the elastic force of air enclosed in a vessel, $v$ its velocity of efflux for the tension of the external air, and $\gamma$ its density, then

$Q\gamma \,.\, \frac{v^2}{2g} = Qp \text{ hyp. log.}\left(\frac{p_1}{p}\right)$, therefore, *the height due to the velocity:*

$$\frac{v^2}{2g} = \frac{p}{\gamma} \text{ hyp. log. } \left(\frac{p_1}{p}\right) = 2,3026 \frac{p}{\gamma} \text{ log. } \left(\frac{p_1}{p}\right).$$

and the velocity itself:

$$v = \sqrt{2g \frac{p}{\gamma} \text{ hyp. log. } \left(\frac{p_1}{p}\right)}.$$

When the tensions $p$ and $p_1$ differ little from each other, when $p_1 - p$ is $< \frac{1}{10} p$, then we may put:

$$hyp.\ log.\frac{p_1}{p}=hyp.\ log.\left(1+\frac{p_1-p}{p}\right)=\frac{p_1-p}{p},\ \text{and hence}$$

$$v=\sqrt{2g\left(\frac{p_1-p}{\gamma}\right)}.$$ log(1+y) - y

But the height of an external column of air which is in equilibrium by its weight with the pressure $p_1-p$, (§ 294), is $h=\frac{p_1-p}{\gamma}$; hence we may put the velocity of efflux $v=\sqrt{2gh}$, and a perfect analogy with the efflux of water will hereby subsist. For high pressure this formula is not of course sufficient, for in this case:

$$hyp.\ log.\left(\frac{p_1}{p}\right)=\frac{p_1-p}{p}-\tfrac{1}{2}\left(\frac{p_1-p}{p}\right)^2\ \text{at least,}$$

hence then more accurately:

$$v=\sqrt{2g\left(\frac{p_1-p}{\gamma}-\tfrac{1}{2}\,\frac{(p_1-p)^2}{p\gamma}\right)}=\sqrt{2g\left(1-\frac{p_1-p}{2p}\right)h},$$

or if we represent the height of the barometer by $b$, $p=b\gamma$, and

$$v=\sqrt{2g\left(1-\frac{h}{2b}\right)h}=\left(1-\frac{h}{4b}\right)\sqrt{2gh}.$$

FIG. 486.

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$$=F\sqrt{2g\,b\ hyp.\ log.\left(\frac{b+h}{b}\right)}.$$

§ 355. The above formulæ do not admit of direct application, because we cannot measure the internal or the external pressure by the length $b+h$, and $b$ of the columns of air. These pressures are generally measured by columns of water or mercury. As regards the quotient $\frac{p_1}{p}=\frac{b+h}{b}$, it is immaterial whether $b$ and

$h$ be expressed in columns of air, water, or mercury, because each reduction of $b$ and $h$ leaves the fraction $\frac{b+h}{b}$ constant, except that the quotient $\frac{p}{\gamma} = b$, is still dependent on the temperature of the effluent air, and varies for different kinds of gas. For atmospheric air, (§ 301), if $p$ represent the pressure of air on one square centimetre, and $\gamma$ the weight of a cubic metre of air, and $t$ the temperature in degrees centigrade, we have

$$\frac{p}{\gamma} = \frac{1+0{,}00367 \, . \, t}{1{,}2572}, \text{ on the other hand, for steam}$$

$$\frac{p}{\gamma} = \frac{1 + 0{,}00367 \, t}{0{,}7857}.$$

If we substitute these values in the general formula for $v$, we shall obtain for atmospheric air : $\frac{p}{\gamma} = b$

$$v = 395 \sqrt{(1+0{,}00367 \, . \, t) \; hyp. \, log. \left(\frac{b+h}{b}\right)} \text{ metres,}$$

or $\frac{h}{b}$ being small :

$$v = 395 \sqrt{(1+0{,}00367 \, . \, t) \frac{h}{b}} \text{ metres, and for steam}$$

$$v = 500{,}6 \sqrt{(1+0{,}00367 \, . \, t) \; hyp. \, log. \left(\frac{b+h}{b}\right)} \text{ metres.}$$

The theoretical discharge as estimated under the external pressure is $Q = Fv$, but if this is to be estimated at the internal pressure, we must then make $Q_1 p_1 = Qp$, hence $Q_1 = \frac{p}{p_1} Q = \frac{bQ}{b+h}$. Reduced to the temperature of zero, the quantity discharged is :

$$Q_2 = \frac{Q}{1 + 0{,}00367 \, . \, t}, \text{ therefore, for atmospheric air}$$

$$= 395 \, F \sqrt{\frac{hyp. \, log. \, (b+h) - hyp. \, log. \, b}{1+0{,}00367 \, . \, t}} \text{ cubic metres.}$$

If equal masses of air of different temperatures issue from different orifices $F$ and $F_1$ at the same tension, we then have :

$$\frac{F_1}{F} = \sqrt{\frac{1 + 0{,}00367 \, t_1}{1 + 0{,}00367 \, t}}.$$

If, for example, $t=0$ and $t_1=150^0$, we then have:

$$F_1 = \sqrt{1,5505} \,.\, F = 1,245\, F.$$

If, therefore, a blast furnace is to be supplied with heated air of $150^0$, we must apply nozzle pipes, which have a one fourth greater transverse section at the discharging orifice than if cold air were to be used.

For Prussian measure, and centigrade scale of temperature:

$$v=1258 \,.\, \sqrt{(1+0,00367\, t)\; hyp.\, log.\left(\frac{b+h}{b}\right)}, \text{ and for steam}$$

$$v=1595 \,.\, \sqrt{(1+0,00367\, t)\; hyp.\, log.\left(\frac{b+h}{b}\right)}.$$

For English measure, and Fahrenheit's scale of temperature:

$$v=1295 \,.\, \sqrt{(1+0,00204\, t)\; hyp.\, log.\left(\frac{b+h}{b}\right)}, \text{ and for steam}$$

$$v=1642 \,.\, \sqrt{(1+0,00204\, t)\; hyp.\, log.\left(\frac{b+h}{b}\right)}.$$

*Example.* In a large reservoir, air of 120° temperature is enclosed, which corresponds to the height of a mercurial manometer of 5 inches, whilst the external barometer stands at 27,2 inches; what quantity of air will flow from this through a round aperture $1\frac{1}{2}$ inch wide ? It is:

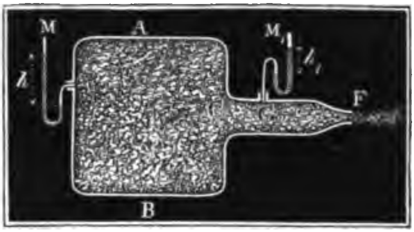
$hyp.\, log.\left(\frac{b+h}{b}\right) = hyp.\, log.\left(\frac{32,2}{27,2}\right) = hyp.\, log.\, 32,2 - hyp.\, log.\, 27,2$ $= 5,77455 - 5,60580 = 0,16875$, hence the velocity of efflux is:

$v = 1258 \,.\, \sqrt{(1 + 0,00367 \,.\, 120)\, 0,16875} = 1258 \,.\, \sqrt{1,4404 \,.\, 016875}$ $= 620,2$ feet. Now the area of the orifice $= \frac{\pi}{4}\left(\frac{1}{8}\right)^2 = \frac{\pi}{256} = 0,01227$ square feet; hence it follows that the discharge $Q = 0,01227 \,.\, 620,2 = 7,61$ cubic feet. Estimated at the interior pressure, it is $= \frac{272}{322} \,.\, 7,61 = 6,43$ cubic feet, and reduced to the mean height of the barometer, 28 inches and 0° temperature, (30 English inches and 32° temperature), the quantity discharged is:

$= 7,61 \,.\, \frac{272}{280} \,.\, \frac{1}{1,4404} = 5,13$ cubic feet.

§ 356. *Efflux of air in motion.*—The formula of efflux given: suppose the pressure $p_1$ or the height of the manometer $h$ to be measured at a place where the air is at rest, or has a very slight motion, but if $p_1$ or $h_1$ is measured at a place where the air is in motion, if, for instance, the manometer $M_1$ communicates with the air in a conducting tube $CF$, Fig. 487,

FIG. 487.




we shall then have to take into account the *vis viva* of the arriving air. If now $c$ be the velocity of the air passing the orifice of the manometer we shall accordingly have to make:

$$Q\gamma \cdot \frac{v^2}{2g} = Q\gamma \cdot \frac{c^2}{2g} + Qp \; hyp. \; log. \left(\frac{p_1}{p}\right),$$

or if $F$ be the transverse section of the orifice, and $G$ that of the tube, or of the air passing the orifice of the manometer, according to the law of Mariotte, $\frac{Gc}{Fv} = \frac{p}{p_1}$, or $Gcp_1 = Fvp$, therefore,

$$c = \frac{F}{G} \cdot \frac{p}{p_1} v, \; Q\gamma \left[1 - \left(\frac{F}{G}\right)^2 \left(\frac{p}{p_1}\right)^2\right] \frac{v^2}{2g} = Qp \; hyp. \; log. \left(\frac{p_1}{p}\right),$$

and the velocity of efflux in question:

$$v = \frac{\sqrt{2g\frac{p}{\gamma} \; hyp. \; log. \left(\frac{p_1}{p}\right)}}{\sqrt{1 - \left(\frac{Fp}{Gp_1}\right)^2}}.$$

The velocity of efflux is, therefore, here exactly like that of water from vessels, the velocity is greater, the greater the ratio $\frac{F}{G}$ of the transverse section of the orifice to that of the tube or the arriving current of air. From this it is evident, that under otherwise similar circumstances the height of the manometer $p_1$ is so much the less the narrower the conducting tube is, or the greater the velocity of the air issuing from it.

*Examples.*—1. A mercurial manometer, placed upon an air tube $3\frac{1}{2}$ inches wide, stands at $2\frac{1}{2}$ inches, while the wind flows from its conical extremity through a round 2 inch wide orifice; with what velocity will the current move? If the external barometer stand at $27\frac{1}{2}$ inches, we shall then have $\frac{p_1}{p} = \frac{27\frac{1}{2} + 2\frac{1}{2}}{27\frac{1}{2}} = \frac{30}{27,5} = \frac{12}{11}$ and $\frac{Fp}{Gp_1} = \left(\frac{2}{3,5}\right)^2 \cdot \frac{11}{12} = \frac{16 \cdot 11}{49 \cdot 12} = \frac{44}{147}$; hence the theoretical velocity of efflux at a temperature of the air 10°:

$$v = \frac{1258 \cdot \sqrt{1,0367 \cdot hyp. \; log. \left(\frac{12}{11}\right)}}{\sqrt{1 - \left(\frac{44}{147}\right)^2}} = \frac{1258 \sqrt{1,0367 \cdot 0,087}}{\sqrt{0,9104}} = 396 \text{ feet.}$$

2. The tension $p_2$ in the air regulator, where the air is without motion, is given by the formula,

$$hyp.\ log.\left(\frac{p_2}{p}\right) = \frac{v^2}{2g}\cdot\frac{\gamma}{p},\ \text{or}\ hyp.\ log.\ p_2 = hyp.\ log.\ p + \frac{hyp.\ log.\left(\frac{p_1}{p}\right)}{1-\left(\frac{Fp}{Gp_1}\right)^2},$$

therefore, in the present case, $= hyp.\ log.\ 27{,}5 + \frac{0{,}087}{0{,}9104} = 3{,}3142 + 0{,}0965 = 3{,}4107$. Hence it follows that $p_2 = 30{,}3$ inches.

§ 357. *Efflux under decreasing pressure.*—If an air reservoir has no influx, whilst an uninterrupted efflux goes on, the density and tension gradually diminish, and hence the velocity of efflux becomes less and less. We may determine in the following manner in what ratio this diminution is to the time and to its discharge.

Let $V$ be the volume of the reservoir, $h_0$ the initial height of the manometer, and $h_n$ the height of the manometer at the end of a certain time $t$, $b$ the height of the external barometer. Then the quantity of air or wind in the reservoir at the commencement reduced to the external pressure $= \frac{V(b+h_0)}{b}$, and at the end of the time $t$, $= \frac{V(b+h_n)}{b}$, and, consequently, the quantity discharged in the time $t$, and at the external pressure is:

$$V_n = \frac{V(b+h_0)}{b} - \frac{V(b+h_n)}{b} = \frac{V(b_0-h_n)}{b};$$

and, inversely, the height of the manometer corresponding to the discharge $V_n$ is:

$$h_n = h_0 - \frac{V_n}{V}\cdot b.$$

If we take four intervals and the initial height of the manometer $h_0$, and at the end of the time $t = h_4$, and

$$h_1 = h_0 - \frac{h_0 - h_4}{4},\ h_2 = h_0 - \tfrac{2}{4}(h_0 - h_4),\ \text{and}$$

$$h_3 = h_0 - \tfrac{3}{4}(h_0 - h_4),$$

we shall then obtain by Simpson's rule the time

$$t = \frac{V(h_0 - h_4)}{12Fb\sqrt{2g\frac{p}{\gamma}}}\left(\frac{1}{\sqrt{hyp.\ log.\left(\frac{b+h_0}{b}\right)}} + \frac{4}{\sqrt{hyp.log.\left(\frac{b+h_1}{b}\right)}}\right.$$

$$+\frac{2}{\sqrt{hyp.\,log.\left(\frac{b+h_2}{b}\right)}}+\frac{4}{\sqrt{hyp.\,log.\left(\frac{b+h_3}{b}\right)}}$$

$$\left.+\frac{1}{\sqrt{hyp.\,log.\left(\frac{b+h_4}{b}\right)}}\right).$$

For moderate pressures or heights of the manometer:

$$hyp.\,log.\left(\frac{b+h}{b}\right)=\frac{h}{b}\left(1-\frac{h}{2b}\right),$$

consequently $\sqrt{hyp.\,log.\left(\frac{b+h}{b}\right)}=\left(1-\frac{h}{4b}\right)\sqrt{\frac{h}{b}}$

and $\frac{1}{\sqrt{hyp.\,log.\left(\frac{b+h}{b}\right)}}=\left(1+\frac{h}{4b}\right)\sqrt{\frac{h}{b}}.$ $\frac{b}{h}$

If we now take $n$ intervals, and therefore the discharge for one interval: $\frac{V_1}{n}=\frac{V(h_0-h_n)}{n\,b}$, we shall then obtain the corresponding element of time: $\tau=\frac{V(h_0-h_n)}{n\,b}:\mu F\sqrt{2g\frac{p}{\gamma}\,hyp.\,log.\left(\frac{b+h}{b}\right)}$

$$=\frac{V(h_0-h_n)}{nb}\;\frac{\left(1+\frac{h}{4b}\right)\sqrt{\frac{b}{h}}}{F\sqrt{2g\frac{p}{\gamma}}}$$

$$=\frac{V(h_0-h_n)\left(h^{-\frac{1}{2}}+\frac{h^{\frac{1}{2}}}{4b}\right)}{n\,.\,F\sqrt{2g\,b\frac{p}{\gamma}}}.$$

Now if we substitute for $h$; $h_0, h_1, h_2 \ldots . h_n$, we shall then obtain the sum of all the

$$\left(\frac{h_0-h_n}{n}\right)h^{-\frac{1}{2}}=2\,(h_0^{\frac{1}{2}}-h_n^{\frac{1}{2}})=2\,(\sqrt{h_0}-\sqrt{h_n}),$$

and the sum of all the

$$\left(\frac{h_0-h_n}{n}\right)h^{\frac{1}{2}}=\tfrac{2}{3}\,(h_0^{\frac{3}{2}}-h_n^{\frac{3}{2}})=\tfrac{2}{3}\,(\sqrt{h_0^3}-\sqrt{h_n^3}),$$

whence the sum of all the small intervals of time, or the whole time in which $h_n$ passes into $h_0$, and the quantity of air $V_n=\frac{V(h_0-h_n)}{b}$ which flows out, is:

$$t = \frac{2\,V}{F\sqrt{2gb\frac{p}{\gamma}}}\left[(\sqrt{h_0} - \sqrt{h_n}) + \frac{1}{12b}\left(\sqrt{h_0^3} - \sqrt{h_n^3}\right)\right], \text{ or}$$

$$= \frac{2\,V}{F\sqrt{2gb\frac{p}{\gamma}}}(\sqrt{h_0} - \sqrt{h_n})\left(1 + \frac{h_0 + \sqrt{h_0 h_n} + h_n}{12\,b}\right),$$

approximately: putting $\sqrt{h_0 h_n} = \frac{h_0 + h_n}{2}$

$$= \frac{2\,V}{F\sqrt{2gb\frac{p}{\gamma}}}(\sqrt{h_0} - \sqrt{h_n})\left(1 + \frac{h_0 + h_n}{8b}\right).$$

*Example.* A 50 feet long and 5 feet wide cylindrical wind-regulator of a blowing machine is filled with air; the height of its manometer $h = 10$ inches, and the thermometer stands at 6°. If now a flow of air takes place in a space where the height of the barometer is 27 inches, through a 1-inch wide round orifice, then the question arises, in what time will the height of the manometer fall to 7 inches, and what will be the corresponding discharge? The volume of the chamber is $= \frac{\pi}{4} \cdot 5^2 \cdot 50 = 1250 \cdot \frac{\pi}{4} = 981{,}75$ cubic feet, hence the discharge, measured at the external pressure, is $V_1 = \left(\frac{h_0 - h_n}{b}\right)V = \left(\frac{10-7}{27}\right) \cdot 981{,}75 = 109{,}08$ cubic feet. Now $\sqrt{2g\frac{p}{\gamma}} = 1258\sqrt{1 + 0{,}00367\,.\,t} = 1258\sqrt{1{,}02202} = 1272$, and $F = \frac{\pi}{4}\left(\frac{1}{12}\right)^2 = \frac{\pi}{576} = 0{,}005454$ square feet, hence the time of efflux in question is $t = \frac{2\,.\,981{,}75}{0{,}005454\,.\,1272}\left(\sqrt{\frac{10}{27}} - \sqrt{\frac{7}{27}}\right)\left(1 + \frac{10+7}{8\,.\,27}\right)$
$= \frac{1963{,}5}{5{,}454\,.\,1{,}272} \cdot 0{,}0994 \cdot 1{,}079 = 30{,}3$ seconds.

§ 358. *Co-efficients of efflux.*—The phenomena of contraction, which we have considered in the efflux of water from vessels, occur also in the efflux of air. If the orifice of efflux be cut in a thin plate, the air passing through it has a smaller transverse section than the orifice, and on this account the discharge is less than the product $Fv$ of the transverse section $F$ of the orifice and the theoretical velocity $v$. Let $\frac{F_1}{F}$ be the ratio of the transverse section $F_1$ of the blast to that of the orifice $F$, $=\mu$, we then have the effective discharge as for water:

$$Q_1 = \mu\,Q = F_1 v = \mu\,Fv = \mu\,F\sqrt{2g\frac{p}{\gamma}\,hyp.\,log.\left(\frac{p_1}{p}\right)}.$$

From the author's reduction of Koch's experiments at pressures

of the manometer of from $\frac{1}{800}$ to $\frac{1}{5}$ of an atmosphere we may take the mean of $\mu = 0{,}58$.

The effective discharge in the issuing of air through short cylindrical adjutages is likewise less than that determined theoretically, we have, therefore, to multiply this latter by a number deduced from experiment, the co-efficient of efflux, $\mu$, in order to obtain the former; only here $\mu$ is not the ratio of the transverse section $\frac{F_1}{F}$, but the ratio $\frac{v_1}{v}$ of the effective velocity of efflux $v_1$ to the theoretical $v$. Koch's experiments give for the above pressures, in the flow of air through cylindrical adjutages, which were nearly all six times as long as wide, as a mean $\mu = 0{,}74$.

Conically convergent adjutages, similar to the nozzles of bellows, give a still greater co-efficient of efflux; a tube of $6^0$ lateral convergence in the experiments of Koch, gave when five times as long as wide, the mean co-efficient $\mu = 0{,}85$.

From this, therefore, the effective discharge for the flow of air through orifices in a thin plate, measured at the external pressure, is

$$Q_1 = 796{,}7\, F\left(1 - \frac{h}{4b}\right)\sqrt{(1 + 0{,}00367\, t)\,\frac{b}{h}} \text{ cubic feet.}$$

for efflux through short cylindrical adjutages:

$$Q_1 = 1016{,}6\, F\left(1 - \frac{h}{4b}\right)\sqrt{(1 + 0{,}00367\, t)\,\frac{h}{b}} \text{ cubic feet,}$$

and through conical adjutages of $6^0$ convergence.

$$Q_1 = 1167{,}3\, F\left(1 - \frac{h}{4b}\right)\sqrt{(1 + 0{,}00367\, t)\,\frac{h}{b}} \text{ cubic feet.}$$

*Example.* If the two orifices of a bellows together possess an area of 3 square inches, if, further, the pressure of the manometer is 3 inches, the external barometer $27\frac{1}{2}$ inches, and the temperature of the air 15°, then is the discharge:

$$Q_1 = 1069 \cdot \tfrac{3}{144}\left(1 - \frac{3}{4 \cdot 27{,}5}\right)\sqrt{(1 + 0{,}00367 \;.\; 15)\,\frac{3}{27{,}5}}$$

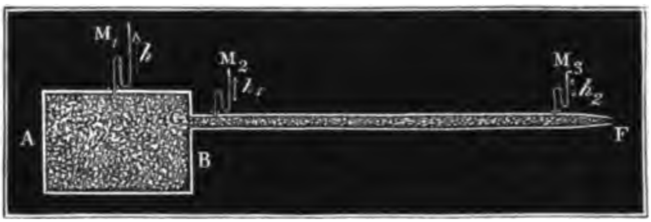
$$= 22{,}27 \cdot \frac{107}{110}\sqrt{1{,}055 \;.\; \tfrac{6}{55}} = 21{,}66\,\sqrt{0{,}1151} = 7{,}34 \text{ cubic feet.}$$

*Remark.* Experiments on the efflux of air have been undertaken by Young, Schmidt, Lagerhjelm, Koch, d'Aubuisson, Buff, and in later time, by Pecqueur, Saint-Venant, and Wantzel. For an account of the experiments of Young and Schmidt, we may refer to Gilbert's "Annalen," vol. 22, 1801, and vol. 6, 1820, and to Poggendorff's "Annalen," vol. 2, 1824; for those of Koch and Buff, to the "Studien des göttingschen Vereines bergmännischer Freunde," vol. 1, 1824; vol. 3, 1833; vol. 4, 1837, and vol. 5, 1838; also in Poggendorff's "Annalen," vol. 27, 1836, and vol. 40, 1837. The experiments of Lagerhjelm are described in the

Swedish work, "Hydrauliska Försök af Lagerhjelm, Forselles och Kallstenius," 1 vol. Stockholm, 1818. D'Aubuisson's experiments are to be found in the "Annales des Mines," vol. 11, 1825; vol. 13, 1826; vol. 14, 1827, and likewise in his "Traité d'Hydraulique." The latest experiments instituted in France are reported in the "Polytechnischen Centralblatt," vol. 6, 1845. Most of these experiments were made with very narrow orifices, and therefore scarcely answer the purpose in practice. The experiments of d'Aubuisson and Koch deserve most consideration; and next to them, perhaps, those of Pecqueur; but the most extensive are those of Koch. The wished-for accordance is hardly to be met with in the results of all these experiments; the co-efficients of efflux found by d'Aubuisson vary considerably from those calculated by Koch. The grounds for my placing the most confidence in the co-efficients of Koch, are given in the "Allgemeinen Maschinenencyclopädie," under the article "Ausflusz," and in a Memoir of mine in Poggendorff's "Annalen," vol. 51, 1840.

§ 359. *Flow through tubes.*—If the air issues through a long tube $CF$, Fig. 488, it has then the resistance of friction to over-

FIG. 488.




come in the same manner as water, this resistance may also be measured by the height of a column of air, which has for expression $h_n = \zeta \cdot \frac{l}{d} \cdot \frac{v^2}{2g}$, where, as in the conducting of water, $v$ represents the velocity, $l$ the length, $d$ the width of the tube, and $\zeta$ a co-efficient of resistance to be determined by experiment.

Numerous experiments of Girard, d'Aubuisson, Buff and Pecqueur, lead to the mean value $\zeta = 0{,}024$. From this, therefore, the resistance generated by the friction of air in tubes may be measured by the height $h_n = 0{,}024 \frac{l}{d} \cdot \frac{v^2}{2g}$ of a column of air, or by the height $h_n = 0{,}0000023 \frac{l}{d} \cdot \frac{v^2}{2g}$ of a column of quicksilver, and the manometer will stand at this much less height at the end of the conducting tube than at the beginning.

If at the end of a conducting tube of the width $d$, the manometer stands at $h_2$, whilst the air flows through an orifice of the

width $d$, then from the preceding, the velocity of discharge will be: See §[illegible]55c

$$v = \frac{\sqrt{2\,g\,\frac{p}{\gamma}\,hyp.\,log.\left(\frac{b+h_2}{b}\right)}}{\sqrt{1-\left(\frac{b}{b+h_2}\right)^2\left(\frac{d_1}{d}\right)^4}};$$

but if $h_1$ be the height of the manometer at the beginning of the conduit, we shall then have:

$$\frac{p}{\gamma}\,hyp.\,log.\left(\frac{b+h_1}{b}\right)=\left[1\mp\left(\frac{b}{b+h_1}\right)^2\left(\frac{d_1}{d}\right)^4+0{,}024\,\frac{l}{d}\left(\frac{d_1}{d}\right)^4\right]\frac{v^2}{2g},$$

because the velocity in the tube $= \frac{d_1^{\,2}}{d^2}\,v$; hence in this case

$$v = \frac{\sqrt{2\,g\,\frac{p}{\gamma}\,hyp.\,log.\left(\frac{b+h_1}{b}\right)}}{\sqrt{1+\left[0{,}024\,\frac{l}{d}\mp\left(\frac{b}{b+h_1}\right)^2\right]\left(\frac{d_1}{d}\right)^4}}.$$

If, lastly, the height of the manometer $h$ is measured in the reservoir at the beginning of the conduit where the air may be regarded as at rest, we then have:

$$v = \frac{\sqrt{2\,g\,\frac{p}{\gamma}\,hyp.\,log.\left(\frac{b+h}{b}\right)}}{\sqrt{1+0{,}024\,\frac{l\,d_1^{\,4}}{d^5}}}.$$

If further, we put the co-efficient of resistance $\zeta$ for entrance into the tube, which when $\mu_1 = 0{,}74$ amounts to $0{,}826$, and further join to it the co-efficient of efflux $\mu$ for the outer adjutage, we then obtain for the velocity:

$$v = \frac{\mu\sqrt{2\,g\,\frac{p}{\gamma}\,hyp.\,log.\left(\frac{b+h}{b}\right)}}{\sqrt{1+\zeta+0{,}024\,\frac{l\,d_1^{\,4}}{d^5}}}$$

$$\text{or} = \frac{1294\,\mu\sqrt{(1+0{,}00367\,t)\;hyp.\,log.\left(\frac{b+h}{b}\right)}}{\sqrt{1+\zeta+0{,}024\,\frac{l\,d_1^{\,4}}{d^5}}}\ \text{feet.}$$

According as the point of the interior orifice lies $s$ lower or higher than the point of the exterior orifice, we have to add $\pm s$ to the quantity under the radical in the denominator. Moreover, other hindrances may present themselves in the tube, such as curvatures, contractions, and widenings, &c. Satisfactory experiments on these obstacles do not exist, but we may assume with great probability that these resistances are not much different from what takes place in the case of water, because the co-efficients of efflux, and the co-efficient of friction are nearly the same for air as for water.

As long, therefore, as no further experiments are made on this subject we may avail ourselves with tolerable safety of the co-efficient of resistance found for water in investigations on the motion and flow of air.

*Example.* In the regulator at the head of a 320 feet long and 4 inch wide wind-conductor, the quicksilver manometer stands at 3,1 inch, whilst the external barometer is at 27,2 inch; further, the width of the orifice of the conically contracted extremity of the conductor is 2 inches, and the temperature of the wind 20°, what quantity of air will this conductor deliver? It will be:

$$1 + \zeta + 0{,}024 \frac{l d_1^4}{d^5} = 1{,}826 + 0{,}024 \cdot \frac{320}{4} \cdot (\tfrac{2}{4})^4 = 1{,}826 + 0{,}024 \cdot \frac{320 \cdot 3}{16}$$

$= 1{,}826 + 1{,}44 = 3{,}266$; further, $(1 + 0{,}00367\, t)\ hyp.\ log. \left(\frac{b+h}{b}\right)$

$$= (1 + 0{,}00367 \cdot 20)\ hyp.\ log. \left(\frac{30{,}3}{27{,}2}\right) = 1{,}0734 \cdot (5{,}7137 - 5{,}6058)$$

$= 1{,}0734 \cdot 0{,}1079 = 0{,}1158$; if now, further, we introduce the co-efficient of efflux $\mu = 0{,}85$, we shall then obtain the velocity of flow:

$$v = \frac{1258 \cdot 0{,}85 \sqrt{0{,}1158}}{\sqrt{3{,}266}} = 201{,}3 \text{ feet; and lastly, the discharge:}$$

$$Q = \frac{\pi d_1^2}{4} \cdot v = \frac{\pi}{4} \cdot \frac{201{,}3}{36} = 4{,}39 \text{ cubic feet.}$$

## CHAPTER VII.

### ON THE MOTION OF WATER IN CANALS AND RIVERS.

§ 360. *Running water.*—The doctrine of the motion of water in canals and rivers, forms the second main division of hydraulics. Water flows either in a natural or in an artificial bed. In the first case, it forms streams, rivers, brooks; in the second

canals, cuts, drains, &c. In the theory of the motion of flowing water this distinction is of little moment.

The bed of a river consists of the *bottom* and the two *banks* or *shores*. The *transverse section* is obtained by a plane at right angles to the direction of motion of the flowing water. Its *perimeter* is that of the transverse section, which again consists of the *air* and the *water section*. A vertical plane in the direction of the flowing water gives the *longitudinal section or profile*. By the *slope* or *declivity* of flowing water is understood the angle of inclination of its surface to the horizon. The *fall*, which is the vertical distance of the two extreme points of a definite length of the fluid surface, serves to assign the angle for a definite length of the flowing stream. For the length of course, $AD = l$, Fig. 489, $BC$ is the bottom of the channel, $DH = h$ the fall, and the angle $DAH = \delta$, the slope $sin. \delta = \frac{h}{l}$ = absolute fall per unit of length.

FIG. 489.

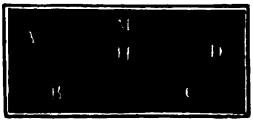

*Remark.* The fall of brooks and rivers is very various. The Elbe, for example, for the extent of a German mile from the Upper Elbe to Podiebrad, has a fall of 57 feet, from thence to Leitmeritz 9 feet, from there to Mühlberg a mean of 5,8, and from thence to Magdeburg 2,5 feet. Mountain brooks have a fall of from 40 to 400 feet per German mile. For further particulars, see "Vergleichende hydrographische Tabellen," &c. von Stranz. Canals and other artificial water conduits have much smaller falls. Here the absolute fall, at most, is 0,001, often 0,0001, and even less. More on this subject will be given in the Second Part.

§ 361. *Different velocities in the transverse section.*—The velocity of water in one and the same transverse section is very different at different points. The adhesion of the water to its bed, and the co-hesion of the particles among each other, cause those lying nearer to the sides of the bed some constraint in their motion, and hence, to flow more slowly than the more remote. For this reason the velocity diminishes from the surface downwards to the bed, and is least near the side or at the bottom. The greatest velocity is found for straight rivers, generally in the middle, or at that part of the free surface of the water where there is the greatest depth. The place where the water attains its maximum velocity is called the *line of current*, and the deepest part of the bed, the *mid-channel*.

The upper surface does not form an exact horizontal line, because the elements lying on the surface of water, flow on with different

velocities with respect to each other, they therefore exert on each other different pressures; the quicker ones a less, and the slower a greater pressure, and thus for the maintenance of relative equilibrium, the quicker elements superpose themselves on the slower. If $v$ and $v_1$ are the velocities of two elements $M$ and $A$, Fig. 490, then according to the doctrine of hydraulic pressure (§ 307) the difference of level of the two elements is:

FIG. 490.



$$MH = h = \frac{v^2}{2g} - \frac{v_1^2}{2g} = \frac{v^2 - v_1^2}{2g}.$$

This difference of level is always very small. If, for example, $v_1 = 0,9\ v$, and $v = 5$ feet, we then have this

$$= (1 - 0,81)\frac{v^2}{2g} = 0,19 \,.\, 0,016 \,.\, 25 = 0,076 \text{ inches} = 0,9 \text{ lines}.$$

For this reason the water stands highest in the current, and lowest at the banks.

In bends, the current is generally near the concave bank.

§ 362. *Permanent motion of water.*—The mean velocity of water in a transverse section is, according to § 308:

$$c = \frac{Q}{F} = \frac{\text{quantity of water per second}}{\text{area of section}}.$$

The mean velocity besides may be further calculated from the velocities $c_1$, $c_2$, $c_3$, &c., of the separate portions of the section, and from the areas $F_1$, $F_2$, $F_3$, &c. It is namely:

$$Q = F_1 c_1 + F_2 c_2 + F_3 c_3 + \dots,$$

and hence also:

$$c = \frac{F_1 c_1 + F_2 c_2 + \dots}{F_1 + F_2 + \dots}.$$

Besides the mean velocity, the mean depth of water has to be introduced, that is, the depth $a$ which a section must have at all points that it may have the same area as it actually has with the variable depths $a_1$, $a_2$, $a_3$, &c. Hence, therefore,

$$a = \frac{F}{b} = \frac{\text{area of section}}{\text{breadth of section}}.$$

If the separate parts of the breadth $b_1$, $b_2$, $b_3$, have the corresponding mean depths $a_1$, $a_2$, $a_3$, &c., Fig. 491, we then have:

FIG. 491.


$$F = a_1 b_1 + a_2 b_2 + \dots,$$

and hence also:

$$a = \frac{a_1 b_1 + a_2 b_2 + \dots}{b_1 + b_2 + \dots}.$$

Lastly:

$$c = \frac{a_1 b_1 c_1 + a_2 b_2 c_2 + \dots}{a_1 b_1 + a_2 b_2 + \dots},$$

and if the portions $b_1$, $b_2$, &c., be of equal size,

$$c = \frac{a_1 c_1 + a_2 c_2 + \dots}{a_1 + a_2 + \dots}.$$

A river or brook is in a state of *permanency* when an equal quantity of water flows through each of its transverse sections in an equal time; when, therefore, $Q$, or the product $Fc$ of the area of the section and the mean velocity throughout the whole extent of the stream is a constant number. Hence this simple law comes out: *in the permanent motion of water, the mean velocities in two transverse sections are to each other inversely as the areas of these sections.*

*Examples.*—1. At the section of a canal, *ABCD*, Fig. 491, it was found that the:

| | | | |
|---|---|---|---|
| Portions of the breadth . . . | $b_1 = 3{,}1$ feet, | $b_2 = 5{,}4$ feet, | $b_3 = 4{,}3$ feet |
| Mean depth . . . . . | $a_1 = 2{,}5$ „ | $a_2 = 4{,}5$ „ | $a_3 = 3{,}0$ „ |
| Corresponding mean velocities . . | $c_1 = 2{,}9$ „ | $c_2 = 3{,}7$ „ | $c_3 = 3{,}2$ „ |

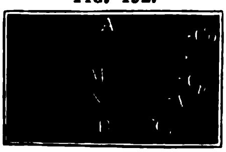
Hence the area of these profiles $F = 3{,}1 \,.\, 2{,}5 + 5{,}4 \,.\, 4{,}5 + 4{,}3 \,.\, 3{,}0 = 44{,}95$ square feet, and the discharge:

$Q = 3{,}1 \,.\, 2{,}5 \,.\, 2{,}9 + 5{,}4 \,.\, 4{,}5 \,.\, 3{,}7 + 4{,}3 \,.\, 3{,}0 \,.\, 3{,}2 = 153{,}665$ cubic feet, and the mean velocity $c = \frac{Q}{F} = \frac{153{,}665}{44{,}95} = 3{,}419$ feet.

2. When a cut is to conduct 4,5 cubic feet of water with a mean velocity $c$ of 2 feet, we must then give to it a transverse section of $\frac{4{,}5}{2} = 2{,}25$ square foot area.—3. If one and the same stream has a mean velocity of $2\frac{1}{4}$ feet at a place 560 feet broad and 9 feet mean depth, it will then have, at a place 320 feet broad and 7,5 feet mean depth, the mean velocity

$$c = \frac{560 \,.\, 9}{320 \,.\, 7{,}5} \,.\, 2{,}25 = \frac{567}{120} = 4{,}725 \text{ feet.}$$

FIG. 492.



§ 363. *Mean velocity.*—If we divide the depths of water at any point of a flowing stream into equal parts, and raise ordinates upon them corresponding to the velocities, we shall then obtain a scale of the velocity

of the current $AB$, Fig. 492. Although it may be granted that the law of this scale, or of the difference of velocity is expressed by some curve, as according to Gerstner by an ellipse, yet it is allowable, without fear of any great error, to substitute for this a straight line, or assume that the velocity diminishes uniformly with the depth, because the diminution of velocity downwards is always very small. From the experiments of Ximenes, Brünnings, and Funk, the mean velocity in a perpendicular $c_m = 0{,}915\ c_0$, where $c_0$ represents the velocity at the surface, or the maximum velocity. The velocity, therefore, diminishes from the surface to the middle $M$

$$\text{by } c_0 - c_m = (1 - 0{,}915)\ c_0 = 0{,}085\ c_0,$$

and, consequently, the velocity below or at the foot of the perpendicular may be put

$$c_u = c_0 - 2 \cdot 0{,}085\ c_0 = (1 - 0{,}170)\ c_0 = 0{,}83\ c_0.$$

If now the whole depth $= a$, we then have, by assuming a straight line for the scale of the velocities, the corresponding velocity for a depth $AN = x$, below the water

$$v = c_0 - (c_0 - c_u)\frac{x}{a} = \left(1 - 0{,}17\frac{x}{a}\right) c_0.$$

Further, let $c_0, c_1, c_2, \ldots$ be the superficial velocities of a whole transverse profile of not very variable depth, we have then the corresponding velocities at a mean depth: $0{,}915\ c_0$, $0{,}915\ c_1$, $0{,}915\ c_2$, and hence the mean velocity in the whole profile:

$$c = 0{,}915\ \frac{(c_0 + c_1 + c_2 + \ldots c_n)}{n}.$$

Lastly, if we assume that the velocity diminishes from the line of current towards the banks, as it does according to the depth, we may then again put the mean superficial velocity

$$\frac{(c_0 + c_1 + \ldots + c_n}{n} = 0{,}915\ c_0,$$

and so obtain the mean velocity in the whole profile:

$$c = 0{,}915 \cdot 0{,}915 \cdot c_0 = 0{,}837 \cdot c_0,$$

*i. e.* from 83 to 84 per cent. of the maximum velocity, or of that of the line of current.

Prony deduced from Du Buat's experiments conducted with very small channels, and for these cases perhaps more correctly:

$$c_m = \left(\frac{2{,}372 + c_0}{3{,}153 + c_0}\right) c_0 \text{ metre} = \left(\frac{\cancel{7{,}71 + c_0}}{\cancel{10{,}25 +}\ c_0}\right) c_0 \text{ feet English.}$$

1:17

For medium velocities of 3 feet it hence follows that $c_m = 0{,}81\ c_0$.

*Example.* In the line of current of a brook the velocity of the water is 4 feet, and the depth 6 feet, we have then the mean velocity at a corresponding perpendicular $c_m = 0{,}915 \ . \ 4 = 3{,}66$ feet, and that at the bottom $= 0{,}83 \ . \ 4 = 3{,}32$ feet; further, the velocity 2 feet below the surface is $v = (1 - 0{,}17 \ . \ \frac{1}{3})\ 4 = (1 - 0{,}057)\ 4 = 3{,}772$ feet; lastly, the mean velocity throughout the profile is, $c = 0{,}837 \ . \ 4 = 3{,}348$ feet, and according to Prony, $c = \frac{11{,}50}{13{,}97} \ . \ 4 = \frac{46}{13{,}97} = 3{,}29$ feet.

*Remark.* This and the following subjects have been fully treated of under the article "Bewegung des Wassers," in the "Allgemeinen Maschinenencyclopädie." New experiments and new views may be found in the following writings: Lahmeyer's "Erfahrungsresultate über die Bewegung des Wassers in Fluszbetten und Kanälen" Brunswick, 1845.

§ 364. *The best form of transverse section.*—The resistance which the bed opposes to the motion of the water in virtue of its adhesion, viscoscity, or friction, increases with the surface of contact between the bed and the water, and therefore with the perimeter $p$ of the water profile, or of the portion of the transverse section which comprises the bed. But as more filaments of water pass through a profile, the greater its area is, so this resistance of a filament increases also inversely as the area, and hence on the whole as the quotient $\frac{p}{F}$ of the perimeter of the water profile, and the area of the whole transverse profile.

That the resistance of friction of a running stream or river may be the smallest possible, we must give to its transverse section that form for which the perimeter $p$ for a given area is a minimum, or the area for a given perimeter a maximum. In enclosed conduits, as, for example, pipes, $p$ is the entire perimeter of the figure formed by the transverse profile. Now of all figures having an equal number of sides, the regular figure, and again, of all regular figures that which has the greater number of sides has for the same area the least perimeter; hence for enclosed conduits, the co-efficient of friction comes out the less, the nearer its transverse profile approaches to a regular figure, and the greater its number of sides; and the circle, which is a regular figure of an infinite number of sides, is in this case the profile which corresponds to the minimum of friction. We must, therefore, in estimating this resistance of friction, leave out of our consideration in the quotient $\frac{p}{F}$ the upper side or surface in contact with the air.

FIG. 493.


The rectangular and trapezoidal sections are those generally applied to canals, cuts, water-courses, &c. A horizontal line $EF$, Fig. 493, passing through the centre $M$ of the square $AC$, divides as well the area as also the perimeter into two equal parts, hence it follows that what is true for the square is also correct for these halves, and accordingly, of all rectangular transverse profiles, the half square $AE$, or that which is twice as broad as it is deep, corresponds to the least resistance of friction. The regular hexagon $ACE$, Fig. 494, may be likewise divided by a horizontal line $CF$ into two equal trapeziums, each of which, like the entire hexagon, has the greatest relative area, and consequently, of all trapezoidal profiles, half the regular hexagon or the trapezium $ABCF$ with the angle of slope $AFM = BCM$ of $60^0$ is that which

FIG. 494.


FIG. 495.


FIG. 496.


when applied gives the least resistance of friction. Half the regular octagon $ADE$, Fig. 495, half the regular decagon, and lastly, the semi-circle $ADB$, Fig. 496, afford under given circumstances the most advantageous transverse profiles for canals. The trapezoidal, or half the regular hexagon, gives a still less resistance than half the square (or rectangle, the ratio of whose sides is 1 to 2,) because the hexagon has a less relative perimeter than the square. Half the regular decagon gives a still less friction, and in general, the minimum of friction corresponds to the semi-circle. The profiles of channels of wood, stone or iron only, are made semi-circular and rectangular; the profiles of canals, on the other hand, which are cut and bricked, are constructed of the trapezoidal figure. Other figures, in consequence of difficulties in the execution, are not easily applicable.

§ 365. In the case where a canal is not walled up, but dug out of loose earth or sand, the angle of $60^0$ slope is too great, and the relative slope $cotg.\ 60^0 = 0{,}57735$ too small, because the banks would not have a sufficient stability; we are, therefore, under the

necessity of applying the trapezoidal profile, for which the inclination of the sides to the base must be still less than $60^0$, perhaps scarcely $45^0$, or even less. For a trapezoidal profile $ABCD$, Fig. 497, which has a perimeter and area equal to that of half the square, the relative slope $= \frac{4}{3}$, and the angle of slope hardly $36^0\ 52'$. If the height $BE$ be divided into three equal parts, the base $BC$ will then have two of them, the parallel line $AD$ ten, and each of the sides $AB = CD =$ five parts. In many cases the slope is made $= 2$, to which belongs an angle of $26^0\ 34'$, and sometimes it is even made still greater.

FIG. 497.


In every case the angle of slope $BAE = \theta$, Fig. 498, or the slope $n = \frac{AE}{BE} = cotang.\ \theta$ may be regarded as a given quantity dependent on the nature of the ground in which the canal is dug, and hence the dimensions of the profile which offers the least resistance have only further to be determined. Let the lower breadth $BC=b$, the depth $BE=a$, and the slope $= n$, we then obtain for the perimeter:

FIG. 498.


$$AB+BC+CD=p=b+2\ \sqrt{a^2+n^2a^2}=b+2\,a\ \sqrt{1+n^2},$$

for the area:

$$F = a\,b + n\,a\,a = a\,(b + n\,a),$$

and hence, inversely, $b = \frac{F}{a} - n\,a$, and the ratio:

$$\frac{p}{F} = \frac{1}{a} + \frac{a}{F}\,(2\ \sqrt{n^2+1} - n).$$

If we substitute for $a$, $a+x$, where $x$ is a small number, we may then put:

$$\frac{p}{F} = \frac{1}{a+x} + \frac{(a+x)}{F}\ (2\ \sqrt{n^2+1} - n)$$

$$= \frac{1}{a}\left(1 - \frac{x}{a} + \frac{x^2}{a^2}\right) + \frac{a+x}{F}\ (2\ \sqrt{n^2+1} - n)$$

$$= \frac{1}{a} + \frac{a}{F}(2\sqrt{n^2+1} - n) + \left(\frac{2\sqrt{n^2+1} - n}{F} - \frac{1}{a^2}\right)x + \frac{x^2}{a^3}.$$

Now that this value may be greater not only for a positive, but also for a negative value of $x$, than the first

$$\frac{1}{a} + \frac{a}{F}(2\sqrt{n^2+1} - n,$$

it is necessary that the member with the factor $x$ should vanish, and therefore that $\frac{p}{F}$ may become a minimum, we must have

$$\frac{2\sqrt{n^2+1}-n}{F} - \frac{1}{a^2} = 0, \textit{ i.e. } a^2 = \frac{F}{2\sqrt{n^2+1}-n},$$

or since:

$$n = \textit{cotang.}\ \Theta \text{ and } \sqrt{n^2+1} = \frac{1}{\sin.\ \Theta},\ a^2 = \frac{F \sin.\ \Theta}{2 - \cos.\ \Theta}.$$

Hence, therefore, the most appropriate form of profile corresponding to a given angle of slope $\Theta$ and a given area is determined by

$$a = \sqrt{\frac{F \sin.\ \Theta}{2 - \cos.\ \Theta}} \text{ and } b = \frac{F}{a} - a\ \textit{cotang.}\ \Theta.$$

*Example.* What dimensions must be given to the transverse profile of a canal, whose banks are to have 40° slope, and which is to conduct a quantity of water $Q$

$Q$ 75 $=$ 25 square

| Angle of slope. | Relative slope. | Dimensions of transverse profile. | | | | Quotient $\frac{p}{F}$ |
|---|---|---|---|---|---|---|
| | | Depth $a$. | Lower breadth $b$. | Absolute slope $na$. | Upper breadth $b+2na$ | |
| 90° | 0 | $0{,}707\sqrt{F}$ | $1{,}414\sqrt{F}$ | 0 | $1{,}414\sqrt{F}$ | $\frac{2{,}828}{\sqrt{F}}$ |
| 60° | 0,577 | $0{,}760\sqrt{F}$ | $0{,}877\sqrt{F}$ | $0{,}439\sqrt{F}$ | $1{,}755\sqrt{F}$ | $\frac{2{,}632}{\sqrt{F}}$ |
| 45° | 1,000 | $0{,}740\sqrt{F}$ | $0{,}613\sqrt{F}$ | $0{,}740\sqrt{F}$ | $2{,}092\sqrt{F}$ | $\frac{2{,}704}{\sqrt{F}}$ |
| 40 | 1,192 | $0{,}722\sqrt{F}$ | $0{,}525\sqrt{F}$ | $0{,}860\sqrt{F}$ | $2{,}246\sqrt{F}$ | $\frac{2{,}771}{\sqrt{F}}$ |
| 36° 52′ | 1,333 | $0{,}707\sqrt{F}$ | $0{,}471\sqrt{F}$ | $0{,}943\sqrt{F}$ | $2{,}357\sqrt{F}$ | $\frac{2{,}828}{\sqrt{F}}$ |
| 35° | 1,402 | $0{,}697\sqrt{F}$ | $0{,}439\sqrt{F}$ | $0{,}995\sqrt{F}$ | $2{,}430\sqrt{F}$ | $\frac{2{,}870}{\sqrt{F}}$ |
| 30° | 1,732 | $0{,}664\sqrt{F}$ | $0{,}356\sqrt{F}$ | $1{,}150\sqrt{F}$ | $2{,}656\sqrt{F}$ | $\frac{3{,}012}{\sqrt{F}}$ |
| 26° 34 | 2,000 | $0{,}636\sqrt{F}$ | $0{,}300\sqrt{F}$ | $1{,}272\sqrt{F}$ | $2{,}844\sqrt{F}$ | $\frac{3{,}144}{\sqrt{F}}$ |
| Semicircle | | $0{,}798\sqrt{F}$ | | | $1{,}596\sqrt{F}$ | $\frac{2{,}507}{\sqrt{F}}$ |

We see from this table that the quotient $\frac{p}{F}$ is least for the semi-circle, namely $= \frac{2{,}507}{\sqrt{F}}$; greater for the semi-hexagon, and greater still for the half square, and the trapezium of $36^0$ 52′, &c.

*Example.* What dimensions must be given to a profile, which has for an area of 40 square feet, a slope of its banks of 35°? From the preceding table, the depth $a = 0{,}697\sqrt{40} = 4{,}408$, the lower breadth $= 0{,}439\sqrt{40} = 2{,}777$ feet, the absolute slope $= 0{,}995\sqrt{40} = 6{,}293$ feet, the upper breadth $= 15{,}363$, and the quotient $\frac{p}{F} = \frac{2{,}870}{\sqrt{40}} = 0{,}4538$.

§ 367. *Uniform motion.*—The motion of water in beds is for a certain tract either *uniform* or *variable;* it is uniform when the mean velocity at all transverse sections of this length remains the same, and therefore, also, the areas of the sections equal; and variable, on the other hand, when the mean velocities, and therefore, also, the areas of the sections vary. We shall treat first of uniform motion.

In the uniform motion of water along the distance $AD=l$, Fig. 489, the whole fall $HD=h$ is expended in overcoming the friction of the water in the bed, because the water flows on with the same velocity with which it arrives, therefore, a height due to a velocity is neither taken up nor set free. If we measure this friction by the height of this column of water, we may then make the fall equal to this height. But the height due to the resistance of friction increases with the quotient $\frac{p}{F}$, with $l$ and with the square of the mean velocity $c$ (§ 329); hence then the formula holds good:

$$1.\ h = \zeta \cdot \frac{lp}{F} \cdot \frac{c^2}{2g},$$

in which $\zeta$ expresses a number deduced from experiment which may be called the *co-efficient of the resistance of friction.*

By inversion it follows:

$$2.\ c = \sqrt{\frac{F}{\zeta \cdot lp} \cdot 2gh}.$$

In determining, therefore, the fall, being given the length, the cross section and the velocity, and inversely, in deducing the velocity from the fall, the length and the cross section, we must know the co-efficient of friction $\zeta$. According to Eytelwein's reduction of the ninety-one observations of Du Buat, Brünings, Funk and Woltmann, $\zeta=0{,}007565$, and hence

$$h=0{,}007565 \cdot \frac{lp}{F} \cdot \frac{c^2}{2g}.$$

If we put $g=9{,}809$ metres or $31{,}25$ feet, we have for the metrical measure

$$h=0{,}0003856\ \frac{lp}{F} \cdot c^2 \text{ and } c=5{,}09 \sqrt{\frac{Fh}{pl}},$$

and for the foot measure:

$$h=0{,}0001245\ \frac{lp}{F} \cdot c^2 \text{ and } c=92{,}5 \sqrt{\frac{Fh}{pl}} \text{ English measure.}$$

For conduit pipes $\frac{lp}{F}=\frac{\pi l d}{\frac{1}{4}\pi d^2}=\frac{4\,l}{d}$, hence this formula gives for pipes $h=0{,}03026\ \frac{l}{d} \cdot \frac{v^2}{2g}$, whilst we have found more correctly for these (§ 331) for mean velocities

$$h = 0{,}025 \frac{l}{d} \cdot \frac{v^2}{2g}.$$

The friction, therefore, as might be expected, is greater in the beds of rivers than in metallic conducting pipes.

*Examples.*—1. What fall must be given to a canal of the length $l = 2600$ feet, lower breadth $b = 3$ feet, upper breadth $b_1 = 7$ feet, and depth $a = 3$ feet, if it is to conduct a quantity of water of 40 cubic feet per second? It is:

$p = 3 + 2\sqrt{2^2 + 3^2} = 10{,}211$, $F = \frac{(7+3)\,3}{2} = 15$, and $c = \frac{40}{15} = \frac{8}{3}$, hence the fall sought, $h = 0{,}000121 \cdot \frac{2600 \cdot 10{,}211}{15} \cdot \left(\frac{8}{3}\right)^2 = \frac{0.3146 \cdot 10{,}211 \cdot 64}{15 \cdot 9} = 1{,}52$ feet.

2. What quantity of water does a canal 5800 feet long, having a 3 feet fall, 5 feet deep, 4 feet lower and 12 feet upper breadth? Here:

$$\frac{p}{F} = \frac{4 + 2\sqrt{5^2 + 4^2}}{5 \cdot 8} = \frac{16{,}806}{40} = 0{,}42015;$$

hence the velocity

$$c = 90{,}9\sqrt{\frac{3}{0{,}42015 \cdot 5800}} = \frac{90{,}9}{\sqrt{0{,}14005 \cdot 5800}} = \frac{90{,}9}{\sqrt{812{,}29}} = \frac{90{,}9}{28{,}5}$$

$= 3{,}19$ feet, and the quantity of water $Q = Fc = 40 \cdot 3{,}19 = 127{,}6$ cubic feet, Prussian measure.

§ 368. *Co-efficients of friction.*—The co-efficient of friction for rivers, brooks, &c., the mean value of which, in the foregoing paragraphs, we have taken at 0,007565, is not constant, but, as in pipes, increases somewhat for small and diminishes for great velocities. We have therefore to put:

$$\zeta = \zeta_1\left(1 + \frac{a}{c}\right) \text{ or } \zeta_1\left(1 + \frac{a}{\sqrt{c}}\right).$$

The author of the work alluded to in § 363, finds from 255 experiments, the greater part of them undertaken by himself, for the Prussian measure $\zeta = 0{,}007409\left(1 + \frac{0{,}0299}{c}\right)$, and hence it follows for the metre $\zeta = 0{,}007409\left(1 + \frac{0{,}00939}{c}\right)$, and for English measure $007409\left(1 + \frac{0{,}0307}{c}\right)$.

It is manifest that these formulæ, for a velocity $c = 1\frac{1}{4}$ feet, give again the above assigned mean co-efficient of resistance $\zeta = 0{,}007565$. The following useful table of the co-efficients of resistance in the metrical measure serves for facilitating calculation.

| Velocity $c$. | 0,1 | 0,2 | 0,3 | 0,4 | 0,5 | 0,6 | 0,7 | 0,8 | 0,9 | Meter. |
|---|---|---|---|---|---|---|---|---|---|---|
| Co-efficient of resistance $\zeta = 0,00$. | 811 | 776 | 764 | 758 | 755 | 753 | 751 | 750 | 749 | |

| Velocity $c$. | 1 | 1,2 | 1,5 | 2 | 3 | Meter. |
|---|---|---|---|---|---|---|
| Co-efficient of resistance $\zeta = 0,00$. | 748 | 747 | 746 | 744 | 743 | |

The following table serves for the Prussian or English measure:

| Velocity $c$. | 0,3 | 0,4 | 0,5 | 0,6 | 0,7 | 0,8 | 0,9 | 1 | 1½ | 2 | 3 | 5 | 10ft. |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Co-efficient of resistance $\zeta = 0,00$. | 815 | 797 | 785 | 778 | 773 | 769 | 766 | 763 | 759 | 752 | 749 | 745 | 743 |

These tables find their direct application in all cases where the velocity $c$ is given and the fall to be found, and where the formula No. 1 of the former paragraph is applicable. But if the velocity $c$ is unknown, and its amount to be determined, these tables will then only admit of a direct application, when we have already an approximate value of $c$. We may set to work in the simplest manner by determining $c$ approximately by the formula $c = 50,9\sqrt{\frac{Fh}{pl}}$, and from this a value of $\zeta$, taken from the table, and the value so obtained put into the formula

$$\frac{c^2}{2g} = \frac{h}{\zeta} \cdot \frac{F}{lp} \text{ or } c = \sqrt{\frac{F}{\zeta l p} \cdot 2gh.}$$

From the velocity $c$, the quantity of water is then given by the formula $Q = Fc$.

If, lastly, the quantity and the fall are given, and, as is often requisite in the construction of canals, it be required to determine the transverse section, we may put $\frac{p}{F} = \frac{m}{\sqrt{F}}$ (*see* Table, § 366) and $c = \frac{Q}{F}$ into the formula $h = 0,007565 \frac{lp}{F} \cdot \frac{c^2}{2g}$, and write, therefore, $h = 0,007565 \frac{m l Q^2}{2 g F^{\frac{5}{2}}}$, and accordingly determine:

$F = \left(0{,}007565 \dfrac{m\, l\, Q^2}{2\, g\, h}\right)^{\frac{2}{5}}$, *i. e.* for the metre $F = 0{,}0431 \left(\dfrac{m l Q^2}{h}\right)^{\frac{2}{5}}$ or the foot measure $F = 0{,}0271 \left(\dfrac{m\, l\, Q^2}{h}\right)^{\frac{2}{5}}$. Hence it follows, approximately, that $c = \dfrac{Q}{F}$; if we take a correspondent value of $\zeta$ from one of the tables, more accurately $F = \left(\zeta \,.\, \dfrac{m\, l\, Q^2}{2\, g\, h}\right)^{\frac{2}{5}}$; and hence, more exact values for $c = \dfrac{Q}{F}$, $p = m \sqrt{F}$, as also for $a$, $b$, &c.

*Examples.*—1. What fall does a canal 1500 feet long, 2 feet lower and 8 feet upper breadth, and 4 feet depth require to give a discharge of 70 cubic feet per second? It is $p = 2 + 2\sqrt{4^2 + 3^2} = 12$, $F = 5 \,.\, 4 = 20$, $c = \dfrac{70}{20} = 3{,}5$, hence $\zeta = 0{,}00748$, and $h = 0{,}00748 \,.\, \dfrac{1500 \,.\, 12}{12} \,.\, \dfrac{3{,}5^2}{2g} = 6{,}732 \,.\, 0{,}196 = 1{,}33$ ft.
2. What discharge does a brook 40 feet broad, $4\frac{1}{2}$ feet mean depth, and 46 feet water profile, if it has a fall of 10 inches for a length of 750 feet? It is about $c = 90{,}9 \,.\, \sqrt{\dfrac{40 \,.\, 4{,}5 \,.\, 10}{46 \,.\, 750 \,.\, 12}} = \dfrac{90{,}9}{\sqrt{230}} = 6$ feet, and hence $\zeta = 0{,}00745$. Hence we obtain, more correctly:
$\dfrac{c^2}{2g} = \dfrac{Fh}{\zeta lp} = \dfrac{4{,}5 \,.\, 40 \,.\, 10}{0{,}00744 \,.\, 46 \,.\, 750 \,.\, 12} = \dfrac{1}{1{,}7112} = 0{,}5844$, and $c = 6{,}03$ feet. Lastly, the corresponding discharge is $Q = 4{,}5 \,.\, 40 \,.\, 6{,}03 = 1085$ cubic feet.
3. A trench 3650 feet long is to be cut, which for a total fall of 1 foot is to carry off a discharge of 12 cubic feet per second, what dimensions are to be given to the transverse profile, if it is to preserve a regular semi-hexagonal figure? Here $m = 2{,}632$ (*see* Table, § 366), hence, approximately, $F = 0{,}0271\,(2{,}632 \,.\, 3650 \,.\, 144)^{\frac{2}{5}} = 7{,}75$ square feet, and $c = \dfrac{12}{7{,}75} = 1{,}548$ feet. Hence $\zeta$ is to be taken $= 0{,}00758$, and $F = \left(0{,}00758 \,.\, 2{,}632 \,.\, \dfrac{3650 \,.\, 144}{62{,}5}\right)^{\frac{2}{5}} = 7{,}92$ square feet. Therefore the depth must be made: $a = 0{,}760 \sqrt{F} = 2{,}14$ feet, the lower breadth $= 0{,}877 \sqrt{F} = 2{,}47$, and the upper breadth $= 2 \,.\, 2{,}47 = 4{,}94$ feet.

*Remark.* Tables for shortening these calculations are given in the "Ingenieur."

§ 369. *Variable motion.*—The theory of the variable motion of water in beds of rivers may be reduced to the theory of uniform motion, provided the resistance of friction for a short length of the river may be considered as constant, and the corresponding height in like manner, as $\zeta = \,.\, \dfrac{l p}{F} \,.\, \dfrac{v^2}{2g}$. But besides this, regard must

be had to the *vis viva* of the water, which corresponds to a change of velocity.

FIG. 499.

Let $ABCD$, Fig. 499, be a short extent of river, of the length $AD = l$, the fall $DH = h$, and let $v_0$ be the velocity of the arriving, and $v_1$ that of the departing water. If we apply the rules of efflux to an element $D$ of the surface, we shall obtain for its velocity $v_1$

$$\frac{v_1^2}{2g} = h + \frac{v_0^2}{2g};$$

as regards an element $E$ below the surface, it is true that on the one side it has a greater pressure height $AG = EH$; but as the down-stream water re-acts with a pressure $DE$, there remains for it only the fall $DH = EH - ED$, as pressure inducing motion, and so, for this or any other element, the formula:

$$h = \frac{v_1^2 - v_0^2}{2g} \text{ answers,}$$

and if further, the resistance due to friction be added, we then obtain:

$$h = \frac{v_1^2 - v_0^2}{2g} + \zeta \cdot \frac{lp}{F} \cdot \frac{v^2}{2g},$$

where $p$, $F$ and $v$ are the mean values of the wetted perimeter, transverse section, and ~~velocity~~. If $F_0$ is the area of the upper, and $F_1$ that of the lower section, we may then put:

$$F = \frac{F_0 + F_1}{2}, \text{ and } Q = F_0 v_0 = F_1 v_1,$$

whence it follows that:

$$\frac{v_1^2 - v_0^2}{2g} = \frac{1}{2g}\left[\left(\frac{Q}{F_1}\right)^2 - \left(\frac{Q}{F_0}\right)^2\right] = \left(\frac{1}{F_1^2} - \frac{1}{F_0^2}\right)\frac{Q^2}{2g}, \text{ and}$$

$$\frac{v^2}{F} = \frac{v_0^2 + v_1^2}{F_0 + F_1} = \left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)\frac{Q^2}{F_0 + F_1}, \text{ we obtain:}$$

$$1.\ h = \left[\frac{1}{F_1^2} - \frac{1}{F_0^2} + \zeta \frac{lp}{F_0 + F_1}\left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)\right]\frac{Q^2}{2g}, \text{ as also}$$

$$2.\ Q = \frac{\sqrt{2gh}}{\sqrt{\frac{1}{F_1^2} - \frac{1}{F_0^2} + \zeta \frac{lp}{F_0 + F_1}\left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)}}.$$

The corresponding fall $h$ may be calculated by means of

the formula 1. from the quantity of water, the length and transverse section of a river or canal; and, inversely, the quantity of water from the fall, the length and the transverse section, by formula 2. To obtain greater accuracy, we may make the calculation for several short portions of the river, and take the arithmetical mean. If the total fall only is known, we must substitute this at once for $h$ in the last formula, and put

$$\frac{1}{F_1^2}-\frac{1}{F_0^2}=\frac{1}{F_n^2}-\frac{1}{F_0^2},$$

where $F_n$ denotes the area of the last section, and in place of

$$\zeta \cdot \frac{lp}{F_0+F_1}\left(\frac{1}{F_0^2}+\frac{1}{F_1^2}\right)$$

the sum of all similar values of the separate lengths of the river.

*Example.* A brook has for a distance of 300 feet a fall of 9,6 inches, the mean perimeter of its water profile is 40 feet, the area of the upper transverse profile 70, that of the lower 60 square feet; what quantity of water does this brook discharge? It is

$$Q = \frac{8,03\ \sqrt{0,8}}{\sqrt{\frac{1}{60^2}-\frac{1}{70^2}+0,007567\ .\ \frac{300\ .\ 40}{130}\left(\frac{1}{60^2}+\frac{1}{70^2}\right)}} =$$

$$= \frac{7,182}{\sqrt{0,0000731\ +\ 0,0003365}} = \frac{7,182}{\sqrt{0,0004096}} = 355 \text{ cubic feet.}$$

The mean velocity is $\frac{2Q}{F_0+F} = \frac{698}{130} = 5,37$ feet; hence, more accurately, $\zeta$ must be taken $= 0,00745$ in place of 0,007565, and therefore more nearly:

$$Q = \frac{7,182}{\sqrt{0,0000731\ +\ 0,0003314}} = 370 \text{ cubic feet.}$$

If the same brook, with the same head of water, had for a length of 450 feet, a fall of 11 inches, and if its upper transverse profile had an area of 50 and its lower of 60 square feet, and the mean perimeter of the profile measured 36 feet, we should then have:

$$Q = \frac{8,03\ \sqrt{0,9167}}{\sqrt{\frac{1}{60^2}-\frac{1}{50^2}+0,00745\ .\ \frac{450\ .\ 36}{110}\left(\frac{1}{60^2}+\frac{1}{50^2}\right)}}$$

$$= 8,03\ \sqrt{\frac{0,9167}{-0,0001222\ +\ 0,0007436}} = 340 \text{ cubic feet.}$$

The mean of these two values is $Q = \frac{370+340}{2} = 355$ cubic feet.

§ 370. In order to obtain a formula for the depth of water, let the upper depth $= a_0$ and the lower $= a_1$, the slope of the bed $= \alpha$, consequently the fall of the bed $= l$ *sin.* $\alpha$. We then obtain the fall of the water $h = a_0 - a_1 + l$ *sin.* $\alpha$, and there results the equation:

$$a_0-a_1-\left(\frac{1}{F_1^2}-\frac{1}{F_0^2}\right)\frac{Q^2}{2g}=\left[\zeta\,\frac{p}{F_0+F_1}\left(\frac{1}{F_0^2}+\frac{1}{F_1^2}\right)\frac{Q^2}{2g}-\sin.\,\alpha\right]l,$$

$$\text{hence } l = \frac{a_0 - a_1 - \left(\frac{1}{F_1^2} - \frac{1}{F_0^2}\right)\frac{Q^2}{2g}}{\zeta \frac{p}{F_0 + F_1}\left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)\frac{Q^2}{2g} - \sin. \alpha}.$$

The length $l$ which corresponds to a difference $a_0 - a_1$ of the depth of water, may be determined by this formula. But if the reverse problem is to be solved, we must do it by the method of approximation, and first determine the distances $l_1$ and $l_2$ corresponding to the assumed depressions $a_0 - a_1$, and $a_1 - a_2$, and from these calculate by a proportion, the depression corresponding to a given distance $l$.*

The formula is further capable of simplification when the breath $b$ of the running water is constant, or may be considered as such. In this case we put:

$$\left(\frac{1}{F_1^2} - \frac{1}{F_0^2}\right)\frac{Q^2}{2g} = \frac{F_0^2 - F_1^2}{F_0^2\, F_1^2} \cdot \frac{Q^2}{2g} = \frac{(F_0 - F_1)\,(F_0 + F_1)}{F_1^2} \cdot \frac{v_0^2}{2g}$$

$$= \frac{(a_0 - a_1)\,(a_0 + a_1)}{a_1^2} \cdot \frac{v_0^2}{2g} \text{ approximately} = 2\,\frac{(a_0 - a_1)}{a_0} \cdot \frac{v_0^2}{2g},$$

and likewise:

$$\frac{p}{F_0 + F_1}\left(\frac{1}{F_0^2} + \frac{1}{F_1^2}\right)\frac{Q^2}{2g} = \frac{p\,(F_0^2 + F_1^2)}{(F_0 + F_1)\,F_1^2} \cdot \frac{v_0^2}{2g} \text{ approximately}$$

$$= \frac{p}{a_0 b} \cdot \frac{v_0^2}{2g}, \text{ hence } l = \frac{(a_0 - a_1)\left(1 - \frac{2}{a_0} \cdot \frac{v_0^2}{2g}\right)}{\zeta \cdot \frac{p}{a_0 b} \cdot \frac{v_0^2}{2g} - \sin. \alpha}, \text{ and hence}$$

$$\frac{a_0 - a_1}{l} = \frac{\zeta \cdot \frac{p}{a_0 b} \cdot \frac{v_0^2}{2g} - \sin. \alpha}{1 - \frac{2}{a_0} \cdot \frac{v_0^2}{2g}}.$$

The difference $(a_0 - a_1)$ of the depth corresponding to a given extent $l$ may be calculated directly by this formula.

*Example.* In a horizontal trench, 5 feet broad and 800 feet long, it is desired to carry off a 20 cubic feet discharge, and to let it flow in at a depth of 2 feet, what depth will the water at the end of the canal have? Let us divide the whole length into two equal portions, and determine from the last formula the fall for each of them.

Here the $\sin. \alpha = 0$, $l = \frac{800}{2} = 400$, and $b = 5$; for the first portion

* See "Ingenieur," Arithmetik, § 16, v.

$v_0 = \frac{20}{2.5} = 2$, hence $\zeta = 0{,}00752$, also $a_0 = 2$; since $p = 8\frac{1}{2}$, it follows that

$$a_0 - a_1 = \left(\frac{0{,}00752 \cdot \frac{8{,}5}{10} \cdot \frac{4}{2g}}{1 - \frac{2}{2} \cdot \frac{4}{2g}}\right) . 400 = \frac{0{,}1588}{0{,}934} = 0{,}170 \text{ feet.}$$

Now, for the second half, $a_1 = 2 - 0{,}170 = 1{,}830$, and $p_1 = 8{,}2$, $v_1 = \frac{20}{9{,}125} = 2{,}192$, and the depression of the second portion:

$$a_1 - a_2 = \left(\frac{0{,}00752 \cdot \frac{8{,}2}{9{,}125} \cdot \frac{2{,}192^2}{2g}}{1 - \frac{2}{1{,}825} \cdot \frac{2{,}192^2}{2g}}\right) . 400 = \frac{0{,}2016}{0{,}916} = 0{,}219 \text{ feet,}$$

hence the whole depression $= 0{,}170 + 0{,}219 = 0{,}389$, and the depth of water at the lower end $= 2 - 0{,}389 = 1{,}611$.

§ 371. *Floods.*—When the depth of water in rivers and canals varies, variations in the velocity and discharge take place likewise. A greater depth of water not only involves a greater section, but also a greater velocity, and hence, for two reasons, a greater quantity of water, and likewise a diminution of the depth of water, gives a diminution of the section and the velocity, and hence also a decrease of the discharge. If the original depth $= a$, and any increased depth $= a_1$, the upper breadth of the canal $= b$, then the augmentation of the section may be put $= b\,(a_1 - a)$, and hence the section afterwards $a_1 - a$, $F_1 = F + b\,(a_1 - a)$, it also follows from this that

$$\frac{F_1}{F} = 1 + \frac{b\,(a_1 - a)}{F}, \text{ and}$$

$$\sqrt{\frac{F_1}{F}} \text{ approximately} = 1 + \frac{b\,(a_1 - a)}{2F}.$$

If further $p$ be the original, $p_1$ the increased perimeter of the water profile, and $\Theta$ the angle of slope of the banks, then

$$p_1 = p + \frac{2\,(a_1 - a)}{sin.\,\Theta}, \text{ hence } \frac{p_1}{p} = 1 + \frac{2\,(a_1 - a)}{p\,sin.\,\Theta}, \text{ and}$$

$$\sqrt{\frac{p_1}{p}} = 1 + \frac{a_1 - a}{p\,sin.\,\Theta}, \text{ as also } \sqrt{\frac{p}{p_1}} = 1 - \frac{a_1 - a}{p\,sin.\,\Theta}.$$

Now the velocity with the first depth of water is

$c=92{,}5\sqrt{\frac{Fh}{pl}}$, and with the second $c_1=92{,}5\sqrt{\frac{F_1}{p_1}\cdot\frac{h}{l}}$,
hence we may put:

$$\frac{c_1}{c}=\sqrt{\frac{F_1}{F}}\cdot\sqrt{\frac{p}{p_1}}=\left(1+\frac{b\,(a_1-a)}{2\,F}\right)\left(1-\frac{a_1-a}{p\,sin.\,\Theta}\right)$$

$$=1+(a_1-a)\left(\frac{b}{2F}-\frac{1}{p\,sin.\,\Theta}\right),$$

therefore the relative change of velocity:

$$1.\quad \frac{c_1-c}{c}=(a_1-a)\left(\frac{b}{2\,F}-\frac{1}{p\,sin.\,\Theta}\right).$$

On the other hand, the ratio of the discharge is:

$$\frac{Q_1}{Q}=\frac{F_1c_1}{Fc}=\left(1+\frac{b\,(a_1-a)}{F}\right)\left[1+(a_1-a)\left(\frac{b}{2\,F}-\frac{1}{p\,sin.\,\Theta}\right)\right]$$

$$=1+(a_1-a)\left(\frac{3\,b}{2F}-\frac{1}{p\,sin.\,\Theta}\right),$$

and the relative increase:

$$2.\quad \frac{Q_1-Q}{Q}=(a_1-a)\left(\frac{3\,b}{2F}-\frac{1}{p\,sin.\,\Theta}\right).$$

Less accurately, but in many cases, especially in broad canals with little slope, we may put $F=ab$, and neglect $\frac{1}{p\,sin.\,\Theta}$, whence it follows more simply that:

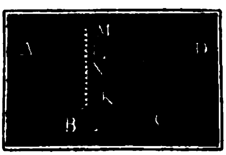
$$\frac{c_1-c}{c}=\tfrac{1}{2}\,\frac{a_1-a}{a},\ \text{and}$$

$$\frac{Q_1-Q}{Q}=\tfrac{3}{2}\,.\,\frac{a_1-a}{a}.$$

***From this, therefore, the relative change of velocity is $\frac{1}{2}$, and the relative change in the quantity of water $\frac{3}{2}$ that of the relative change in the depth of water.***

*Examples.*—1. When the head of water increases $\frac{1}{10}$ of its original amount, the velocity is then $\frac{1}{20}$, and the quantity $\frac{3}{20}$ greater than its original value.—2. When the depth diminishes 8 per cent., the velocity then diminishes 4, and the quantity 12 per cent.—3. From the more correct formula:

FIG. 500.




$$\frac{Q_1-Q}{Q}=(a_1-a)\left(\frac{3\,b}{2\,F}-\frac{1}{p\,sin.\,\Theta}\right)$$

a scale of the depth of water $KM$, Fig. 500, may be constructed, on which the discharge of a canal corresponding to any depth $KL$, may be read off, when

the quantity of water for a certain mean depth is once known. If $b = 9$ feet, $b_1 = 3$, $a = 3$, and $\Theta = 45°$, we then have $F = \frac{(9+3)\,3}{2} = 18$ square ft., $p = 3 + 2.3\sqrt{2} = 11,485$ and $sin.\ \Theta = \sqrt{\tfrac{1}{2}} = 0,707$, hence:

$$\frac{Q_1 - Q}{Q} = \left(\frac{3\ .\ 9}{2\ .\ 18} - \frac{1}{11,485\ .\ 0,707}\right)(a_1 - a) = (0,750 - 0,123)\ (a_1 - a)$$

$= 0,627\ (a_1 - a)$. If the quantity corresponding to a mean head of water $Q = 40$ cubic feet, we then have $Q_1 = 40 + 40\ .\ 0,627\ (a_1 - a) = 40 + \frac{a_1 - a}{0,04}$. If $a_1 - a = 0,04$ feet $= 5,76$ lines, it follows that $Q_1 = 41$; $a_1 - a = 0,08$ feet $= 11,52$ lines, we then have $Q_1 = 42$ cubic feet; if, further, $a_1 - a = -0,04$, then is $Q_1 = 39$ cubic feet, &c. A scale, therefore, whose intervals are $LM = LN = 5,76$ lines, gives the discharge accurately to a cubic foot. Of course the accuracy is the less, the more the head of water differs from a mean value.

*Remark.* The conducting and carrying off of water in canals, as well as the subject of weirs and dams, will be fully treated of in the Second Part.

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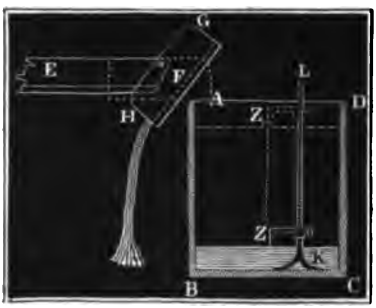
# CHAPTER VIII.

## HYDROMETRY, OR THE DOCTRINE OF THE MEASUREMENT OF WATER.

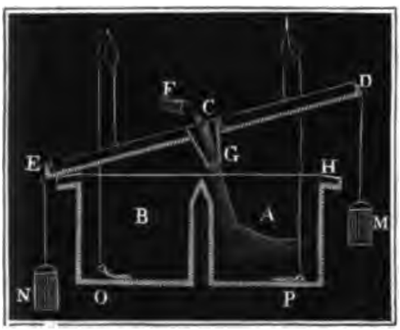
§ 372. *Gauges.*—The quantity, which a stream discharges in a certain time, is determined either by a gauge, by an apparatus of efflux, or by an hydrometer. The most simple way of measuring water is by the gauge, *i. e.* by the use of a graduated vessel, but this method is only applicable to small discharges, carried off by pipes or small brooks, or drains. The guage vessel is generally made of wood, and of a rectangular form, and to increase its strength is bound round with iron-hooping. In the "Ingenieur" we have shown how the exact capacity of this vessel may be measured. The water is conducted into it by a trough $EF$, Fig. 501, at whose extremity there is a double valve $GH$, by which the water may be made to flow at will into the vessel $AC$, or by the side of it. To obtain the exact depth of the body of water

FIG. 501.



in the vessel, a scale $KL$ is further applied. If before measurement, the index $Z$ be moved down to the surface of the water, already in the vessel, and merely covering the bottom, and the head of water read off from the scale, we shall obtain the height $ZZ_1$ of the gauged water by subtraction of this from the head of water which the scale indicates when the index hand $Z_1$ is brought into contact with the surface of water at the end of the observation. Before measurement, the valve must be so placed that the water may flow off outside the vessel. When we are convinced that the efflux in the trough is in a state of permanency, and, watch in hand, have noted a certain moment, the valve must then be turned, so that the water may run into the guage vessel, and after it is either partly or entirely filled, a second interval is noted by the watch, and the valve again brought into its first position. From the mean section $F$ of the vessel, and the depth $ZZ_1 = a$ of the body of water, the whole quantity $= Fa$ may be estimated, and again from the time of filling $t$, given by the difference of the times observed, the quantity of water per second $Q = \frac{Fa}{t}$.

FIG. 502.




*Remark.* To determine a variable quantity of efflux at each period of the day, we may make use of the apparatus represented in Fig. 502, as applicable especially in salt works. There are here two guage vessels, $A$ and $B$, which alternately fill and empty themselves, and the water which is conducted by the pipe $F$ passes through a short pipe $CG$, which is rigidly connected with a lever $DE$, revolving about $C$. When one vessel $A$ becomes filled, the water then flows through a short tube $H$ into the little vessel $M$, this draws the lever down again on one side, and the pipe $CG$ comes into such a position that the water is conducted into $B$. The drawing up of the valves $O$ and $P$ takes place by means of strings passing over pulleys, whose extremities are connected with the lever and sustained by iron balls, which impart a final impulse to the descent of the lever. The vessels $M$ and $N$ have small efflux orifices, by which they empty themselves after each reversion of the lever. An apparatus is besides applied, by which the number of strokes may be read off at any time.

§ 373. *Efflux regulators.*—Small and medium discharges are very frequently determined by means of their flow through a

definite orifice, and under a known head. From the area $F$ of the orifice, the head of water $h$, and the efflux co-efficient $\mu$, the discharge per second $Q = \mu F \sqrt{2gh}$ is given. The Poncelet orifices are those best adapted for this purpose, because the co-efficients of efflux of these under different heads of water are known with great accuracy (§ 316), still they are applicable only to certain mean discharges. The author availed himself of four such orifices for his measurements, one of five, one of ten, one of fifteen, one of twenty centimetres depth, but all of twenty centimetres width. These orifices are cut out of brass plate, and fixed to a wooden frame $AC$, Fig. 503, which is fastened by four strong iron screws to each wall. In many cases indeed greater orifices, the co-efficients of efflux for which are not so accurately determined, and sometimes weirs must be used which admit generally of a still less accuracy. In all cases, however, the rule holds good, that we must endeavour to get as complete and perfect a contraction as possible, and for this reason must give to the orifice, if it is in a thick plate, a slope outward. The corrections which must be applied for incomplete and partial contraction, have been sufficiently distinguished in paragraphs 319, 320, &c. To measure the water of a brook we must set the frame with its orifice, and wait for the moment when the head of water is permanent. For the measurement of the head of water we must avail ourselves of the index scale, Fig. 500, or of the moveable scale $EF$, Fig. 505. If we would note the efflux directly from the apertures of sluices, it is better to fix before hand a pair of brass scales $BC$ and $DE$, Fig. 504, with their indices $F$ and $G$ to the slide, and to the sluice-board $A$, in order to read off more safely the height of the aperture. It is generally better for the purpose of measuring water, to put on a new sluice board with its guide, and with the

FIG. 503.

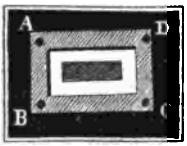



FIG. 504.

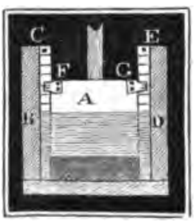
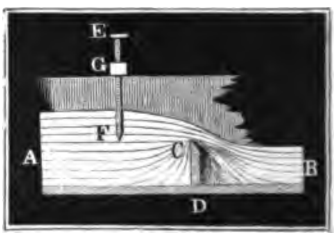
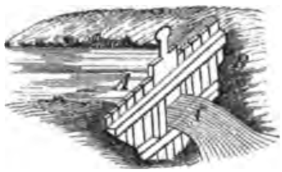



FIG. 505.



requisite slope outwards. The simplest means of measuring water in a channel, consists in putting in a broad $CD$, with its upper edge sloped off, Fig. 505, and measuring the fall produced by it. If the channel is long, and there is little rise, it is generally some time before the condition of permanency takes place, and it is for this reason good, before measurement, to put on a second board, so as to impede the efflux of water for a long time, in order to accelerate the rise to a height corresponding to a state of permanency.

FIG. 506.



To measure the quantity of water of a brook, we may dam it up with posts and boards as in Fig. 506, and let the water $C$ run off through an aperture, or we may avail ourselves of a simple overfall or weir, but of this we shall treat in the second part.

§ 374. But as it is often long before a state of permanency occurs in water dammed up by this construction, we may adopt with advantage the following method, first proposed by Prony. We may first close entirely the aperture by a sluice-board, and let the water rise to some height, or as high as circumstances will admit, then draw it so far up that more water may flow in than out, and measure the heads of water at equal and very short intervals; lastly, the aperture of the sluice must be again perfectly closed, and the time $t_1$ in which the water rises to the first height, further noted. In each case, then, during the whole time of observation $t + t_1$, as much water flows in as out, and hence the quantity flowing in, in the time $t + t_1$, may be expressed by the quantity flowing out in the time $t$. If the heads of water during the depressions are $h_0$, $h_1$, $h_2$, $h_3$, and $h_4$, we have then the mean velocity of efflux:

$$v = \frac{\sqrt{2g}}{12}\,(\sqrt{h_0} + 4\sqrt{h_1} + 2\sqrt{h_2} + 4\sqrt{h_3} + \sqrt{h_4}),\ (see\ \S\ 351)$$

and if the area of the aperture $= F$, we have then the quantity of efflux in the time $t$:

$$V = \frac{\mu F t \sqrt{2g}}{12}\,(\sqrt{h_0} + 4\sqrt{h_1} + 2\sqrt{h_2} + 4\sqrt{h_3} + \sqrt{h_4}),\text{ and}$$

the quantity flowing in per second:

$$Q = \frac{V}{t + t_{,}} = \frac{\mu F t \sqrt{2g}}{12\,(t+t_1)}\,(\sqrt{h_0} + 4\sqrt{h_1} + 2\sqrt{h_2} + 4\sqrt{h_3} + \sqrt{h_4}).$$

*Example.* To measure the water of a brook used for the driving of a water-wheel, which has been dammed up by a sluice, Fig. 506, after opening the rectangular aperture, the following is observed: the original head of water is 2 feet, after 30″ 1,8 feet, after 60″ 1,55 feet, after 90″ 1,3 feet, after 120″ 1,15 feet, after 150″ 1,05 feet, and after 180″ 0,9 feet, breadth of the aperture 2 feet, depth ½ foot, time of rising to the first height with closed aperture = 110″. The mean velocity of efflux is:

$$v = \frac{8{,}03}{18}\,(\sqrt{2} + 4\sqrt{1{,}8} + 2\sqrt{1{,}55} + 4\sqrt{1{,}3} + 2\sqrt{1{,}15} + 4\sqrt{1{,}05} + \sqrt{0{,}9})$$

$= 0{,}440\,(1{,}414 + 5{,}364 + 2{,}490 + 4{,}561 + 2{,}145 + 4{,}099 + 0{,}949)$ $= 0{,}440 \,.\, 21{,}022 = 9{,}248$ feet; but now $F = 2 \,.\, \frac{1}{2} = 1$ square foot, hence is follows that the theoretical discharge is $= 9{,}248$ cubic feet. If the co-efficient of efflux is taken $= 0{,}61$, we finally obtain the quantity of water sought:

$$Q = \frac{0{,}61 \,.\, 180}{180 + 110} \,.\, 9{,}248 = 3{,}495 \text{ cubic feet.}$$

§ 375. *The "pouce d'eau," or water-inch.*—To measure small discharges, we avail ourselves of the flow through round 1 inch wide orifices, in a thin plate, under a given pressure. The discharge given through such an aperture under the least pressure, or when the surface is only a line above the uppermost position of the orifice, is called an *inch of water*. The French assume for the water-inch (old Paris measure) 15 pints, or 19,1953 cubic metres of water in the 24 hours; therefore in 1 hour 0,7998, and in 1 minute 0,0133 cubic metres; yet older data, by Mariotte, Couplet, and Bossut, vary not a little from the above. According to Hagen, an inch of water (Prussian measure) delivers in 24 hours 520 cubic feet, therefore in a minute, 0,3611 cubic feet. The double water modulus of Prony, which corresponds to an orifice of 2 centimetres diameter, with a pressure of 5 centimetres, and discharges 20 cubic metres of water in 24 hours, has not been adopted. The apparatus by which water is measured by the inch is represented in Fig. 507. The water to be measured flows through the tube $A$ into a box; from this it passes through holes below in the partition $CD$ into the box $E$, and from this through a horizontal row of round orifices $F$, of exactly 1 inch width,

FIG. 507.

and cut in tin plate, into the reservoir $G$. That the surface of water may stand a line above the heights of these orifices, it is necessary that there be a sufficient number of them, and that a part of them be closed by stoppers. For great discharges the whole water is divided, and in this way a part, only one-tenth, is measured. This division may be accomplished easily, by first conducting the water into a reservoir, with a certain number of orifices at the same level, and only to receive the quantity delivered by one orifice in the apparatus represented above.

*Remark* 1. We may apply also cocks and other regulating apparatus to the measurement of water, if we know the co-efficient of resistance for each position. If $h$ is the head of water, $F$ the transverse section of the pipe, and $\mu$ the co-efficient of efflux, for a cock quite opened, we then have the discharge $Q = \mu F \sqrt{2gh}$, as inversely, $\mu = \frac{Q}{F\sqrt{2gh}}$ and $\frac{1}{\mu^2} = \left(\frac{F}{Q}\right)^2 . 2gh$. If now we put the co-efficient of resistance corresponding to a position of the cock, and taken from the tables already given $= \zeta$, we then have the corresponding discharge:

$$Q_1 = F\sqrt{\frac{2gh}{\frac{1}{\mu^2} + \zeta}} = \frac{\mu F\sqrt{2gh}}{\sqrt{1+\mu^2\zeta}} = \frac{Q}{\sqrt{1+\zeta\left(\frac{Q}{F}\right)^2 . \frac{1}{2gh}}}.$$

For convenience sake, we may construct for ourselves a table, so that we can find at a glance the discharge corresponding to a position of the cock, or the position of the cock corresponding to a given dischar e. If, for example, $\mu = 0{,}7$ and $F = 5$ square inches, we have then:

$$Q_1 = \frac{0{,}7 \;.\; 4 \;.\; 12 \;.\; 8{,}03\sqrt{h}}{\sqrt{1 + 0{,}49 \,.\, \zeta}} = 269{,}8\sqrt{\frac{h}{1 + 0{,}49\,\zeta}} \text{ cubic inches,}$$

or if $h$ is constantly 1 foot, $Q_1 = \frac{269{,}8}{\sqrt{1 + 0{,}49\,\zeta}}$. If now the positions of the cock are at 5°, 10°, 15°, 20°, 25°, &c., the co-efficients of resistance, 0,057; 0,293; 0,758; 1,559; 3,095, the discharges corresponding to these are: 262,1; 248,4; 226,8; 200,0; 167,4 cubic inches.

FIG. 508.

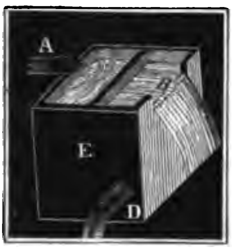
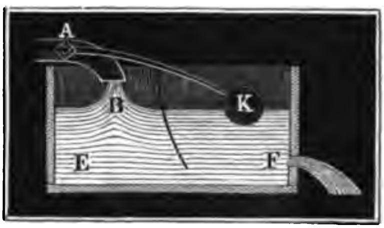



FIG. 509.




*Remark* 2. To regulate the flow through an orifice $D$, Fig. 508, we may apply a weir $B$ that the excess of water from the pipe $A$ may flow over, and that a

constant pressure may be maintained in the reservoir *DE*. That there may be no loss of water, a cock or a valve *A*, Fig. 509, is applied, which is regulated by a float *K* acting upon a lever, so that as much water only flows in through *B* as flows out through *F*.

§ 376. *Floating bodies.*—The discharge of large brooks, canals, and rivers, can be determined only by an hydrometer indicating the velocity. Of such instruments floating bodies are the most simple. We may use any floating body for this purpose, but it is better to have bodies of a moderate size, which are only a little specifically lighter than water. Substances of about $\frac{1}{10}$ of a cubic foot contact are large enough. Very large ones do not easily assume the velocity of water, and very small ones again, especially when much above the water, are easily disturbed in their motion by accidental circumstances, sometimes by the air on the surface of the fluid. Often, plain pieces of wood are sufficient; it is better, however, if they have a coating of some bright varnish, and better still if the floats are hollow, such as glass flasks, tin balls, &c., because these may be filled at will with water. Swimming balls are most frequently used. They are from 4 to 12 inches diameter, and made of brass, and painted over with some light oil-colour, to make them more visible to the eye, and have an opening with a neck, that they may be filled with water and stopped. A floating ball, such as *A*, Fig. 510, gives only the velocity at the surface, and often only that of the main current; but by suspending two balls one to the other, *A* and *B*, Fig. 511, we may determine the velocity at different depths. In this case, the one ball *B*, which swims under water, is quite filled with the fluid; the other, however, which swims on the surface, is only filled just enough to make it float a little above the surface. Both balls are connected with each other by a string or wire, or by a light wire chain. The velocity $c_0$ of the surface is first determined by the single ball, and then the mean velocity of the two observed by the connection of balls. If now the velocity at the depth of the

FIG. 510.

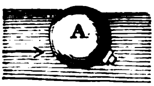
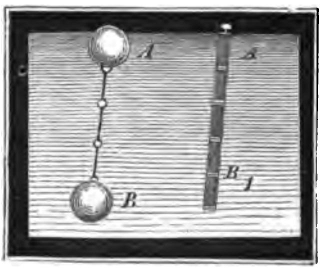



FIG. 511.



second ball be denoted by $c_1$, we may then put $c = \frac{c_0 + c_1}{2}$, and hence, inversely, $c_1 = 2c - c_0$. Whilst now both balls are connected with one another by longer and longer wires, we may in this manner find the velocities at greater depths. The mean velocity $c$ of a perpendicular is otherwise given if the second ball is allowed to swim a little above the bottom, and $c$ is made $= 2c_1 - c_0$; still more accurately, however, if for $c_1$ the mean of all the velocities observed in a perpendicular be taken.

To find the mean velocity in a perpendicular, the floating staff, $A_1 B_1$, represented in Fig. 512, is used. This is particularly convenient for measurements in canals and cuts when it is composed of short pieces screwed together. The floating staff which the author uses, is composed of 15 hollow portions, each 1 decimetre in length. That this may swim pretty upright, the lowermost piece is loaded with shot, so that the top just rises above the water. The number of pieces composing the staff, depends, of course, on the depth of the canal.

Both with the floating staff as well as the connection of balls, it may be observed that the velocity at the surface, when the motion of the water in beds is unimpeded, is greater than at the bottom, because the top of the staff swims in advance of the bottom, and the upper ball in advance of the lower. In contraction only, for example, when the water is dammed up by piles, &c., does the contrary take place.

*Remark.* As a rule, especially with large and floating bodies, as ships, &c., the velocity of the swimming body is somewhat greater than that of the water; not so much because these bodies in swimming float down an inclined plane formed by the surface of the water, but because they take none, or scarcely any, part in the irregular intimate motion of the water; still, the variation for small floating bodies is so slight that it may be neglected.

§ 377. The velocity of a floating ball is found by noting the time $t$ with a good seconds watch, or a half-second pendulum (§ 247), which it takes while floating on the water to describe a measured distance $s$, marked out on the banks. Then the required velocity of the ball is $c = \frac{s}{t}$. That the time $t$ corresponding to the space described along the bank may be accurately found, it is necessary, with the assistance of a cross line or lines, to erect at

the opposite bank two signal staffs $C$ and $D$, perpendicular at $A$ and $B$, Fig. 512. If we place ourselves behind $A$, we may then note the moment when the float $K$, dropped in a little above $A$, comes into the line $AC$, and if behind $B$, we may then also observe the time by a watch held in the hand, when the float reaches the line $BD$, and we then find by subtraction of the times of observation, the required time $t$ corresponding to the describing of the space $s$. Besides the mean velocity $c$ of the water, the area $F$ of the transverse profile is further required for determining the quantity of water $Q = F c$. To find this area, it is necessary to know the breadth and the mean depth of the water. The depths are measured by a sounding rod $AB$, Fig. 513, having a rhomboidal section, and a board $B$ at the foot; for greater depths we may also use a sounding chain, at whose extremity there is an iron plate, which in sinking rests on the bottom. The breadth and the abscissæ corresponding to the measured depths, or the distances from the banks in canals and small brooks $EFG$ Fig. 514, are found by stretching across a measuring chain $AB$, or the placing of a rod right across the running water. For broad rivers this is determined by a measure table $M$, which is placed at a proper distance $AO$, from the section $EF$, Fig. 515, which is to be measured. If $ao$ is the distance $AO$ between $A$ and $O$,

FIG. 512.


FIG. 513.


FIG. 514.

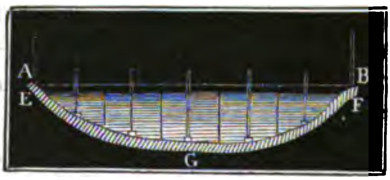
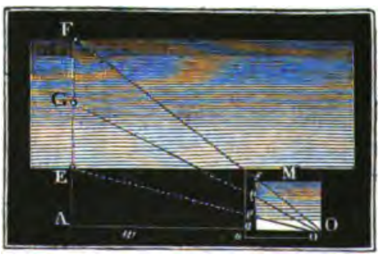



FIG. 515.



reduced to the table, and if $ao$ is placed in the direction of $AO$, and thereby also the direction of the breadth $af$ made parallel to the line of breadth $AF$ marked out, then each line of vision will intersect in the direction of the points $E$, $F$, $G$, &c., in the profile, the corresponding points $e$, $f$, $g$ on the table, and $ae$, $af$, $ag$, &c., are the distances $AE$, $AF$, $AG$, &c., in the reduced measure. It is not, therefore, necessary on putting in the sounding rod, and measuring the depths by it, to measure the distances of the corresponding points of the banks, if the engineer standing by the measure table looks at the sound on its being put in, in the line $EF$.

If now the breadth $EF$, Fig. 514, of a transverse profile, consist of parts $b_1$, $b_2$, $b_3$, &c., and the mean depths within those parts $a_1$, $a_2$, $a_3$, and the mean velocities $c_1$, $c_2$, $c_3$, &c., we have then the area of the profile:

$$F = a_1 b_1 + a_2 b_2 + a_3 b_3 + \ldots,$$

the discharge:

$$Q = a_1 b_1 c_1 + a_2 b_2 c_2 + a_3 b_3 c_3 + \ldots,$$

and, lastly, the mean velocity:

$$c = \frac{Q}{F} = \frac{a_1 b_1 c_1 + a_2 b_2 c_2 + \ldots}{a_1 b_1 + a_2 b_2 + \ldots}.$$

*Example.* In a tolerably straight and uniform extent of river, we have at the middle points of portions of the breadth:

| | | | | | |
|---|---|---|---|---|---|
| | 5 feet, | 12 feet, | 20 feet, | 15 feet, | 7 feet, |
| The depths . . . . . | 3 „ | 6 „ | 11 „ | 8 „ | 4 „ |
| The mean velocities . . . | 1,9 „ | 2,3 „ | 2,8 „ | 2,4 „ | 2,1 „ |

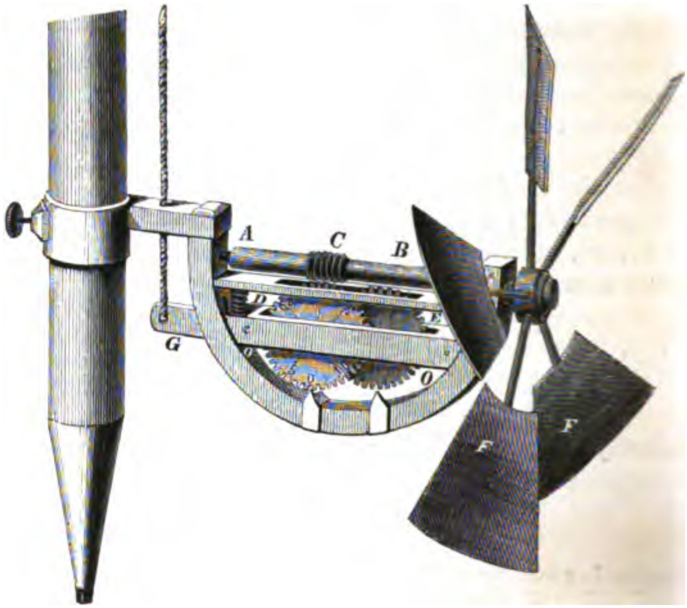
Hence we may put:

The area of the profile $F = 5 \,.\, 3 + 12 \,.\, 6 + 20 \,.\, 11 + 15 \,.\, 8 + 7 \,.\, 4 = 455$ square feet. The quantity of water $Q = 15 \,.\, 1{,}9 + 72 \,.\, 2{,}3 + 220 \,.\, 2{,}8 + 120 \,.\, 2{,}4 + 28 \,.\, 2{,}1 = 1156{,}9$ cubic feet. The mean velocity is $c = \frac{1156{,}9}{455} = 2{,}54$ feet.

§ 378. *The Tachometer.*—The most preferable hydrometer is the tachometer of Woltmann, Fig. 516. It consists of a horizontal axle $AB$, with from two to five vanes $F$, placed at an inclination to the direction of the axis, and gives, when immersed in the water and held at right angles to the ~~duration~~ of direction. motion, by the number of its revolutions in a certain time, the velocity of the running water. To read off the number of these revolutions, the axle has a few turns of a screw $C$, and these work

into the teeth of a wheel $D$, upon whose lateral surfaces number are engraved, and give, by means of an index, the number of revolutions of the wheel. But to be able to register a great number of revolutions, upon the axle of this toothed wheel, there is a contri-

FIG. 516.




vance which works into the teeth of the wheel $E$, by which, like the hands of a watch, several, for example, five or tenfold revolutions may be read off. If, for example, each of the two toothed wheels has fifty teeth, and the trundle ten, then the second wheel revolves one tooth whilst the first advances five teeth, or the vanes make five revolutions, if the index of the first wheel points to $27 = 25 + 2$, and that of the second to 32, the corresponding number of revolutions of the vanes is accordingly: $= 32 \cdot 5 + 2 = 162$. The entire instrument is screwed to a staff having a tin vane attached, to admit of easy immersion in the water, and of being kept opposed to the current. But that the wheel-work may only revolve during the time of observation, the axis is connected with a lever $GO$, which is pressed down by a spring, so that the teeth of the first wheel are thrown into gear with the screw only when the lever is drawn up by a string.

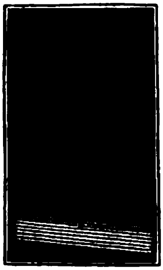
The number of revolutions of a wheel in a certain time, for example, in a second, is not exactly proportional to the velocity of the water, hence we cannot put $v = au$, where $u$ is the number of revolutions, $v$ the velocity, and $a$ a number deduced from experiments; but rather: $v = v_0 + a\, u$, or more correctly $v = v_0 + a\, u + \beta\, u^2 \ldots$, or still more correctly:
$v = a\, u + \sqrt{v_0^2 + \beta\, u^2}$, where $v_0$ is the velocity, at which the water is no longer able to turn the wheel, and $a$ and $\beta$ are co-efficients from experiment. The constants $v_0$, $a$ and $\beta$, are to be determined for each instrument in particular. With their assistance the velocity is known from a single observation, nevertheless it is always safer to make at least two, and to substitute the mean value as the correct one.

*Example.* If for a sail-wheel $v_0 = 0{,}110$ feet, $a = 0{,}480$, and $\beta = 0$, therefore $v = 0{,}11 + 0{,}48\, u$, and we have by an observation with this instrument found the number of revolutions 210 in 80'', then the corresponding velocity is:

$$v = 0{,}11 + 0{,}48 \cdot \frac{210}{80} = 0{,}11 + 1{,}26 = 1{,}37 \text{ feet.}$$

*Remark* 1. The constants $v_0$, $a$ and $\beta$ depend principally on the magnitude of the angle of impact, *i. e.*, on the angle which the plane of the vane makes with the direction of motion of the water, and therefore, also, with the direction of the axis of the wheel. To observe with tolerable accuracy small velocities, it is well to have a large angle of impulse, *i. e.*, one of 70°. For the rest, it is desirable to have vanes of different sizes and with different angles of impulse, and to use the vane with small angles of impulse for great velocities, and a smaller one for shallow water.

FIG. 517.



*Remark* 2. To find the velocity of the surface of water, a small tin wheel may be used, as represented in Fig. 517, and its under part allowed to dip into the water. The number of its revolutions may be determined by a system of wheels, as in the tachometer.

§ 379. To find the constant or co-efficient of the tachometer, it is necessary to set this instrument in a stream, whose velocity is known, and to note the corresponding number of revolutions. Although as many observations only are required, as there are constants, it is still safer to have as many observations as possible, and especially for very different velocities, because we may then apply the method of least squares, and thereby eliminate the effect of accidental errors of observation. For the rest, the velocity of the water may be found by the floating ball, or by receiving the water in a gauge vessel, and dividing

the measured discharge by the transverse section. In using floating balls, the air should be still, and the tract of water straight and uniform. The tachometer is to be held at several places of the space described by the floating ball, and it is also requisite for accuracy, that the diameter of the ball should be equal to that of the tachometer.

The second method of determination has several advantages when the water in which the instrument is immersed is received into a gauge vessel. For this purpose, and especially for adjusting the hydrometer, it is well if the Engineer can erect a proper hydraulic observatory, consisting of a vessel of efflux, a gauge reservoir, and a channel of communication between the two. With such an arrangement, we may impart to the water any arbitrary velocity, because we can not only regulate the entrance into the channel, but also the motion by means of boards placed in according to will. During observations we must keep the tachometer at different parts of the transverse section of the channel, measure the depth of this section by a scale, and, lastly, gauge the water running through in a definite time, in the lower reservoir (§ 372). We obtain the area $F$ of the transverse profile by multiplication of the mean depth with the mean breadth, and the quantity of water $Q$ is found from the mean transverse section $G$ of the gauge measure, and the height ($s$) of the quantity which has flowed in during the time by the formula $Q=\frac{Gs}{t}$; but the mean velocity of the water: $v=\frac{Q}{F}=\frac{Gs}{Ft}$ follows from $Q$ and $F$.

The corresponding number of revolutions $u$ of the wheel is the mean of all the revolutions which are obtained when the instrument is immersed at different breadths and depths of the measured profile.

If from a series of experiments we have found the mean velocities $v_1$, $v_2$, $v_3$, &c., and the corresponding number of revolutions, we then obtain by substitution in the formula $v=v_0+\alpha u$, or in the more correct one: $v=\alpha u+\sqrt{v_0^2+\beta u^2}$ as many equations of condition for the constants $v_0$, $\alpha$, $\beta$, as there have been observations made, and we may from these find the constants, if we either apply the method given in the "Ingenieur" § 17, or if these equations are divided into as many groups as there are unknown constants, and these added together for as many equations of condition as are requisite for determining $v_0$, $\alpha$, and as it may be $\beta$.

*Remark.* If we adopt the more simple formula with 2 constants, we may then, after the method of least squares, put:

$$v_0 = \frac{\Sigma (y)^2 \Sigma (x) - \Sigma (xy) \Sigma (y)}{\Sigma (x^2) \Sigma (y^2) - [\Sigma (xy)]^2} \text{ and } a = \frac{\Sigma (x^2) \Sigma (y) - \Sigma (xy) \Sigma (x)}{\Sigma (x^2) \Sigma (y^2) - [\Sigma (xy)]^2},$$

where $x = \frac{1}{v}$ and $y = \frac{u}{v}$, and the sign $\Sigma$ represents the sum of all successive similar values, for example, $\Sigma (x) = \frac{1}{v_1} + \frac{1}{v_2} + \frac{1}{v_3} + \ldots,$

$$\Sigma (xy) = \frac{1}{v_1} \cdot \frac{u_1}{v_1} + \frac{1}{v_2} \cdot \frac{u_2}{v_2} + \frac{1}{v_3} \cdot \frac{u_3}{v_3} + \ldots$$

*Example.* For a small tachometer the velocities are: 0,163; 0,205; 0,298; 0,366; 0,610 metres, the number of revolutions per second: 0,600; 0,835; 1,467; 1,805; 3,142 required to determine the constants corresponding to this wheel. From the formula given in the remark it follows, that:

$$\Sigma (x) = \frac{1}{0{,}163} + \frac{1}{0{,}205} + \ldots = 18{,}740,\ \Sigma (y) = \frac{0{,}600}{0{,}163} + \ldots = 22{,}759$$

$$\Sigma (x^2) = \left(\frac{1}{0{,}163}\right)^2 + \left(\frac{1}{0{,}205}\right)^2 + \ldots = 82{,}846,\ \Sigma (y^2) = 105{,}223, \text{ and}$$

$$\Sigma (xy) = \frac{0{,}600}{(0{,}163)^2} + \frac{0{,}835}{(0{,}205)^2} + \ldots = 80{,}961,$$

$$v_0 = \frac{105{,}223 \cdot 18{,}740 - 80{,}961 \cdot 22{,}759}{82{,}846 \cdot 105{,}223 - (80{,}961)^2} = \frac{129{,}5}{2162} = 0{,}060 \text{ and}$$

$a = \frac{368{,}3}{2162} = 0{,}1703$, hence for this instrument the formula $v = 0{,}060 + 0{,}1703\, u$.

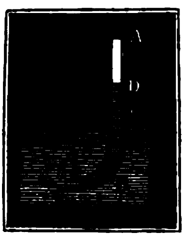
If in this we put $u = 0{,}6$, we then obtain:

$v = 0{,}060 + 0{,}102 = 0{,}162$; further, $u = 0{,}835$,
$v = 0{,}060 + 0{,}142 = 0{,}202$; further, $u = 1{,}467$,
$v = 0{,}060 + 0{,}249 = 0{,}309$; $u = 1{,}805$,
$v = 0{,}060 + 0{,}307 = 0{,}367$; lastly $u$, $= 3{,}142$,
$v = 0{,}060 + 0{,}535 = 0{,}595$;

therefore the calculated values agree very well with the observed.

§ 380. *Pitot's tube.*—Other hydrometers are not so satisfactory as the tachometer, for they either admit of less accuracy, or they are more complicated in their use. The most simple instrument of this kind is *Pitot's tube.* In its simplest form it consists of a bent glass tube *ABC,* Fig. 518, which is held in the water in such a manner that its lower part stands horizontally, and is opposed to the water. By the percussion of the water, a column of water is sustained in this tube, which stands above the level of the exterior fluid surface, and the elevation *DE* of this column is greater, the greater the percussion or the velocity of the water generating it; this elevation or difference of level may hence serve inversely for a measure of the velocity of the water. Let this eleva-

FIG. 518.



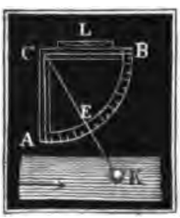
tion $DE$ above the external surface of water $= h$, and the velocity $= v$, then $h = \frac{v^2}{2g\mu^2}$, where $\mu$ is a number derived from experiment, and we have inversely, $v = \mu \sqrt{2gh}$, or more simply: $v = \psi \sqrt{h}$. To find the constant $\psi$, the instrument is immersed at a place in the water where the velocity $v_1$ is known; if the elevation is here $= h_1$, we then have the constant $\psi = \frac{v_1}{\sqrt{h_1}}$, which is to be applied in other cases, where the velocity is to be determined with this instrument.

FIG. 519.


To facilitate the reading off of the height $h$, the instrument consists of two tubes, as shown in Fig. 519, and from the one a small tube $F$ is directed against the stream, from the other two small tubes $G$ and $G_1$ at right angles to the direction of the stream, both tubes are connected with a single cock $H$, by which the water can be retained in them. When the instrument is drawn out of the water, we may then conveniently read off on a scale attached to both the tubes, the difference $CD = h$ of the two columns of water. That the water in the tube may not oscillate much, it is necessary to make the exterior orifices of the tubes narrow, and that the closing of them may take place quickly and safely; the cock is provided with an arm and an even rod $HK$, which terminates above, near the handle of the instrument.

§ 381. *Hydrometric pendulum.* — The hydrometric pendulum has been used in preference by Ximenes, Michelotti, Gerstner and Eytelwein for the measurement of the velocity of running water. This instrument consists of a quadrant $AB$, Fig. 520, divided into degrees and smaller parts, and a metallic or ivory ball $K$ of from two to three inches diameter, suspended by a thread from the centre $C$, the velocity of the water is given by the angle $ACE$, at which the thread when stretched by the ball deviates from the vertical, when the plane of the instrument is brought into the direction of the stream, and

FIG. 520.



the ball submerged in the water. As the angle rarely amounts to forty or more degrees, this instrument has often the form of a right angled triangle given to it, and the divisions made on its horizontal cathetus. For the placing of the index or zero line in the vertical, it is best to use a spirit level on the horizontal arm of the instrument, or the ball itself may serve for this purpose, by letting it be suspended out of the water, and the instrument revolve until the thread coincides with the zero line of the division.

For velocities under four feet we may use the ivory ball, but for greater velocities the hollow metal ball. On account of the constant undulations of the ball in the direction of the motion of the water, as also at right angles to the direction of the current, the reading off is somewhat difficult, and leaves a good deal of uncertainty, for which reason this instrument cannot be relied upon for the more exact numbers.

The dependence between the angle of deviation and the velocity of the water may be determined in the following manner when the ball is not very deeply immersed. From the weight $G$ of the ball and from the impulse of the water $P = \mu Fv^2$, increasing simultaneously with the square of the velocity $v$ and the section of the ball $F$, the resultant $R$, whose direction the thread assumes, follows, and is determined by the angle of deviation $\beta$, for which the

$tang.\ \beta = \frac{P}{G} = \frac{\mu Fv^2}{G}$, hence also inversely:

$v^2 = \frac{G\ tang.\ \beta}{\mu F}$, and $v = \sqrt{\frac{G}{\mu F}} \cdot \sqrt{tang.\ \beta}$, i. e. $v = \psi \sqrt{tang.\ \beta}$,

if $\psi$ represents a co-efficient derived from experiment, which must be obtained before use, according to the above-mentioned instructions.

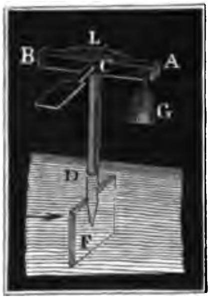
§ 382. *Rheometer.* — The remaining hydrometers, such as Lorgna's water lever, Ximenes's water vane, Michelotti's hydraulic balance, Brünning's tachometer, Poletti's rheometer, are more complicated in their use, and not altogether to be relied on. The principle of all these instruments is the same, they are composed of a surface of impulse and a balance, and the last serves for the purpose of giving the percussion $P$ of the water against the former, but since this $= \mu Fv^2$, we then have inversely:

$v = \sqrt{\frac{P}{\mu F}} = \psi \sqrt{P}$, where $\psi$ denotes a constant deduced from

experiment dependent on the magnitude of the surface of impact $F$.

The *rheometer* which was lately proposed by Poletti, and does not materially differ from the hydrometric balance of Michelotti, consists of a lever $AB$, Fig. 521, turning about a fixed axis $C$, and an arm $CD$ to which the surface of impulse, or according to Poletti, a mere impulse-staff is screwed. To maintain equilibrium with the percussion of the water against the surface, the boxes suspended at the extremity $A$ of the lever are loaded with weight or shot, and to put the empty balance in equilibrium in still water, a counterpoise is placed at $B$, which makes up the outermost end of the arm $CB$. From the weight put on $G$, the impulse $P$ is found by means of the arm $CA = a$, and $CF = b$ from the formula $Pb = Ga$, whence, therefore,

FIG. 521.




$$P = \frac{a}{b}\, G, \text{ and } v = \sqrt{\frac{P}{\mu F}} = \sqrt{\frac{a\, G}{\mu\, b\, F}} = \psi\, \sqrt{G},$$

where $\psi$ is a constant derived from experiment.

*Remark.* With respect to the last hydrometer, ample details will be found in Eytelwein's "Handbuch der Mechanik fester Körper und der Hydraulik;" further, in Gerstner's "Handbuch der Mechanik," vol. 2; in Brünning's "Treatise on the velocity of running water;" in Venturoli's "Elementi di Meccanica e d'Idraulica," vol. 2. Concerning Poletti's hydrometer, we must refer to Dingler's "Polytechn. Journal," vol. 20, 1826. The hydrometer described in Stevenson's treatise on Marine Surveying and Hydrometry is the tachometer of Woltmann, *see* Dingler's "Journal," vol. 65, 1842.

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# CHAPTER IX.

## ON THE IMPULSE AND RESISTANCE OF FLUIDS.

§ 383. *Impulse and resistance of water.*—Water or any other fluid imparts a shock to a rigid body, when it meets it in such a manner that its condition of motion is thereby altered. The resistance which water opposes to the motion of a body, does not essentially differ from impulse. The investigation of these two

forms the third principal division of hydraulics. We distinguish from each other:

1. The impulse of an isolated stream.
2. The impulse of a limited stream.
3. The impulse of an unlimited stream.

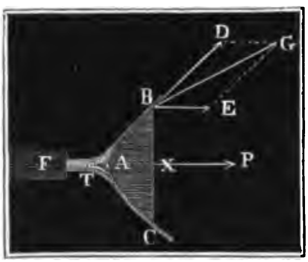
An impulse of the first kind takes place when a body, for instance, the float board of an over-shot water-wheel, is opposed to a stream of water issuing from a reservoir; an impulse of the second kind occurs where water, in a canal or in a water-course, impinges against a body which entirely fills up its transverse section, as for instance, against the float board of an under-shot wheel; the third kind, lastly, presents itself, when running water strikes against a body immersed in it, whose transverse section is only a very small part of that of the current of water, as for instance, against the float boards of a floating mill-wheel.

We must distinguish the impulse of water against a body at rest and against a body in motion, and further, the impulse against a curved and against a plane surface, and in this last again, between the perpendicular and the oblique impulse.

Let us consider at once the general case, namely, the impulse of an isolated stream against a surface of rotation which moves in its proper axis and in the direction of motion of the stream.

§ 384. *Impact of isolated streams.*—Let $BAC$, Fig. 522, be a surface of rotation, $AX$ its axis, and $FA$ a fluid stream meeting it in this direction. Let the velocity of the water $= c$, that of the surfaee $= v$, and the angle $BTX$, which the tangent $DT$ at the extremity $B$ of the generating curve or of each of the filaments of water $BT$ leaving the surface, includes with the direction of the axis $BE = a$; lastly, let us further assume that the water in running off from the surface loses nothing in *vis viva* by friction. The water strikes against the surface with the relative velocity $c-v$, and hence leaves the surface with this, and therefore quits it in the tangential directions $TB$, $TC$, &c. From the tangential velocity $BD = c - v$, and the angle velocity of the axis $BE = v$, the absolute velocity $BG = c_1$ of the water after impinging against the surface is found by the known formula:

FIG. 522.




$$c_1 = \sqrt{(c-v)^2 + 2\,(c-v)\,v\,cos.\,a + v^2}.$$

But now a quantity of water $Q$ is able to produce by virtue of its *vis viva* the mechanical effect $\frac{c^2}{2g} \cdot Q\gamma$, if its velocity $c$ is fully imparted; accordingly the residuary effect of the water:
$= \frac{c_1^2}{2g} \cdot Q\gamma$; consequently the mechanical effect distributed over the surface is:

$$Pv = \frac{c^2}{2g} Q\gamma - \frac{c_1^2}{2g} Q\gamma = \frac{c^2 - c_1^2}{2g} \cdot Q\gamma$$

$$= \frac{[c^2 - (c-v)^2 - 2(c-v) v . \cos. a - v^2]}{2g} Q\gamma$$

$$= \frac{2cv - 2v^2 - 2(c-v) v \cos. a}{2g} Q\gamma, \text{ i. e.}$$

$$Pv = (1 - \cos. a) \frac{(c-v) v}{g} Q\gamma,$$

and the force or the impulse of the water in the direction of its axis is:

$$P = (1 - \cos. a) \frac{(c-v)}{g} Q\gamma.$$

If the surface meets the water with the velocity $v$, we then have:

$$P = (1 - \cos. a) \cdot \frac{(c+v)}{g} Q\gamma,$$

and if this is without motion, therefore, $v = 0$, the impulse or hydraulic pressure of the axis comes out:

$$P = (1 - \cos. a) \frac{c}{g} \cdot Q\gamma.$$

*It follows from this, that the impulse of one and the same mass of water under otherwise similar circumstances is proportional to the relative velocity $c \mp v$ of the water.*

From the area $F$ of the transverse section of the fluid stream, it follows that the quantity discharged is $Q = Fc$; hence

$$P = (1 - \cos. a) \frac{(c \mp v) c}{g} F\gamma;$$

and for $v = 0$:

$$P = (1 - \cos. a) \frac{c^2}{g} F\gamma.$$

*For an equal transverse section of the stream, the impulse against a surface at rest increases therefore as the square of the velocity of the water.*

§ 385. *Impulse against plane surfaces.*—The impulse of one and the same fluid stream depends principally on the angle $a$, under which the water, after the impulse, leaves the axis; it is nothing if this angle $= 0$; and on the other hand, a maximum, namely, $= 2\,\frac{(c+v)}{g}\,Q\,\gamma$, if this angle is $180^0$, therefore its cosine $= -1$, where the water, as represented in Fig. 523, leaves the surface in a direction opposite to that in which it impinges. This is generally greater for concave surfaces than for convex, because the angle is there oblique, therefore the cosine negative and $1 - cos.\ a$ becomes $1 + cos.\ a$.

FIG. 523.

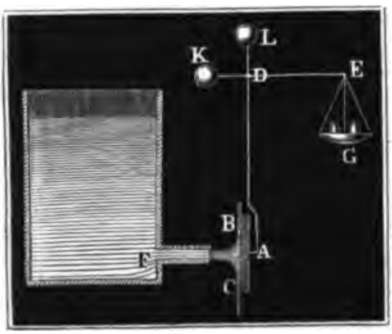

FIG. 524.


Most frequently the surface, as represented in Fig. 524, is plane, and hence $a = 90^0$, therefore $cos.\ a = 0$, and the impulse $P = \frac{(c+v)}{g}\,.\,Q\,\gamma$; for a surface at rest:

$$P = \frac{c}{g}\,Q\gamma = \frac{c^2}{g}\,F\gamma = 2\,.\,\frac{c^2}{2g}\,F\gamma = 2\,F\,h\,\gamma.$$

*The normal impulse of water against a plane surface is therefore equivalent to the weight of a column of water which has for base the transverse section $F$ of the stream, and for altitude, twice the height due to the velocity,* $2\,h = 2\,.\,\frac{c^2}{2g}$.

FIG. 525.




The experiments made on this subject by Michelotti, Vince, Langsdorf, Bossut, Morosi, and Bidone, have nearly led to the same results when the transverse section of the impinged surface was at least six times as great as that of the stream, and when this surface was twice as far from the plane of the orifice as

the thickness of the stream. The apparatus which was usen for this purpose consisted of a lever, similar to that of Poletti's rheometer, which received upon one side the impulse of the water, and whilst its other side was kept in equilibrium by weights. The instrument which Bidone made use of is represented in Fig. 525. $BC$ is the surface impinged on by the stream $FA$, $G$ is the scale-pan for the reception of the weights, $D$ the axis of rotation, $KL$ counter-weights.

*Remark.* The latest and most extensive experiments on the percussion of water are those of Bidone. *See* "Memorie de la Reale Accademia delle Scienze di Torino," vol. 40, 1838. They were performed with a velocity of at least 27 feet, and on brass plates of from 2 to 9 inches diameter. In general, Bidone found that the normal impulse against a plane surface was somewhat greater than $2\,F h \gamma$, yet this variation is perhaps to be attributed to an augmentation of the leverage which is produced by the falling back of the water. *See* Duchemin's "Recherches expérim. sur les lois de la résistance des fluides." When the impinged surface was quite near the orifice, Bidone found that $P$ was only $1{,}5\,F h \gamma$; when, further, the surface had a transverse section equal to that of the stream, in which case the water only deviated by an acute angle $a$, then, after Du Buat and Langsdorf, $P$ was only $= F h \gamma$. Lastly, it has been deduced by Bidone and others that the impulse is in the first moment nearly as great again as the permanent impulse.

§ 386. *Maximum effect of impulse.*—The mechanical effect of impulse:

$$Pv = (1 - cos.\ a)\,\frac{(c - v)\,v}{g}\,Q\,\gamma$$

depends principally on the velocity $v$ of the impinged surface; it is, for example, nothing, not only for $v=c$, but also for $v=0$; hence there is a velocity for which the effect of the impulse is a maximum. It is manifest that it only depends on $(c-v)\,v$ becoming a maximum. If we consider $c$ as half the perimeter of a rectangle, and $v$ as its base we have then its height $= c-v$ and its area $=(c-v)\,v$. But of all rectangles the square is that which has for a given perimeter $2\,c$ the greatest area, hence also $(c - v)\ v$ is a maximum, when $c-v=v$, i. e. $v=\frac{c}{2}$, *and we therefore obtain the maximum value of the mechanical effect of the impulse when the surface moves from it with half the velocity of the water, and indeed*

$$Pv=(1-cos.\ a)\,.\,\tfrac{1}{2}\,.\,\frac{c^2}{2g}\,.\,Q\gamma=(1-cos.\ a)\,.\,\tfrac{1}{2}\,Qh\gamma.$$

If now $a=180^0$, therefore the motion of the water be reversed by the impulse, we then have the effect equal to $2\,.\,\tfrac{1}{2}\,Qh\gamma = Qh\gamma$. But if $a=90^0$, *i. e.* if it impinges against a plane surface,

this effect is then only $\frac{1}{2}$ $Qh\gamma$, therefore, in the last case, the half only of the whole disposable effect, or that which corresponds to the *vis viva* of the water, is gained or brought to bear upon the surface.

*Examples.*—1. If a stream of water, of 40 square inches transverse section, delivers a quantity of 5 cubic feet per second, and strikes normally against a plane surface, and escapes with a 12 feet velocity, the effect of impulse is then:

$$P = \frac{(c-v)}{g} Q\gamma = \left(\frac{5 \,.\, 144}{40} - 12\right) \,.\, 0{,}032 \,.\, 5 \,.\, 66 = 6{,}032 \,.\, 312{,}5 = 60 \text{ lbs.},$$

and the mechanical effect brought to bear upon the surface $Pv = 60 \times 12 = 720$ ft. lbs.

The greatest effect is for $v = \frac{c}{2} = \frac{1}{2} \,.\, \frac{5 \,.\, 144}{40} = 9$ feet, and indeed:

$$= \frac{1}{2} \,.\, \frac{c^2}{2g} \,.\, Q\gamma = \frac{1}{2} \,.\, 18^2 \,.\, 0{,}016 \,.\, 5 \,.\, 66 = 81 \,.\, 0155 \,.\, 62{,}5 = 784{,}68 \text{ ft. lbs.};$$

the corresponding impulse, or hydraulic pressure $= \frac{784{,}68}{9} = 87{,}18$ lbs.—2. If a stream $FA$, Fig. 525, of 64 square inches section, strikes with a 40 feet velocity against an immoveable cone, having an angle of convergence $BAC = 100^\circ$, then is the hydraulic pressure in the direction of the stream:

$$P = (1 - \cos. a)\, \frac{c}{g}\, Q\gamma = (1 - \cos. 50^\circ)\, 40 \,.\, 0{,}031 \,.\, \frac{64}{144} \,.\, 40 \,.\, 62{,}5$$

$$= (1 - 0{,}64279) \,.\, 1815 \,.\, \frac{3520}{3} = 0{,}35721 \,.\, 1815 = 648{,}33 \text{ lbs.}$$

FIG. 525.

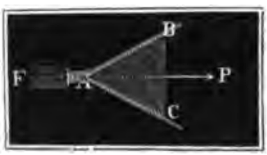
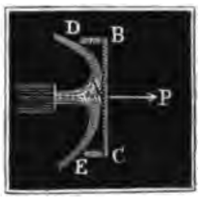



FIG. 526.




§ 387. *Impulse of a limited stream.*—If we add borders $BD$, $CE$, to the perimeter of a plane surface $BE$, Fig. 526, which project from the side impinged upon by the water, then will the water deviate from its direction at an obtuse angle, in a similar manner as from concave surfaces, and hence the impulse will be greater than for plane surfaces. The effect of this impulse depends principally on the height of the border and the ratio of the transverse section between the stream and the part confined. In an experiment, where the stream was 1 inch thick, the cylindrical enclosure 3 inches wide and $3\frac{1}{2}$ lines deep, the water ran off almost in a reversed direction, and the impulse

amounted to 3,93 $\frac{c^2}{2g} F\gamma$; in every other case this force was less. In consequence of the friction of the water at the surface and the sides, the theoretical maximum value never reaches $4 \frac{c^2}{2g} F\gamma$.

In the impulse of a limited stream $FAB$, Fig. 528, a rising at the edges takes place; this rising occupies only a portion of the perimeter, and extends itself, on the other hand, simultaneously to the impinged surface and the fluid stream. The impinging water takes the direction of the unbordered portion of the perimeter, and here, therefore, becomes deflected 90 degrees, whence the formula above found for the isolated stream $P = \frac{(c-v)}{g} Q\gamma$ holds good; yet this may also be deduced in the following manner. If we assume that the velocity $c$ of the arriving water by the impulse against its surface is changed into the velocity $v$ of the surface, we may then also assume that a loss of mechanical effect $\frac{(c-v)^2}{2g} Q\gamma$ (similar to that in § 337), expended in the division of the water, is connected with it. But now the effect due to the *vis viva* of the arriving water $= \frac{c^2}{2g} Q\gamma$ and to that of the water going on $= \frac{v^2}{2g} Q\gamma$, hence it follows that the mechanical effect imparted to the surface is:

FIG. 528.


$$Pv = [c^2 - (c-v)^2 - v^2] \frac{1}{2g} Q\gamma = \frac{(c-v)v}{g} Q\gamma.$$

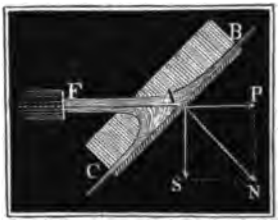
*Remark.* This formula will be found applicable hereafter, when we come to the theory of water-wheels.

§ 388. *Oblique impulse.*—In oblique impulse against a plane surface, we must distinguish whether the water flows away in one, two, or in all directions in the plane. If, as in the impact of limited water, the surface $AB$, Fig. 529, is confined at three sides, so that the water can run off only in one direction, we have

then the hydraulic pressure against the surface in the direction of the stream $P = (1 - cos.\ a) \frac{(c - v)}{g} Q \gamma$.

FIG. 529.


FIG. 530.




But if the impinged plane $BC$, Fig. 530, is only bordered on two oppositely situated sides, the stream then divides itself into two unequal portions; the greater portion $Q_1$ takes the small deflexion $a$, and the lesser $Q_2$, the greater deflexion $180 - a$; hence, the whole impulse in the direction of the stream is:

$$P = (1 - cos.\ a) \cdot \frac{c - v}{g} Q_1 \gamma + (1 + cos.\ a) \cdot \frac{c - v}{g} Q_2 \gamma$$

$$= \left(\frac{c - v}{g}\right) \gamma \left[(1 - cos.\ a) Q_1 + (1 + cos.\ a) Q_2\right].$$

Now the equilibrium of the two portions of the stream requires that the pressures

$$\frac{(c-v)}{g} \gamma (1 - cos.\ a) Q_1 \text{ and } \frac{(c-v)}{g} \gamma (1 + cos.\ a) Q_2$$

between them should be equal; hence, also:

$(1 - cos.\ a) Q_1 = (1 + cos.\ a) Q_2$, or since $Q_1 + Q_2 = Q$,

$$(1 - cos.\ a) Q_1 = (1 + cos.\ a) (Q - Q_1), \text{ i.e.,}$$

$$Q_1 = \left(\frac{(1 + cos.\ a)}{2}\right) Q \text{ and } Q_2 = \left(\frac{1 - cos.\ a}{2}\right) Q,$$

so that the whole impulse in the direction of the stream is finally:

$$P = \frac{(c-v)}{g} \gamma \cdot 2 (1 - cos.\ a) \frac{(1 + cos.\ a) Q}{2} = \frac{(c-v) \gamma}{g} (1 - cos.\ a^2) Q,$$

*i.e.*, $P = \frac{c - v}{g} sin.\ a^2 \cdot Q \gamma$.

Besides the *parallel impulse* $P$, acting in the direction of the stream, we distinguish, further, the *lateral impulse* $S$, acting at right angles to the direction of the stream, and the *normal impulse* $N$, composed of these two, and at right angles to the surface. In every case $P = N\ sin.\ a$ and $S = N\ cos.\ a$; hence,

inversely, $N = \frac{P}{sin. a} = \frac{c-v}{g} sin. a\, Q\gamma$ and $S = \frac{c-v}{2g} sin. 2a\,.\, Q\gamma.$

FIG. 531.


*The normal impulse, therefore, increases as the sine, the parallel impulse as the square of the sine of the angle of incidence, and the lateral impulse as double the same angle.* Lastly, if the inclined surface impinged on is not bordered, then the water can spread over it in all directions; the impulse is then greater, because of all the angles by which the filaments of water are deflected, $a$ is the least; and hence, each filament which does not move in the normal plane, exerts a greater pressure than the filament in this plane. Let us assume that a portion $Q_1$ corresponding to the sectors $AOB$ and $DOE$, Fig. 531, is deflected by the angles $a$ and $180^0 - a$, and another $Q_2$, corresponding to the sectors $AOD$ and $BOE$, by $90^0$, and that both portions exert a parallel impulse, we may then put:

$$P = \frac{c-v}{g} Q_1 \gamma\, sin. a^2 + \frac{c-v}{g} Q_2 \gamma,\ Q_1\, sin. a^2 = Q_2, \text{ and } Q_1 + Q_2 = Q;$$

hence it follows, that $Q_1 (1 + sin. a^2) = Q$, and the whole parallel impulse $P = \left(\frac{c-v}{g}\right) \frac{2\, Q\gamma\, sin. a^2}{1 + sin. a^2} = \frac{2\, sin. a^2}{1 + sin. a^2} \,.\, \frac{c-v}{g} \,.\, Q\gamma.$

Although this hypothesis is only approximately correct, it tolerably well agrees, nevertheless, with the latest experiments of Bidone.

§ 389. *Action of an unlimited stream.*—If a body moves progressively in an unlimited fluid, or if a body is put into a fluid which is in motion, it then suffers a pressure which is dependent on the form and dimensions of this body, as well as on the density and on the velocity of the one or the other mass, and in the one case is called the *resistance*, and in the other the *impulse* of the fluid. This hydraulic pressure arises principally from the inertia of the water, whose condition of motion is altered by striking against the solid body, and also, further, from the force of cohesion of the particles of water, which are hereby partially separated from one another, or pushed aside. If a body $AC$ moves against running water, Fig. 532, it pushes away before it a certain quantity with an augmented pressure. Whilst this mass of water, by the further

advance of the body, always increases on the one side, on the other a constant flowing away takes place, while the particles

FIG. 532.

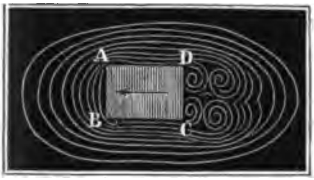
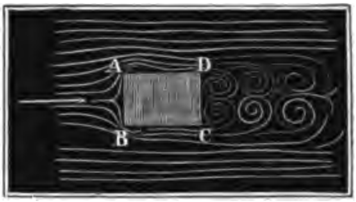



FIG. 533.




lying near the anterior surface assume a motion in the direction of this surface. If the moving mass of water strikes against a body at rest, Fig. 533, then is there likewise an increased pressure produced in front of it, which causes the particles before the body to deviate from their original direction, and to run off at the surface *AB*. When these particles have reached the limits of the surface, they then turn and flow away by the lateral surfaces until they come to the back, when they then again immediately unite, but assume an eddying motion. It is manifest that the general circumstances of motion of the particles surrounding the body are the same in the impact of moving water as in the resistance of a body moving in water, except that in the eddies a difference so far takes place, that with short bodies the eddy in the latter case occupies a less space than in the former. In both cases the velocity of the particles increases more and more from the middle of the anterior surface to its limits, attains its maximum at the commencement of the lateral surfaces, where, for the most part, a contraction takes place, gradually diminishes in the water which passes away at the sides, and lastly, attains its minimum when the water reaches the back and passes into a whirling motion.

§ 390. *Theory of impulse and resistance.*—The normal pressure varies at different points of the body; it is greatest at the middle of the anterior, and least at the middle of the posterior surface, and, next to that, at the parts of the sides nearest this; because, in respect to the body, there is at the one place rather a flow to, and at the other a flow from these surfaces. If the body be symmetrical, as we shall suppose it to be, with respect to the direction of motion, then the aggregate pressures in this direction counteract each other, and hence only the pressures in the direction of motion are to be taken into account. But now the pressures

on the posterior surface are opposed to those on the anterior; hence the *resultant impulse or resistance of the water may be equated to the difference of pressure of the anterior and posterior surfaces.*

If we cannot assign the amount of these pressures *à priori*, we may, nevertheless, from the great similarity of the circumstances to the impulse of isolated streams, assume that at least the general law for the impulse of unlimited water does not differ from that of the impulse of isolated streams. If, therefore, $F$ is the area of a surface, which is impinged on by an unlimited current whose density is $\gamma$, with a velocity $v$, then the corresponding impulse or hydraulic pressure may be put $P = \zeta \frac{v^2}{2g} F \gamma$, where $\zeta$ represents a number deduced from experiment, dependent on the form of the surface. But this expression is not only applicable to action against the anterior, but also to that against the posterior surface, only that in this last, when the water has a tendency to flow away, it consists of a draught or negative pressure. If now $F h \gamma$ is the hydrostatic pressure (§ 276) against the front and back surface of a body, the whole pressure against the front is: $P_1 = F h \gamma + \zeta_1 \cdot \frac{v^2}{2g} F \gamma$, and that against the back: $P_2 = F h \gamma - \zeta_2 \cdot \frac{v^2}{2g} F \gamma$, and the resultant impulse or resistance of the water is then found:

$$P = P_1 - P_2 = (\zeta_1 + \zeta_2) \cdot \frac{v^2}{2g} F \gamma = \zeta \cdot \frac{v^2}{2g} F \gamma, \text{ if } \zeta_1 + \zeta_2 = \zeta.$$

This general formula for the impulse of unlimited water is applicable to the percussion of the wind or to the resistance of the air. Besides the difference of aërodynamic pressure at the front and back, there is further a difference of aërostatic pressure, because the air in front, in consequence of its greater elasticity, has a greater density ($\gamma$) than that at the back. For this reason, in high velocities, as those of cannon-balls, the co-efficient of the resistance of air is greater than that of water.

*Remark.* The adhesion of a certain quantity of air or water to the body, is a peculiar phenomenon of the impulse or resistance of an unlimited medium (water or air), whose influence is particularly remarkable in the variable motion of bodies, as, for example, in the oscillations of the pendulum. For a ball, the air or water adhering to the moving body is equal to 0,6 of the volume of the ball. For a prismatic body moved in the direction of its axis, the ratio of this volume

$= 0{,}13 + 0{,}705 \frac{\sqrt{F}}{l}$, where $l$ is the length and $F$ the transverse section of the body. These relations, discovered by Du Buat, have been fully confirmed by the later observations of Bessel, Sabine, and Baily.

§ 391. *Impulse and resistance against surfaces.*—The co-efficient of resistance $\zeta$, or the number with which the height due to the velocity is to be multiplied to obtain the height of a column of water measuring this hydraulic pressure, varies for bodies of different figures, and only for plates which are at right angles to the direction of motion is it nearly a definite quantity. According to the experiments of Du Buat and those of Thibault, we may put $\zeta = 1{,}85$ for the impulse of air or water against a plane surface at rest, and on the other hand assume, but with less accuracy, for the resistance of air or water against a surface in motion $\zeta = 1{,}40$. In both cases about two-thirds of the whole effect are expended on the front, and one-third on the back. The resistance which the air opposes to a surface revolving in a circle, has been found by Borda, Hutton, and Thibault to vary a good deal, but may be expressed by a mean of $\zeta = 1{,}5$. If the surface does not stand at right angles to the direction of the motion, but makes with it an acute angle $a$, we may then, with Duchemin, substitute for $\zeta$, $\frac{2\,\zeta\, sin.\, a^2}{1 + sin.\, a^2}$ with tolerable correctness.

The impulse and resistance of unlimited media are also augmented when the surfaces are hollowed out or have projecting edges at their perimeters, but we have arrived at no general results on this subject.

*Example.* If the wind impinges with a 20 feet velocity against a firmly fixed wind-wheel, which consists of four wings, of which each has an area of 200 square feet and 75° inclination to the direction of the wind, then is the impinging force of the wind in its direction, or in that of the axis of the wheel:

$$P = 1{,}85 \,.\, \frac{2\,(sin.\, 75)^2}{1 + (sin.\, 75)^2} \,.\, \frac{20^2}{2g} \,.\, 4 \,.\, 200 \,.\, 0{,}086 = 1{,}85 \,.\, 0{,}965 \,.\, 6{,}21 \,.\, 800 \,.\, 0{,}086$$

$= 762{,}7$ ft. lbs., when the density of the wind is (from § 301) taken at 0,086 lbs.

*Remark.* Views, with respect to the impulse and resistance of unlimited fluids, entirely at variance with these, are put forward in the above-mentioned work of Duchemin. It is there maintained, for instance, that the impulse and resistance against the front surface of a thin plate amounts to $2 \,.\, \frac{v^2}{2g} Fh$, and is not negative at the back, that the impulse $= 0{,}136 \frac{v^2}{2g} F\gamma$, and the resistance $= 0{,}746 \frac{v^2}{2g} F\gamma$. It would be too circumstantial here to give a detail of the reasons why the author cannot

agree with the views of Duchemin, but more with reference to this will be found in Poncelet's "Introduction à la mécanique industrielle," 2nd edition, 1841.

§ 392. *Impulse and resistance to bodies.*—The impulse and resistance of water to prismatic bodies, whose axis coincides with the direction of motion, diminishes when the length of the body is considerable. From the experiments of Du Buat and Duchemin, the impulse of the front surface is invariable, and only the effect against the back surface variable. To this corresponds the co-efficient $\zeta_1 = 1{,}186$, for the total effect, however, with the relative lengths

$$\frac{l}{\sqrt{F}} = 0, \quad 1, \quad 2, \quad 3,$$

$$\zeta = 1{,}86;\ 1{,}47;\ 1{,}35;\ 1{,}33.$$

For still greater ratios between the length $l$ and the mean breadth $\sqrt{F}$ of the body $\zeta$ diminishes, owing to the friction of the water at the lateral surfaces of the body. From the resistance of the water, reverse relations take place. Here, from Du Buat, for the effect on the front surface, $\zeta_1 = 1$ invariably; for the total effect, however, with

$$\frac{l}{\sqrt{F}} = 0, \quad 1, \quad 2, \quad 3,$$

$\zeta = 1{,}25;\ 1{,}28;\ 1{,}31;\ 1{,}33$, so that, for a prism which is 3 times as long as broad, the impulse is the same as the resistance.

The experiments undertaken by Borda, Hutton, Vince, Desaguilliers, Newton, and others, with angular and with round bodies, leave still much uncertainty. In what relates to spheres, it appears that for moderate velocities the mean co-efficient for motion in air or water $= 0{,}6$. For a greater velocity and for motion in air, according to Robins and Hutton, for the velocities

$$v = 1, \quad 5, \quad 25, \quad 100,\ 200,\ 300,\ 400,\ 500,\ 600 \text{ metr.}$$
$$\zeta = 0{,}59;\ 0{,}63;\ 0{,}67;\ 0{,}71;\ 0{,}77;\ 0{,}88;\ 0{,}99;\ 1{,}04;\ 1{,}01.$$

Duchemin and Piobert have given particular formulæ for the rate of increase of these co-efficients.

For the impulse of water against a sphere, Eytelwein found $\zeta = 0{,}7886$.

*Remark.* Poncelet, in his work above cited, and Duchemin and Thibault in their "Recherches expérimentales," have treated very fully of these circumstances. In the

Second Part we shall treat of the resistance to floating bodies, especially to ships, &c., as also the impact of the wind on wheels, &c.

*Example.* If, according to Borda, we put the resistance and impact at right angles to the axis of a cylinder at half as great as that against a parallelepiped which has the same dimensions, we then obtain for the resistance $\zeta = \frac{1}{2} \,.\, 1{,}28 = 0{,}64$ and the impact $= \frac{1}{2} \,.\, 1{,}47 = 0{,}735$. If we apply these values to the human body, whose section has an area of some 7 square feet, we then find for the resistance and impulse of air against it, the values:

$P = 0{,}64 \,.\, 0{,}016 \,.\, 7 \,.\, 0{,}086\, v^2 = 0{,}00616\, v^2$, and

$P = 0{,}735 \,.\, 0{,}016 \,.\, 7 \,.\, 0{,}086\, v^2 = 0{,}00708\, v^2$. Hence the resistance of air for a velocity of 5 feet is only $0{,}00616 \,.\, 25 = 0{,}154$ lbs.; and the corresponding mechanical effect per second $= 5 \,.\, 0{,}154 = 0{,}77$ ft. lbs.; for a velocity of 10 feet this resistance is four times, and the effect expended eight times as great, and for a velocity of 15 feet, these numbers are respectively 9 and 27. If a man moves against wind having a 50 feet velocity, with a 5 feet velocity, he has then a resistance $0{,}00708 \,.\, 55^2 = 21{,}42$ lbs. to overcome, corresponding to the relative velocity $50 + 5 = 55$ feet, and thereby to produce the mechanical effect of $21{,}42 \,.\, 5 = 107{,}1$ ft. lbs.

§ 393. *Motion in resisting media.*—The laws of the motion of a body in a resisting medium are rather complex, because we have here to deal with a variable force, *i. e.* one increasing with the square of the velocity. From the force $P_1$ which urges the body forward, and from the resistance $P_2 = \zeta \,.\, \frac{v^2}{2g} F\gamma$, which the medium opposes to the motion, the motive force is:

$$P = P_1 - P_2 = P_1 - \zeta \,.\, \frac{v^2}{2g} F\gamma,$$

but since the mass of the body $= M = \frac{G}{g}$, the accelerating force is:

$$p = \frac{P}{M} = \left(P_1 - \zeta\, \frac{v^2}{2g} F\gamma\right) : M = \left(\frac{P_1 - \zeta\, \frac{v^2}{2g} F\gamma}{G}\right) . g,$$

or if we represent $\frac{F\gamma}{2gP_1}$ by $\frac{1}{w^2}$.

$p = \left[1 - \zeta \left(\frac{v}{w}\right)^2\right] \frac{P_1}{G}\, g$. But the velocity $v$ is accelerated in the instant of time $\tau$ by $\kappa = p\,\tau$, hence:

$\kappa = \left[1 - \zeta \left(\frac{v}{w}\right)^2\right] \frac{P_1}{G}\, g\,\tau$, and inversely:

$$\tau = \frac{G_1}{P_1} \,.\, \frac{\kappa}{g\left[1 - \zeta \left(\frac{v}{w}\right)^2\right]}.$$

Now to find the time corresponding to a given change of velocity, let us divide the difference $v_n - v_0$, of the final and initial velocity into $n$ parts, let any such part $\frac{v_n - v_0}{n} = \kappa$, and let us calculate the velocities :

$$v_1 = v_0 + \kappa,\ v_2 = v_0 + 2\kappa,\ v_3 = v_0 + 3\kappa,\ \&c.,$$

and substitute these values in the formula of Simpson. In this manner, by taking four parts we shall obtain the time sought

$$1.\ t = \frac{G}{P_1} \cdot \frac{v_n - v_0}{12g}\left(\frac{1}{1-\zeta\left(\frac{v_0}{w}\right)^2} + \frac{4}{1-\zeta\left(\frac{v_1}{w}\right)^2} + \right.$$

$$\left. + \frac{2}{1-\zeta\left(\frac{v_2}{w}\right)^2} + \frac{4}{1-\zeta\left(\frac{v_3}{w}\right)^2} + \frac{1}{1-\zeta\left(\frac{v_4}{w}\right)^2}\right).$$

Further, the small space described in any instant $\tau$ (§ 19), is $\sigma = v\,\tau$, or since $\tau = \frac{\kappa}{p}$, $\sigma = \frac{v\kappa}{p}$, therefore,

$dS = v\,dt$
$dS = \frac{v\,dv}{k}$

$s = \frac{v\,\kappa}{1-\zeta\left(\frac{v}{w}\right)^2} \cdot \frac{G}{P_1 g}$. By the application of Simpson's rule, we shall now find the space which is described while the velocity $v$ passes into that of $v_n$.

$$2.\ s = \frac{G}{P_1} \cdot \frac{v_n - v_0}{12g}\left(\frac{v_0}{1-\zeta\left(\frac{v_0}{w}\right)^2} + \frac{4v_1}{1-\zeta\left(\frac{v_1}{w}\right)^2} + \right.$$

$$\left. + \frac{2v_2}{1-\zeta\left(\frac{v_2}{w}\right)^2} + \frac{4v_3}{1-\zeta\left(\frac{v_3}{w}\right)^2} + \frac{v_4}{1-\zeta\left(\frac{v_4}{w}\right)^2}\right).$$

Of course the accuracy is greater, when we take six, eight, or more parts. This formula takes into account the variability of the co-efficients of resistance, which in considerable velocities is necessary. For the free descent of bodies in air or water $P_1 = G$, and for motion on a horizontal plane $P_1 = 0$, is more correctly equal to the friction $fG$. Since this is a resistance, we have then to introduce it as negative into the calculation, whence

$$P = -(P_1 + P_2),\ \text{and}\ p = -\left[1 + \zeta\left(\frac{v}{w}\right)^2\right]\frac{P_1}{G}\,g.$$

As it cannot be a question here of an increase, but only of a

$dt = \frac{G}{P_1 g} \cdot \frac{dv}{1 - \frac{S v^2}{\omega^2}}$ See § 393 Place $\frac{G}{P_1 g} = B$

$\& \ \frac{S}{\omega^2} = a^2 \ \therefore \int dt = B \frac{dv}{1 - a^2 v^2} = B \int \frac{dv}{1 - a^2 v^2}$ „

$\frac{1}{1 - a^2 v^2} = \frac{A}{1 + av} + \frac{A_1}{1 - av}$ „ $1 = A - Aav + A_1 + A_1 av$

$1 = A + A_1 \ \therefore A = 1 - A_1$ „ $0 = A_1 a - Aa$

$\therefore A = A_1 \ \therefore A_1 = \frac{1}{2}; A = \frac{1}{2}$

$\int dt = B \int^v \frac{dv}{1 - a^2 v^2} = B \left[ \int_0^v \frac{\frac{1}{2}\, dv}{1 + av} + \int_0^v \frac{\frac{1}{2}\, dv}{1 - av} \right]$

$= B \cdot \frac{1}{2a} \left[ \int_0^v \frac{dv}{\frac{1}{a} + v} + \int_0^v \frac{dv}{\frac{1}{a} - v} \right] = \frac{B}{2a} \left[ l\left(\frac{1}{a} + v\right) - l\frac{1}{a} - l\left(\frac{1}{a} - v\right) \right.$

$\left. + l \cdot \frac{1}{a} \right] = \frac{B}{2a} \left[ l(1 + av) - l(1 - av) \right] = \frac{B}{2a}\, l\left(\frac{1 + av}{1 - av}\right) = t$

$\therefore \frac{2at}{B} = l\left(\frac{1 + av}{1 - av}\right)$ „ $\varepsilon^{\frac{2at}{B}} = \frac{1 + av}{1 - av}$ „

$\varepsilon^{\frac{2at}{B}} - \varepsilon^{\frac{2at}{B}} av = 1 + av$ „ $\left(a + a\varepsilon^{\frac{2at}{B}}\right) v = \varepsilon^{\frac{2at}{B}} - 1$

$v = \left( \frac{\varepsilon^{\frac{2a}{B}} - 1}{\varepsilon^{\frac{2a}{B}} + 1} \right) \frac{1}{a}$ „ Substituting the ...

$2\sqrt{\frac{S}{\omega^2}} \frac{P_1 t}{G}$ $\qquad 2\sqrt{\frac{S F \gamma}{2 g P}} \cdot \frac{P_1 t}{G}$

$v = \frac{\varepsilon^{2\sqrt{\frac{S}{\omega^2}} \cdot \frac{P g t}{G}} - 1}{\varepsilon^{2\sqrt{\frac{S}{\omega^2}} \cdot \frac{P g t}{G}} + 1} \cdot \frac{\omega}{\sqrt{S}} = \frac{\varepsilon^{2\sqrt{\frac{S F \gamma}{2 g P}} \cdot \frac{P g t}{G}} - 1}{\varepsilon^{2\sqrt{\frac{S F \gamma}{2 g P}} \cdot \frac{P g t}{G}} + 1} \sqrt{\frac{2 g P_1}{S F \gamma}}$

$= \frac{\varepsilon^{\mu t} - 1}{\varepsilon^{\mu t} + 1} \sqrt{\frac{2 g F}{S F \gamma}}$ value of $v$ given in

§ 393 Remark

$ds = \frac{G}{P_1 g} \cdot \frac{v\,dv}{1 - \xi\frac{v^2}{w^2}}$ ,, See §393 put $\frac{G}{P_1 g} = B$ ...

$\int ds = B\int \frac{v\,dv}{1 - a^2 v^2}$ ,, As before, $\frac{v}{a^2 \ldots} = \frac{A}{1 \ldots}$

& we have $A - Aav + A_1 + A_1 av$ ,, $0 = A_+ \ldots$

$A_1 = -A$ $\qquad 1 = A_1 a - Aa$ ,, $A_1 = \frac{1}{a} + A = -A$

$\therefore \frac{1}{a} = -2A \therefore A = -\frac{1}{2a}$ ,, $A_1 = \frac{1}{2a}$

$\int ds = B\left[\int_0^{v_1} \frac{-1}{2a} \cdot \frac{dv}{1 + av} + \int_0^{v} \frac{1}{2a} \cdot \frac{dv}{1 - av}\right] = -B \cdot \frac{1}{2a^2}\left[l\left(\frac{1}{a} + v\right) \ldots\right.$

$\left. + l\left(\frac{1}{a} - v\right) - l\frac{1}{a}\right] = -B \cdot \frac{1}{2a^2}\left[l(1 + av) + l(1 - av)\right] = -B \ldots l$

Substituting for $B, a$ & $v$

$S = -\frac{G}{P_1 g} \cdot \frac{2P_1 g}{2Fy\xi}\, l\left[1 - \frac{\xi F y}{2gP_1} \cdot \frac{2gP_1}{\xi F y}\left(\frac{\varepsilon^{pt} - 1}{\varepsilon^{pt} + 1}\right)^2\right]$

$= -\frac{G}{Fy\xi} \cdot l\left[1 - \frac{(\varepsilon^{pt} - 1)^2}{(\varepsilon^{pt} + 1)^2}\right] = -\frac{G}{Fy\xi} \cdot l\left[\frac{(\varepsilon^{pt} + 1)^2 - (\ldots)}{(\varepsilon^{pt} + 1)^2}\right]$

$= -\frac{G}{Fy\xi}\, l\left[\frac{4\varepsilon^{pt}}{(\varepsilon^{pt} + 1)^2}\right] = \frac{G}{Fy\xi}\left[l(4\varepsilon^{pt}) + l\left(\frac{\ldots}{\varepsilon^{\ldots}}\right)\right]$

$= \frac{G}{Fy\xi} \cdot l\left[\frac{(\varepsilon^{pt} + 1)^2}{4\varepsilon^{pt}}\right] = S$ which is given

in §393 Remark.

diminution of velocity, we have then to substitute in the above formula $v_0 - v_n$ for $v_n - v_0$.

In the case, where the body is urged by a force, by its weight for instance, the motion approximates more and more to a uniform one, so that after the lapse of a certain time, it may be considered as such, although not so in reality. The accelerating force $p = 0$, when

$$\zeta \cdot \frac{v^2}{2g} F\gamma = P_1, \text{ when, therefore, } v = \sqrt{\frac{2g P_1}{\zeta F\gamma}}.$$

The velocity of a falling body approximates, therefore, to this limit more and more, without ever actually attaining it.

*Example.* Piobert, Morin, and Didion found, for a parachute whose depth was 0,31 that of the diameter of its opening, $\zeta = 1,94 \cdot 1,37 = 2,66$. Hence, from what height will a man, of 150 lbs. weight, be able to descend with a similar parachute, of 10 lbs. weight and 60 square feet transverse section, without acquiring a greater velocity than that which he would have acquired by jumping from a 10 feet height, without a parachute? The last velocity is $v = 8,03\sqrt{10} = 25,4$ feet, the force is $P_1 = G = 150 + 10 = 160$ lbs., the surface $F = 60$ square feet, the density $\gamma = 0,0859$, and the co-efficient of resistance $\zeta = 2,66$, hence:

$$\frac{1}{w^2} = \frac{60 \cdot 0,0859}{64,4 \cdot 160} = 0,000515, \text{ and } \zeta \cdot \frac{v^2}{w^2} = 2,66 \cdot 0,000515 \cdot 25^2 = 0,85625.$$

If, therefore, we take 6 parts, we then obtain for these:

$1 - \zeta \cdot \frac{v^2}{w^2} = 0,97621;\ 0,90486;\ 0,78593;\ 0,61944;\ 0,40537;\ 0,14375$, and for

$\dfrac{v}{1 - \zeta \frac{v^2}{w^2}} = 0;\ 4,268;\ 9,210;\ 15,905;\ 26,910;\ 51,393$, and $173,913$; from Simpson's rule the mean value is:

$$= (1 \cdot 0 + 4 \cdot 4,268 + 2 \cdot 9,210 + 4 \cdot 15,905 + 2 \cdot 26,910 + 4 \cdot 51,393 + 1 \cdot 173,913) : 3 \cdot 6$$

$= \frac{532,42}{18} = 29,58$; and from this the space of descent sought:

$$s = \frac{v_n - v_0}{g} \text{ times the mean of } \frac{v}{1 - \zeta \cdot \frac{v^2}{w^2}} = \frac{25 - 0}{32,25} \cdot 29,58 = 22,9 \text{ feet.}$$

The corresponding time of descent is, since the mean value of $\dfrac{1}{1 - \zeta \frac{v^2}{w^2}}$

$$= (1 \cdot 0 + 4 \cdot 1,024 + 2 \cdot 1,105 + 4 \cdot 1,272 + 2 \cdot 1,614 + 4 \cdot 2,467 + 1 \cdot 6,957) : 18$$

$$= 1,747,\ t = \frac{25}{32,2} \cdot 1,747 = 1,35 \text{ sec.}$$

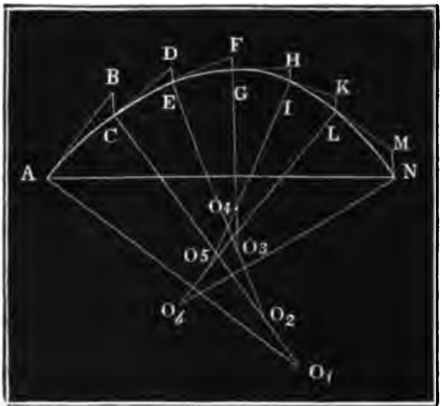
*Remark.* For a constant co-efficient of resistance, the higher calculus gives us:

$$v = \left(\frac{e^{\mu t} - 1}{e^{\mu t} + 1}\right) \sqrt{2g \frac{P}{\zeta F\gamma}} \text{ and } s = \frac{G}{\zeta F\gamma} L n \left(\frac{(e^{\mu t} + 1)^2}{4 e^{\mu t}}\right),$$

where $\mu = \sqrt{2 g \zeta \frac{P F \gamma}{G^2}}$, $e$ the base of the hyperbolic system of powers, and $L$ the hyperbolic logarithm.

§ 394. *Projectiles.*—We have already investigated the motion of projectiles in vacuo (§ 38), and found this motion to be parabolic, we may now obtain a more exact knowledge of motion in a resisting medium, and consider that, for instance, of a shot. In no case is the path $AGN$, Fig. 534, of a body passing through the air a symmetric curve; the portion $GN$ in which the body descends is rather shorter, and, therefore, less inclined than the portion $AG$ in which the body ascends, because the resistance of the air operating in the direction of motion tends always to shorten the portions of its path $AC$, $CE$, $EG$, &c., more and more; if, therefore, the first portion of the path $AC$, for motion in the air is only a little shorter than it would be in vacuo, the last portion $LN$ is considerably shorter in the first motion than it is in the last. The construction of the path in a resisting medium by means of circles of curvature may be accomplished in the following manner.

FIG. 534.




From the initial velocity $v_1$, and the angle of elevation $BAN = \alpha_1$, it follows that $\angle ABC = 90 - \alpha_1$, and $sin. ABC = cos. \alpha_1$, from § 40 the radius of curvature

$$O_1 A = O_1 C = r_1 = \frac{v_1^2}{g \, cos. \alpha_1},$$

hence with this we may approximately describe the portion of arc $AC$. If now we assume the angle subtended at the centre $AO_1C = \phi^0_1$, therefore $AC = s_1 = r_1 \phi_1$, we then obtain for the succeeding particle of space $CE$ the angle of inclination $\alpha^0_2 = \alpha^0_1 - \phi^0_1$. Let,

further, the height of fall $BC = h_1$, and the measure of the retardation due to the air's resistance $\zeta \cdot \frac{v_1^2}{2g} F\gamma$ being

$$\zeta \cdot \frac{v_1^2}{2g} \cdot \frac{F\gamma}{G} = \mu v_1^2, \text{ therefore } \zeta \cdot \frac{F\gamma}{2G} = \mu,$$

from the principle of living forces, we then obtain for the velocity $v_2$ at the initial point of the second portion of arc :

$$\frac{v_2^2}{2g} = \frac{v_1^2}{2g} - h_1 - \mu\left(\frac{v_1^2 + v_2^2}{2g}\right)s_1, \text{ or } (1 + \mu s_1)\frac{v_2^2}{2g}$$

$$= (1 - \mu s_1)\frac{v_1^2}{2g} - h_1, \text{ and hence } v_2 = \sqrt{\frac{(1 - \mu s_1)v_1^2 - 2gh_1}{1 + \mu s_1}}.$$

Since now the height of fall $h_1 = \frac{1}{2} g\tau^2 = \frac{1}{2} g \left(\frac{s_1}{v_1}\right)^2$, it follows that

$$v_2 = \sqrt{\frac{(1 - \mu s_1)v_1^2 - \left(\frac{gs_1}{v_1}\right)^2}{1 + \mu s_1}} = v_1 \sqrt{\frac{1 - \mu r_1 \phi_1 - \frac{\phi_1^2}{cos.\, a_1^2}}{1 + \mu r_1 \phi_1}}.$$

If we substitute these values of $a_2$ and $v_2$ in the equation: $r_2 = \frac{v_2^2}{g\ cos.\ a_2}$, we then obtain the radius of curvature $O_2C = O_2E$ of the succeeding portion of arc $CE$, and if we assume an angle of revolution $CO_2E = \phi_2$, it again follows from this that the angle of inclination in the vicinity of $E$: $a_3 = a_2 - \phi_2$, and the velocity at this point

$$v_3 = v_2 \sqrt{\frac{1 - \mu r_2 \phi_2 - \frac{\phi_2^2}{cos.\, a_2^2}}{1 + \mu r_2 \phi_2}}.$$

It is therefore easy to see how the entire path of the projectile may be successively composed of circular arcs.

*Example.* A cast-iron ball, of 4 inches diameter, is shot off at an angle of elevation of $50^0$ with a velocity of 1000 feet, required its path, if only approximately. The radius of curvature of the first portion of arc is $r_1 = \frac{v_1^2}{g\ cos.\ a} = \frac{1000000}{32{,}2\ cos.\ 50^0} = 48339$ ft. As the density of the air $= 0{,}0859$, and that of cast-iron $= 470$ lbs., we have then $\mu = \zeta \cdot \frac{F\gamma}{2G} = \zeta \cdot \frac{3 \,.\, 3 \,.\, 0{,}0859}{4 \,.\, 470} = 0{,}00041122 \,.\, \zeta$; now for $v = 1000$, $\zeta = 0{,}90$, hence $\mu$ 0,0003701. If in the first place we take an arc of $1^{\circ}$ only, we then obtain the velocity at the end of it:

$$v_2 = 1000 \sqrt{\frac{1 - 0{,}0003701 \,.\, 48339 \,.\, 0{,}017453 - (0{,}017453 \div cos.\ 50^{\circ})^2}{1 + 0{,}0003701 \,.\, 48339 \,.\, 0{,}017453}}$$

$= 747.3$ feet.

and the radius of curvature for a second portion of arc:

$$r_2 = \frac{(747,3)^2}{32,2 \; cos. \; 49^0} = 28035 \text{ feet.}$$

For $v_2 = 747,3$ feet, $\zeta = 0,81$, therefore $\mu = 0,0003331$. If, therefore, we describe with the last radius, an arc $\phi_2 = 2°$, the velocity at its ending point will be $v_3 = 564,6$ feet. For a third arc $Q_3$, the radius of curvature $r_3 = 14420$ feet, and if, therefore, we assume $\zeta = 0,75$, we shall then obtain at the end of a length of arc of 4°, the velocity $v_4 = 401,3$ feet. The radius of curvature for a fourth arc may be likewise found by assuming $\zeta = 0,72$, and we shall then obtain the velocity $v_5$ at the end of an arc of 8°, from which a fifth radius of curvature $r_5$ may be calculated. Proceeding in this manner, we shall obtain, by degrees, the collective elements for the construction of the line of projection in question.

# INDEX.

D.



E.



F.

END OF VOL. I.

# ERRATA.

---

Page 3, line 7, *for* 6 feet *read* 6 feet per second.
„ 8, „ 5, *for* 8,01 $\sqrt{s}$ *read* 8,03 $\sqrt{s}$.
„ 8, „ 6, *for* 2,480 $\sqrt{s}$ *read* 2,489 $\sqrt{s}$.
„ 9, „ 22, *for* height of velocity *read* height due to the velocity, and so on.
„ 11, „ 5, *for* where *a* *read* where $\alpha$.
„ 17, „ 4, *for* 31,25 *read* 32,2, *for* 36,25 *read* 38,7.
„ 18, „ 5, *for* relationship *read* ratio.
„ 29, line 2 from bottom, *for* rotation *read* relation.
„ 36, „ 2 from bottom, *for* 746 *read* 708.
„ 37, „ 2, *for* 727,48 *read* 7055,4.
„ 49, „ 2 from bottom, *for* weight *read* weights as.
„ 65, „ 2, *for* arm *read* arms.
„ 73, „ 8 from bottom, *for* enumerated *read* enunciated.
„ 79, „ 1, *for* diameter *read* radius, and in the following page.
„ 108, „ 11, *for* common $X\overline{X}$ *read* $X\overline{X}$ common.
„ 165, „ 4, *for* diameters *read* radii.
„ 169, „ 6, ditto ditto.
„ 177, „ 12, *for* English *read* Prussian.
„ 178, „ 17, *for* thus $r_3$ *read* in this.
„ 204, „ 4 from bottom, *for* less on the side *read* on the least side.
„ 204, „ 1, *for* straight lines *read* of the straight line *AD*.
„ 211, „ 10, *for* as prismatic *read* as for prismatic.
„ 217, „ 17, *for* flexion *read* flexure.
„ 224, „ 16 and 17 from bottom, *for* tension *read* torsion.
„ 234, „ 1, *for* vives vivæ *read* vires vivæ.
„ 242, „ 7, *for* dermine *read* determine.
„ 252, „ 3 from bottom, *for* acceleration rotatory *read* rotatory acceleration.
„ 271, „ 7, *for* this *read* the.
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## A TREATISE

ON

# METALLURGY
# AND THE CHEMISTRY OF THE METALS.


BY DR. E. RONALDS,

LECTURER ON CHEMISTRY AT THE MIDDLESEX HOSPITAL.

AND DR. THOMAS RICHARDSON,

OF NEWCASTLE-ON-TYNE.


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THE applications of the abundant supply of metallic ores, and the production from them of the several useful metals, although carried on to a vast extent in this country, have never been made the subject of a distinct Treatise in our language. No work exists to which the term *Metallurgy*, can with propriety, be applied.

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science will admit, has never yet been offered to the English public.

In endeavouring to fill up this gap in our literature, the plan we propose to follow is, perhaps, not so strictly scientific as that adopted by the Germans and French, the public of which countries have been familiar for nearly a century with scientific treatises on this important subject; but we believe that it will be more practically useful in this country, and more in accordance with the view we take of including Metallurgy as a branch of Chemical Technology. It also justifies us in excluding those metals of which no technical use has yet been made, or which will find a more appropriate place in some other division of our subject, while we properly include platinum, rhodium, iridium, osmium, nickel, manganese, uranium, tungsten, and chromium, which have been wholly, or partly, omitted by former authors.

Our arrangement will, therefore, embrace the following particulars:

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The following order will be observed in treating the special part of the subject, and the description of the metallurgical processes concerned in the production of each metal will be followed by an account of the manufacture of those of its compounds which deserve such notice from the extent of their production or general usefulness.

| | |
|---|---|
| Iron. | Iridium. |
| Tin. | Osmium. |

| | |
|---|---|
| Zinc. | Rhodium. |
| Cadmium. | Mercury. |
| Bismuth. | Antimony. |
| Copper. | Arsenic. |
| Lead. | Nickel. |
| Silver. | Cobalt. |
| Gold. | Chromium. |
| Platinum. | |

It is impossible to overstate the value of this branch of knowledge, embracing as it does one of the most important sources of the national wealth, in the development of which so much capital, enterprize, and skill, have been, and are being, daily expended, which profitably employs so many thousands of our fellow-countrymen, and which has so materially aided in raising this country to the rank it enjoys among the nations of the earth. The following table of the value of some of the important metals produced in this country, and for which we are indebted to the Editor of the "Mining Journal" will confirm the above statement:

| | |
|---|---|
| Iron | £8,400,000 |
| Copper | 1,200,000 |
| Lead | 920,000 |
| Tin | 390,000 |
| Manganese | 60,000 |
| Silver | 70,000 |
| Zinc | 8,000 |
| Antimony, Bismuth, Arsenic | 25,000 |

Although numerous improvements have been introduced in the processes for smelting ores, it is acknowledged on every hand that the progress in this branch of industry has not kept pace with some other chemical manufactures, and no doubt can exist that it is to be solely attributed to a deficiency of correct scientific knowledge. Had we possessed a *Bergacadamie*, like that at Freiberg, or an *Ecole des Mines*, as exists in Paris, such a complaint would not have been heard, nor should we have been so dependent upon the cheapness of our fuel and the facility of our transport.

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